ON UNSUPERSTABLE THEORIES IN GDST

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Abstract. We study the κ-Borel-reducibility of isomorphism relations of complete first-order theories by using coloured trees. Under some cardinality assumptions, we show the following: For all theories T and T', if T is classifiable and T' is unsuperstable, then the isomorphism of models of T' is strictly above the isomorphism of models of T with respect to κ-Borel-reducibility.

§1. Introduction. The interaction between Generalized Descriptive Set Theory (GDST) and Classification theory has been one of the biggest motivation to study the Borel reducibility in the Generalized Baire spaces. One of the main questions is to determine if there is a counterpart of Shelah's Main Gap Theorem in the Generalized Baire Spaces (provable in ZFC). In [9] Mangraviti and Motto Ros study this for classifiable shallow theories. In [6] Hyttinen, Weinstein (né Kulikov),[1] and Moreno showed the consistency of a counterpart of Shelah's Main Gap Theorem in the Borel reducibility hierarchy of the isomorphism relations (see preliminaries); indeed, it can be forced.

Fact 1.1 (Hyttinen–Kulikov–Moreno [6, Theorem 7]). *Suppose that* $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *There is a forcing notion* $\mathbb{P}$ *which forces the following statement*:

"If T_1 *is a classifiable theory and* T_2 *is not, then the isomorphism relation of* T_1 *is Borel reducible to the isomorphism relation of* T_2, *and there are* 2^κ *equivalence relations strictly between them."*

In the same article the authors proved the following in ZFC.

Fact 1.2 (Hyttinen–Kulikov–Moreno [6, Corollary 2]). *Suppose that* $\kappa = \kappa^{<\kappa} = \lambda^+$ *and* $\lambda^\omega = \lambda$. *If* T_1 *is classifiable and* T_2 *is stable unsuperstable, then the isomorphism relation of* T_1 *is Borel reducible to the isomorphism relation of* T_2.

In this article we will extend Fact 1.2 to unsuperstable theories, i.e., the unstable case.

Theorem A. *Suppose that* $\kappa = \kappa^{<\kappa} = \lambda^+$ *is such that* $\lambda^\omega = \lambda$. *If* T_1 *is classifiable and* T_2 *is unsuperstable, then the isomorphism relation of* T_1 *is Borel reducible to the isomorphism relation of* T_2.

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[1] Kulikov's last name changed to Weinstein.

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To prove Theorem A we will use the coloured trees tools developed in [5] by Hyttinen and Weinstein (né Kulikov), and the tools used by Shelah in [11], to construct models of unsuperstable theories. In [5] Hyttinen and Weinstein (né Kulikov) used the coloured trees to construct models of an already fixed stable unsuperstable theory in the context of the Generalized Baire spaces. In [11] Shelah used ordered trees with $\omega + 1$ levels to construct non-isomorphic models of unsuperstable theories.

The objective of Hyttinen and Weinstein (né Kulikov) was to use elements of κ^κ to construct models of the theory $T_{\omega+\omega}$, which is a stable unsuperstable theory. The difficulties with this construction appear when we want to apply it to unstable theories. Hyttinen and Weinstein (né Kulikov) constructed coloured trees for all the elements of κ^κ, such that the classes of the isomorphism of coloured trees are characterized by the classes of the equivalence modulo non-stationary. This is relevant when we construct a Borel reduction. In [4], similar trees were used to construct models of stable unsuperstable theories. In [4] the authors used the isolation of types on stable theories (see F^f_ω in [12, Chapter 4]). This is a limitation for unstable theories.

On the other hand, the objective of Shelah was to use stationary sets to construct as many models as possible for unsuperstable theories. Even though for each unsuperstable theory, Shelah constructs 2^κ models, this construction does not define a Borel reduction. The problem comes when the ordered trees are constructed.

In Section 3 we will combine Hyttinen–Kulikov's construction with Shela's construction. We use coloured trees to construct ordered trees; by doing this, we ensure that the construction of the models will define a continuous reduction. To construct the ordered trees from coloured trees we will use similar ideas to ones used by Abraham in [1] to construct a rigid Aronszajn tree.

In [2] Fernandes, Moreno, and Rinot showed that the isomorphism relation of unsuperstable theories can be forced to be analytically complete for κ a successor cardinal. We will extend this result to inaccessible cardinals.

THEOREM B. *Suppose that $\kappa = \kappa^{<\kappa}$ is an inaccessible cardinal. There exists a $< \kappa$-closed κ^+-cc forcing extension in which: If T is unsuperstable, then the isomorphism relation of T is analytically complete.*

1.1. Organization of this paper. In Section 2 we recall the notion of ordered trees of Shelah [11] and the notion of a $(<\kappa, bs)$-stable (κ, bs, bs)-nice linear order. The notion of colorable orders is introduced and its properties are studied. The notion of colorable linear orders is introduced to construct ordered trees in Section 3. In this section we prove the existence of a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order, which is crucial for constructing ordered trees from the coloured trees of Hyttinen–Kulikov [5].

In Section 3 we recall the notion of coloured trees of Hyttinen–Kulikov [5]. We use a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order to construct an ordered coloured tree A^f. An ordered coloured tree is both, an ordered tree as in [11] and a coloured tree as in [5]. We prove that $f =^\beta_\omega g$ holds if and only if $A^f \cong A^g$.

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In Section 4 we use the ordered coloured trees to construct generalized Ehrenfeucht–Mostowski models. In this section we prove Theorem A and Theorem B.

1.2. Preliminaries. During this paper we will work under the general assumption that κ is a regular uncountable cardinal that satisfies $\kappa = \kappa^{<\kappa}$ and for all $\gamma < \kappa$, $\gamma^{\omega} < \kappa$. We will work only with first-order countable complete theories in a countable language, unless something else is stated.

Let us recall some definitions and results on Generalized Descriptive Set Theory (from now on GDST); for more on GDST see [4]. We will only review the definitions and results that are relevant for the article.

The generalized Baire space is the set κ^{κ} endowed with the bounded topology; in this topology the basic open sets are of the form

$$[\zeta] = \{\eta \in \kappa^{\kappa} \mid \zeta \subseteq \eta\},$$

where $\zeta \in \kappa^{<\kappa}$. The collection of κ-Borel subsets of κ^{κ} is the smallest set that contains the basic open sets and is closed under union and intersection both of length κ. A κ-Borel set is any set of this collection.

A function $f : \kappa^{\kappa} \to \kappa^{\kappa}$ is κ-Borel, if for every open set $A \subseteq \kappa^{\kappa}$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of X. Let E_1 and E_2 be equivalence relations on κ^{κ}. We say that E_1 is *κ-Borel reducible to* E_2 if there is a κ-Borel function $f : \kappa^{\kappa} \to \kappa^{\kappa}$ that satisfies

$$(\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2.$$

We call f a reduction of E_1 to E_2 and we denote this by $E_1 \hookrightarrow_B E_2$. We will use this notation instead of $(\leq_B)$, because we will deal with the equivalence relations $=^{\beta}_S$ (Definition 1.3) and the notation could become heavy for the reader. In case f is continuous, we say that E_1 is continuously reducible to E_2 and we denote it by $E_1 \hookrightarrow_c E_2$.

A subset $X \subseteq \kappa^{\kappa}$ is a $\Sigma^1_1(\kappa)$ set of κ^{κ} if there is a closed set $Y \subseteq \kappa^{\kappa} \times \kappa^{\kappa}$ such that the projection $\mathrm{pr}(Y) := \{x \in \kappa^{\kappa} \mid \exists y \in \kappa^{\kappa},\ (x, y) \in Y\}$ is equal to X. These definitions also extend to the product space $\kappa^{\kappa} \times \kappa^{\kappa}$. An equivalence relation E is Σ^1_1-complete if E is a $\Sigma^1_1(\kappa)$ set and every $\Sigma^1_1(\kappa)$ equivalence relation R is Borel reducible to E.

The generalized Cantor space is the subspace 2^{κ}. Since in this article we will only work with κ-Borel and $\Sigma^1_1(\kappa)$ sets, we will omit κ, and refer to them as Borel and Σ^1_1.

Definition 1.3. Given $S \subseteq \kappa$ and $\beta \leq \kappa$, we define the equivalence relation $=^{\beta}_S \subseteq \beta^{\kappa} \times \beta^{\kappa}$, as follows:

$$\eta =^{\beta}_S \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

We will denote by $=^{\beta}_{\mu}$ the relation $=^{\beta}_S$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \mu\}$. Let us denote by CUB the club filter on κ and $=^{\beta}_{CUB}$ the relation $=^{\beta}_S$ when $S = \kappa$.

Definition 1.4. Let $\mathcal{L} = \{Q_m \mid m \in \omega\}$ be a countable relational language. Fix a bijection π between $\kappa^{<\omega}$ and κ. For every $\eta \in \kappa^{\kappa}$ define the structure $\mathcal{A}_{\eta}$ with

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domain κ as follows. For every tuple $(a_1, a_2, \dots, a_n)$ in κ^n,

$$(a_1, a_2, \dots, a_n) \in Q_m^{\mathcal{A}_\eta} \Leftrightarrow Q_m \text{ has arity } n \text{ and } \eta(\pi(m, a_1, a_2, \dots, a_n)) > 0.$$

DEFINITION 1.5. Assuming T is a first-order theory in a relational countable language, we define the isomorphism relation, $\cong_T \ \subseteq \kappa^\kappa \times \kappa^\kappa$, as the relation

$$\{(\eta, \xi) | (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

§2. Ordered trees.

2.1. Background. In [11], Shelah used ordered trees to construct non-isomorphic models. That construction was focused on obtaining non-isomorphic models. This is the reason why we have to modify the trees to adapt the construction to the generalized Cantor space and such that for all $f, g \in 2^\kappa$, f and g are $=^2_\omega$-equivalent if and only if the constructed models are isomorphic. Let us start by reviewing the trees used by Shelah.

Let γ be a countable ordinal, and we will denote by K^γ_{tr} the class of ordered trees with $\gamma + 1$ levels.

DEFINITION 2.1. Let K^γ_{tr} be the class of models $(A, \prec, (P_n)_{n \leq \gamma}, <, h)$, where:

(1) There is a linear order $(I, <_I)$ such that $A \subseteq I^{\leq \gamma}$.
(2) A is closed under initial segment.
(3) $\prec$ is the initial segment relation.
(4) $h(\eta, \xi)$ is the maximal common initial segment of η and ξ.
(5) Let $lg(\eta)$ be the length of η (i.e., the domain of η) and $P_n = \{\eta \in A \mid lg(\eta) = n\}$ for $n \leq \gamma$.
(6) For every $\eta \in A$ define $Suc_A(\eta)$ as $\{\xi \in A \mid \eta \prec \xi \wedge lg(\xi) = lg(\eta) + 1\}$. $<$ is $\bigcup_{\eta \in A}(< \restriction Suc_A(\eta))$, i.e., if $\xi < \zeta$, then there is $\eta \in A$ such that $\xi, \zeta \in Suc_A(\eta)$.
(7) For every $\eta \in A \backslash P_\gamma$, $< \restriction Suc_A(\eta)$ is the induced linear order from I, i.e.,
$$\eta^\frown\langle x \rangle < \eta^\frown\langle y \rangle \Leftrightarrow x <_I y.$$
(8) If η and ξ have no immediate predecessor and $\{\zeta \in A \mid \zeta \prec \eta\} = \{\zeta \in A \mid \zeta \prec \xi\}$, then $\eta = \xi$.

To construct the models of unsuperstable theories, Shelah study the types of the ordered trees. To do this study, the notions of κ-representation and CUB-invariant are crucial.

DEFINITION 2.2 (κ-representation). Let A be an arbitrary set of size at most κ. The sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha < \kappa} A_\alpha = A$.

DEFINITION 2.3 (CUB-invariant). A function $\mathcal{H}$ is CUB-invariant if the following holds:

- The domain of $\mathcal{H}$ is the class of κ-representations of the models of some model class K, where K contains only models of size at most κ.
- If $\mathbb{I}_1$ and $\mathbb{I}_2$ are κ-representations of $\mathcal{I}_1, \mathcal{I}_2 \in K$, respectively, and $\mathcal{I}_1 \cong \mathcal{I}_2$, then $\mathcal{H}(\mathbb{I}_1) =^2_{CUB} \mathcal{H}(\mathbb{I}_2)$.

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Let us define for every $\mathcal{H}$ CUB-invariant and $A \in K^{\omega}_{tr}$, $\mathcal{H}(A)$ as the $=^2_{CUB}$-equivalence class of any $\mathbb{A}$, κ-representation, i.e., $[\mathcal{H}(\mathbb{A})]_{=^2_{CUB}}$.

We will use some properties of formulas and types. For any $\mathcal{L}$-structure $\mathcal{A}$ we denote by *at* the set of atomic formulas of $\mathcal{L}$ and by *bs* the set of basic formulas of $\mathcal{L}$ (atomic formulas and negation of atomic formulas). For all $\mathcal{L}$-structures $\mathcal{A}$, $a \in \mathcal{A}$, and $B \subseteq \mathcal{A}$ we define

$$tp_{bs}(a, B, \mathcal{A}) = \{\varphi(x, b) \mid \mathcal{A} \models \varphi(a, b), \varphi \in bs, b \in B\}.$$

In the same way $tp_{at}(a, B, \mathcal{A})$ is defined.

DEFINITION 2.4. Let $\mathcal{A}$ be a model, $a \in \mathcal{A}$, and $B, D \subseteq \mathcal{A}$. We say that $tp_{bs}(a, B, \mathcal{A})$ (bs,bs)-splits over $D \subseteq \mathcal{A}$ if there are $b_1, b_2 \in B$ such that $tp_{bs}(b_1, D, \mathcal{A}) = tp_{bs}(b_2, D, \mathcal{A})$ but $tp_{bs}(a^\frown b_1, D, \mathcal{A}) \neq tp_{bs}(a^\frown b_2, D, \mathcal{A})$.

DEFINITION 2.5. Let $|A| \leq \kappa$, for a κ-representation $\mathbb{A}$ of A. Define $Sp_{bs}(\mathbb{A})$ as the set

$$\{\delta < \kappa \mid \delta \text{ a limit ordinal}, \exists a \in A\, [\forall \beta < \delta\ (tp_{bs}(a, A_\delta, A)\ (\text{bs,bs})\text{-splits over } A_\beta)]\}.$$

REMARK 2.6. The function Sp_{bs} is CUB-invariant; this was stated in [11, Remark 1.10A] and proved in [8, Lemma 8.6 and page 232 above Definition 8.8]. This is generally true under the assumption that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$, which is one of our cardinal assumptions on κ above.

DEFINITION 2.7.

- Let $\mathcal{A}$ be a model of size at most κ. We say that A is (κ, bs, bs)-nice if $Sp_{bs}(\mathbb{A}) =^2_{CUB} \emptyset$.
- $A \in K^{\omega}_{tr}$ of size at most κ, is locally (κ, bs, bs)-nice if for every $\eta \in A \backslash P^A_\omega$, $(Suc_A(\eta), <)$ is (κ, bs, bs)-nice, $Suc_A(\eta)$ is infinite, and there is $\xi \in P^A_\omega$ such that $\eta \prec \xi$.
- $A \in K^{\omega}_{tr}$ is $(< \kappa, bs)$-stable if for every $B \subseteq A$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, A) \mid a \in A\}|.$$

In [11], Shelah used $(<\kappa, bs)$-stable locally (κ, bs, bs)-nice ordered trees to construct the models of unsuperstable theories. In [8] Hyttinen and Tuuri give a very good example of a $(<\kappa, bs)$-stable (κ, bs, bs)-nice linear order, which is crucial for the construction of ordered trees.

DEFINITION 2.8 (Hyttinen–Tuuri [8, Definition 3.2]). Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$ for which $\{n \in \omega \mid f(n) \neq 0\}$ is finite. If $f, g \in \mathcal{R}$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

FACT 2.9 (Hyttinen–Tuuri [8, Lemma 8.17]).

- *The linear order $\mathcal{R}$ is $(< \kappa, bs)$-stable and (κ, bs, bs)-nice.*
- *There is a κ-representation $\langle R_\alpha \mid \alpha < \kappa \rangle$ and a club $C \subseteq \kappa$ such that for all $\delta \in C$ and $\nu \in \mathcal{R}$ there is $\beta < \delta$ which satisfies the following*:

$$\forall \sigma \in R_\delta [\sigma > \nu \Rightarrow \exists \sigma' \in R_\beta\ (\sigma \geq \sigma' \geq \nu)].$$

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2.2. Colorable orders. As it was mentioned in the previous subsection, the linear order plays a crucial role when we construct the ordered trees and therefore the models. For our purpose, constructing ordered trees from coloured trees, we will need to choose the right linear orders. The linear orders that we will use are the colorable linear orders.

DEFINITION 2.10. Let I be a linear order of size κ. We say that I is κ-colorable if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$, $|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa$.

We say that F is a κ-coloration of I, if F witnesses that I is a κ-colorable linear order.

Notice that I is a κ-colorable order if every type over a small set is realizable if and only if it is realizable by κ many elements. Under the assumption $\kappa^{<\kappa} = \kappa$ the saturated model of DLO of size κ is κ-colorable but it is not $(< \kappa, bs)$-stable (DLO is unstable). Clearly κ-colorable orders make us think of saturation. The interesting κ-colorable orders are those in which not all the types over small sets are realizable.

Although the saturated model of DLO of size κ (assuming it exists due to the cardinal assumptions) is κ-colorable, we cannot use it for our purpose. We need a $(< \kappa, bs)$-stable linear order. We will construct a κ-colorable linear order that is $(< \kappa, bs)$-stable; therefore, it is not κ-saturated (i.e., there are types over small sets that are not realized).

We will modify the order of Definition 2.8 to construct a $(< \kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.

DEFINITION 2.11. Let $\mathbb{Q}$ be the linear order of the rational numbers. Let $\kappa \times \mathbb{Q}$ be ordered by the lexicographic order, and let I^0 be the set of functions $f : \omega \to \kappa \times \mathbb{Q}$ such that $f(n) = (f_1(n), f_2(n))$, for which $\{n \in \omega \mid f_1(n) \neq 0\}$ is finite. If $f, g \in I^0$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

LEMMA 2.12. *There is a κ-representation $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ such that for all limit $\delta < \kappa$ and $\nu \in I^0$ there is $\beta < \delta$ which satisfies the following*:

I. $\forall \sigma \in I^0_\delta \ [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma \geq \sigma' \geq \nu)]$.

II. *If $\nu \notin I^0_\delta$, then* $\forall \sigma \in I^0_\delta \ [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma > \sigma' > \nu)]$.

PROOF. Let us start by defining the representation κ-representation $\langle I^0_\alpha \mid \alpha < \kappa \rangle$. For all $\gamma < \kappa$, let us define $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ by

$$I^0_\gamma = \{\nu \in I^0 \mid \nu_1(n) < \gamma \text{ for all } n < \omega\}$$

it is clear that $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation.

Let us show item II, i.e., $\nu \notin I^0_\delta$.

Suppose $\nu \notin I^0_\delta$. Let $\beta < \delta$ be $max\{\nu_1(i) \mid i < n\}$, where n is the least number such that $\nu_1(n) \geq \delta$.

CLAIM 2.12.1. *β is as wanted, i.e.,*

$$\forall \sigma \in I^0_\delta \ [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma > \sigma' > \nu)].$$

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PROOF. Let us suppose $\sigma \in I^0_\delta$ is such that $\sigma \geq \nu$. By the definition of I^0, there is $n < \omega$ such that $\sigma(n) > \nu(n)$ and n is the minimum number such that $\sigma(n) \neq \nu(n)$. Since $\sigma \in I^0_\delta$, for all $m \leq n$, $\nu_1(m) \leq \sigma_1(m) < \delta$. Thus for all $m \leq n$, $\nu_1(m) < \beta$.

Let us divide the proof in two cases, $\sigma_1(n) = \nu_1(n)$ and $\sigma_1(n) > \nu_1(n)$.

Case 1. $\sigma_1(n) = \nu_1(n)$.

By the density of $\mathbb{Q}$ there is r such that $\sigma_2(n) > r > \nu_2(n)$. Let us define σ' by

$$\sigma'(m) = \begin{cases} \nu(m), & \text{if } m < n, \\ (\nu_1(n), r), & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\sigma > \sigma' > \nu$. Since $\nu_1(m) < \beta$ for all $m \leq n$, $\sigma' \in I^0_\beta$.

Case 2. $\sigma_1(n) > \nu_1(n)$.

Let us define σ' by

$$\sigma'(m) = \begin{cases} \nu(m), & \text{if } m < n, \\ (\nu_1(n), \nu_2(n) + 1), & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\sigma > \sigma' > \nu$. Since $\nu_1(m) < \beta$ for all $m \leq n$, $\sigma' \in I^0_\beta$. ⊣

The previous claim proves item II. From the proof of this claim we can see that $\sigma \neq \sigma'$.

To prove item I, it is enough to prove the case $\nu \in I^0$.

Suppose $\delta < \kappa$ is a limit and $\nu \in I^0_\delta$. It is clear that there is $\beta < \delta$ such that $\nu \in I^0_\beta$ and the result follows. ⊣

Now let us used the order I^0 to construct a $(<\kappa, bs)$-stable, (κ, bs, bs)-nice, and κ-colorable linear order. Let us construct the linear orders $\langle I^i \mid i < \kappa \rangle$ by induction, such that for all $i < j$, $I^i \subseteq I^j$. Suppose $i < \kappa$ is such that I^i has been defined. For all $\nu \in I^i$ let ν^{i+1} be such that

$$\nu^{i+1} \models tp_{bs}(\nu, I^i \backslash \{\nu\}, I^i) \cup \{\nu > x\}. \tag{1}$$

Notice that ν^{i+1} is a copy of ν that is smaller than ν. Let $I^{i+1} = I^i \cup \{\nu^{i+1} \mid \nu \in I^i\}$.

Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, I^j has been defined, we define I^i by $I^i = \bigcup_{j<i} I^j$.

For all $i < \kappa$, let us define the κ-representation $\langle I^i_\alpha \mid \alpha < \kappa \rangle$ by induction as follows:

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa \rangle$ has been defined. For all $\alpha < \kappa$,

$$I^{i+1}_\alpha = I^i_\alpha \cup \{\nu^{i+1} \mid \nu \in I^i_\alpha\}.$$

Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, $\langle I^j_\alpha \mid \alpha < \kappa \rangle$ has been defined, we define $\langle I^i_\alpha \mid \alpha < \kappa \rangle$ by

$$I^i_\alpha = \bigcup_{j<i} I^j_\alpha.$$

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Finally, let us define I as

$$I = \bigcup_{j<\kappa} I^j$$

and the κ-representation $\langle I_\alpha \mid \alpha < \kappa\rangle$ as

$$I_\alpha = I^\alpha_\alpha.$$

The linear order I can be constructed in a non-inductive way. For every ν in I^0 we define a linear order L_ν, and we use I^0 to glue all these linear orders. To show this construction in more detail (it will be useful in the proof of Lemma 2.23) and be able to prove the main result of this section, we will need to develop the theory of I.

DEFINITION 2.13 (Generator). For all $\nu \in I$ let us denote by $o(\nu)$ the least ordinal $\alpha < \kappa$ such that $\nu \in I^\alpha$. Let us denote the generator of ν by $Gen(\nu)$ and define it by induction as follows:

- $Gen^i(\nu) = \emptyset$, for all $i < o(\nu)$.
- $Gen^i(\nu) = \{\nu\}$, for $i = o(\nu)$.
- For all $i \geq o(\nu)$,

$$Gen^{i+1}(\nu) = Gen^i(\nu) \cup \{\sigma \in I^{i+1} \mid \exists \tau \in Gen^i(\nu)\, [\tau^{i+1} = \sigma]\}.$$

- For all $i < \kappa$ limit,

$$Gen^i(\nu) = \bigcup_{j<i} Gen^j(\nu).$$

Finally, let

$$Gen(\nu) = \bigcup_{i<\kappa} Gen^i(\nu).$$

Notice that $o(\nu)$ is a successor ordinal for all ν. For clarity purposes let us fix the following notation.

Notation. For all $i < \kappa$ and $\sigma \in I^i$, we have defined σ^{i+1} (see (1) above) as the element generated by σ in I^{i+1}. We will also denote by $(\sigma)^{i+1}$ the element σ^{i+1}. This is to avoid a saturated notation, such as σ'^{i+1} when we work with the element generated by σ' in I^{i+1}.

FACT 2.14. *Suppose $\nu \in I$. For all $\sigma \in Gen(\nu)$, $\sigma \neq \nu$, there is $n < \omega$ and a sequence $\{\sigma_i\}_{i\leq n}$ such that the following holds*:

- $\sigma_0 = \nu$.
- *For all $j < n$,*

$$\sigma_{j+1} = (\sigma_j)^{o(\sigma_{j+1})}.$$

- $\sigma = \sigma_n = (\sigma_{n-1})^{o(\sigma)}$.

PROOF. Let $\sigma \neq \nu$ be such that $\sigma \in Gen(\nu)$. From Definition 2.13, we know that $\sigma \in Gen^{o(\sigma)}(\nu)$ and $Gen^{o(\sigma)}(\nu) \subseteq I^{o(\sigma)}$. Let us proceed by induction on $o(\sigma)$. Notice that $o(\nu) < o(\sigma)$, so the induction starts with the case $o(\sigma) = o(\nu) + 1$. Since $o(\sigma)$ is a successor ordinal, the limit step of the induction is not required.

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For the case $o(\sigma) = o(\nu) + 1$ it is easy to see from Definition 2.13 that $\sigma = \nu^{o(\nu)+1}$. Thus $\sigma_0 = \nu$ and $\sigma = \sigma_1 = (\sigma_0)^{o(\sigma_1)}$ is the desire sequence, and $n = 1$.

Let $o(\sigma) = i + 1 > o(\nu) + 1$ be such that for any $\tau \in Gen^i(\nu)$ there are $n < \omega$ and a sequence $\{\tau_j\}_{j \leq n}$ such that the following holds:

- $\tau_0 = \nu$;
- for all $j < n$,

$$\tau_{j+1} = (\tau_j)^{o(\tau_{j+1})};$$

- $\tau = \tau_n = (\tau_{n-1})^{o(\tau)}$.

We know that $\sigma \in Gen^{o(\sigma)}(\nu) = Gen^{i+1}(\nu)$. By Definition 2.13, there is $\tau \in Gen^i(\nu)$ such that $\tau^{i+1} = \sigma$. We conclude that $n + 1 < \omega$ and the sequence $\{\sigma_i\}_{i \leq n+1}$ defined by:

- $\sigma_0 = \tau_0 = \nu$;
- for all $j \leq n$, $\sigma_j = \tau_j$;
- $\sigma = \sigma_{n+1} = (\tau_n)^{i+1}$

are as wanted. ⊣

For every $\nu \in I$, $\sigma \in Gen(\nu)$, and $\sigma \neq \nu$, we call the sequence $\{\sigma_i\}_{i \leq n}$ of the previous fact, *the road from ν to σ*. It is clear that for all $\nu \in I \backslash I^0$, there is $\nu' \in I^0$ such that $\nu \in Gen(\nu')$. Notice that for all $\nu \in I$, if $\sigma \in Gen(\nu)$, then ν and σ have the same type of basic formulas over $I^{o(\nu)} \backslash \{\nu\}$. Even more, if $\{\sigma_i\}_{i \leq n}$ is the road from ν to σ, then for all $i < n$, σ_i and σ have the same type of basic formulas over $I^{\gamma} \backslash \{\sigma_i\}$, where $o(\sigma_{i+1}) = \gamma + 1$. Let us define the road from ν to ν by $\{\nu\}$.

It is clear that I is the orders $Gen(\nu)$, for $\nu \in I^0$, glued by I^0. Let us show the non-inductive construction of I in more detail.

Let us fix $\nu \in I^0$, $\sigma \in Gen(\nu)$, and let $\{\nu_i\}_{i \leq n}$ be the road from ν to σ. Let us define $f_\sigma : \omega \to \kappa$ by

$$f_\sigma(i) = \begin{cases} o(\nu_i), & \text{if } i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for all $\sigma, \sigma' \in Gen(\nu)$, f_σ and $f_{\sigma'}$ are equal if and only if the road from ν to σ is the same road from ν to σ'. Thus $f_\sigma = f_{\sigma'}$ if and only if $\sigma = \sigma'$. Since the road from ν to σ is finite, $\{i < \omega \mid f_\sigma(i) \neq 0\}$ is finite.

Let $\sigma, \sigma' \in Gen(\nu)$, and i the least number such that $f_\sigma(i) \neq f_{\sigma'}(i)$. By the construction of I, $\sigma > \sigma'$ holds if and only if one of the following holds:

- $f_\sigma(i) = 0$.
- $f_\sigma(i) > f_{\sigma'}(i)$.

From the previous discussion on the functions f_σ, we can conclude that for all $\nu, \nu' \in I^0$, the orders and $(Gen(\nu'), <)$ are isomorphic. Even more, this holds for all $\sigma, \sigma' \in I$.

DEFINITION 2.15 (Generator order). Let Gen be the set of functions $f : \omega \to \kappa$ such that the following holds:

- $f(0) = 0$.
- For all $n < \omega$, $f(n)$ is either 0 or a successor ordinal.

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- There is $n < \omega$ such that for all $m > n$, $f(m) = 0$.
- $f \restriction n+1 \setminus \{0\}$ is strictly increasing.

Let $f, g \in Gen$ and i the least number such that $f(i) \neq g(i)$. Let us define $<_{Gen}$ as follows: $g <_{Gen} f$ if and only if one of the following holds:

- $f(i) = 0$.
- $g(i) < f(i)$.

From the discussion above, it is clear that for all $v \in I^0$, $(Gen(v), <)$ and $Gen, <_{Gen}$ are isomorphic. Therefore I is isomorphic to $I^0 \times Gen$ with the lexicographic order. Notice that I is the orders $L_v = \{v\} \times Gen$ glued by I^0; in particular L_v and $Gen(v)$ are isomorphic.

Now we proceed with the study of other properties of I. All the properties of I that we will prove, can be proved using $I^0 \times Gen$. Nevertheless, we will use the inductive construction in the proofs, to provide an intuitive point of view.

FACT 2.16. *Let i, δ, v be such that $v \in I^i_\delta$. Then for all $\sigma \in Gen(v)$, $\sigma \in I^{o(\sigma)}_\delta$. In particular for all $j < \kappa$*

$$\sigma \notin I^j_\delta \Rightarrow \sigma \notin I^j.$$

PROOF. It follows from the construction of $I^{o(\sigma)}$ and the κ-representation $\langle I^{o(\sigma)}_\alpha \mid \alpha < \kappa \rangle$. ⊣

FACT 2.17. *For all $v, \sigma \in I$, $\sigma \in Gen(v)$, if $\sigma' \in I$ is such that $v \geq \sigma' \geq \sigma$, then $\sigma' \in Gen(v)$.*

PROOF. If $v = \sigma$, the result follows. Thus we only need to prove the case $v \neq \sigma$. Let us suppose towards contradiction that $\sigma' \notin Gen(v)$.

Case $o(v) = o(\sigma')$. Since v and σ have the same type of basic formulas over $I^{o(v)} \setminus \{v\}$, v and σ have the same type of basic formulas over $I^{o(\sigma')} \setminus \{v\}$. Since $v \geq \sigma' \geq \sigma$, $v = \sigma'$ a contradiction.

Case $o(\sigma') < o(v)$. Since $v \geq \sigma'$, there is $v' \neq \sigma'$ such that $v' > v$, $o(v') = o(\sigma')$ and $v \in Gen(v')$. Thus v', σ', and σ satisfy $v' \geq \sigma' \geq \sigma$, $o(v') = o(\sigma')$, and $\sigma \in Gen(v')$. The result follows from the previous case.

Case $o(v) < o(\sigma')$. There is $\sigma^0 \in I$ such that $\sigma^0 > \sigma'$, $o(\sigma^0) = o(v)$ and $\sigma' \in Gen(\sigma^0)$. If $v \geq \sigma^0 \geq \sigma$, then the result follows from the previous cases. Therefore, we are only missing the case $\sigma^0 \geq v \geq \sigma' \geq \sigma$. Since σ^0 and σ' have the same type of basic formulas of basic formulas over $I^{o(\sigma^0)} \setminus \{\sigma^0\}$, $\sigma^0 = v$ and $\sigma' \in Gen(v)$ a contradiction. ⊣

From the previous fact we can conclude that for all $v, \sigma \in I$ such that $\sigma \in Gen(v)$, v and σ have the same type of basic formulas over $I \setminus Gen(v)$.

LEMMA 2.18. *For all $i < \kappa$, $\delta < \kappa$ a limit ordinal, and $v \in I^i$, there is $\beta < \delta$ that satisfies the following*:

I. $\forall \sigma \in I^i_\delta \, [\sigma > v \Rightarrow \exists \sigma' \in I^i_\beta \, (\sigma \geq \sigma' \geq v)]$.

II. *In particular, for all $i < \kappa$, $\delta < \kappa$ a limit ordinal, and $v \in I^i \setminus I^i_\delta$, there is $\beta < \delta$ that satisfies the following*:

$$\forall \sigma \in I^i_\delta \, [\sigma > v \Rightarrow \exists \sigma' \in I^0_\beta \, (\sigma > \sigma' > v)].$$

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PROOF. Notice that if $\nu \in I^i_\delta$, then there is $\theta < \delta$ such that $\nu \in I^i_\theta$ and the result follows for $\beta = \theta$. So we only have to prove the lemma when $\nu \in I^i \backslash I^i_\delta$ (the second part of the lemma).

We will proceed by induction over i. The case $i = 0$ is precisely Lemma 2.12 II. Let us suppose $i < \kappa$ is such that for all limit ordinal $\delta < \kappa$ and $\nu \in I^i \backslash I^i_\delta$, there is $\beta < \delta$ that satisfies II. Let $\delta < \kappa$ be a limit ordinal and $\nu \in I^{i+1} \backslash I^{i+1}_\delta$. We have two cases, $\nu \in I^i$ and $\nu \in I^{i+1} \backslash I^i$.

Case $\nu \in I^i$. By the induction hypothesis, we know that there is $\beta < \delta$ such that II holds. Let us prove that this β is the one we are looking for. Let $\sigma \in I^{i+1}_\delta$ be such that $\sigma > \nu$. The subcase $\sigma \in I^i_\delta$ follows from the way β was chosen.

Subcase $\sigma \in I^{i+1}_\delta \backslash I^i_\delta$. By the construction of I^{i+1}, there is $\sigma_0 \in I^i_\delta$ such that $\sigma = (\sigma_0)^{i+1}$ (so $\sigma_0 > \sigma$). Thus $\sigma_0 > \sigma > \nu$, and by the way β was chosen, there is $\sigma' \in I^0_\beta$ such that $\sigma_0 > \sigma' > \nu$. Since σ_0 and σ have the same type of basic formulas over $I^i \backslash \{\sigma_0\}$, $\sigma > \sigma' > \nu$ as we wanted.

Case $\nu \in I^{i+1} \backslash I^i$. By the construction of I^{i+1}, there is $\nu_0 \in I^i$ such that $(\nu_0)^{i+1} = \nu$. Since $\nu \in Gen(\nu_0)$ and $\nu \in I^{i+1} \backslash I^{i+1}_\delta$, by Fact 2.16 $\nu_0 \in I^i \backslash I^i_\delta$. Thus, by the previous case, there is $\beta < \delta$ such that for all $\sigma \in I^{i+1}_\delta$,

$$\sigma > \nu_0 \Rightarrow \exists \sigma' \in I^0_\beta \, (\sigma > \sigma' > \nu_0).$$

Let us show that this β is as wanted.

CLAIM 2.18.1. *If* $\sigma \in I^{i+1}_\delta$ *is such that* $\sigma > \nu$, *then* $\sigma > \nu_0$.

PROOF. Let us suppose, towards contradiction, that there is $\sigma \in I^{i+1}_\delta$ such that $\nu_0 > \sigma > \nu$. Since ν_0 and ν have the same type of basic formulas over $I^i \backslash \{\nu_0\}$, $\sigma \in I^{i+1}_\delta \backslash I^i$. Therefore, there is $\sigma_0 \in I^i$ such that $(\sigma_0)^{i+1} = \sigma$. Since $\sigma \in Gen(\sigma_0)$ and $\sigma \in I^{i+1}_\delta$, $\sigma_0 \in I^i_\delta$. We conclude that $\sigma_0 \neq \nu_0$. Finally, σ_0 and σ have the same type of basic formulas over $I^i \backslash \{\sigma_0\}$, which implies $\nu_0 > \sigma_0 > \sigma > \nu$. This contradicts the fact that ν_0 and ν have the same type of basic formulas over $I^i \backslash \{\nu_0\}$. ⊣

From the previous claim, we know that for all $\sigma \in I^{i+1}_\delta$, $\sigma > \nu$ implies $\sigma > \nu_0$. By the way β was chosen we conclude that for all $\sigma \in I^{i+1}_\delta$, $\sigma > \nu$ implies the existence of $\sigma' \in I^0_\beta$ such that $\sigma > \sigma' > \nu_0 > \nu$, as we wanted.

Let us proceed with the limit case. Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, for all limit ordinal $\delta < \kappa$, and $\nu \in I^j \backslash I^j_\delta$, there is $\beta < \delta$ such that II holds for j. Let $\delta < \kappa$ be a limit ordinal and $\nu \in I^i \backslash I^i_\delta$. Since i is a limit, $o(\nu) < i$, by the induction hypothesis, there is β such that II holds for $o(\nu)$.

CLAIM 2.18.2. β *is as wanted.*

PROOF. Let $\sigma \in I^i_\delta$ be such that $\sigma > \nu$.

Case $\sigma \in I^{o(\nu)}_\delta$. This case follows from the way β was chosen.

Case $\sigma \in I^i_\delta \backslash I^{o(\nu)}_\delta$. There is $\sigma_0 \in I^{o(\nu)}_\delta$ such that $\sigma \in Gen(\sigma_0)$, with road to σ equal to $\{\sigma_i\}_{i \leq n}$ such that $\sigma_1 \notin I^{o(\nu)}$. Therefore σ_0 and σ have the same type of basic formulas over $I^\gamma \backslash \{\sigma_0\}$, where $o(\sigma_1) = \gamma + 1$. In particular σ_0 and σ have the same type of basic formulas over $I^{o(\nu)} \backslash \{\sigma_0\}$. By the way β was chosen, there is

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$\sigma' \in I^0_\beta \subseteq I^{o(\nu)}_\beta$ such that $\sigma_0 > \sigma' > \nu$. Since σ_0 and σ have the same type of basic formulas over $I^{o(\nu)} \backslash \{\sigma_0\}$, $\sigma > \sigma' > \nu$ as wanted. ⊣

⊣

As it can be seen in the previous lemma, the witness σ' can be chosen in I^0_β, when $\nu \notin I^i_\delta$.

LEMMA 2.19. *For all $\delta < \kappa$ limit, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following*:

$$\forall \sigma \in I_\delta \, [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \, (\sigma \geq \sigma' \geq \nu)].$$

PROOF. Let $\delta < \kappa$ be a limit ordinal, and $\nu \in I$. We have three different cases: $\nu \in I_\delta$, $\nu \in I^{o(\nu)}_\delta \backslash I_\delta$, and $\nu \notin I^{o(\nu)}_\delta$.

Case $\nu \in I_\delta$. Since δ is a limit, $o(\nu) < \delta$ and there is $\theta < \delta$ such that $\nu \in I^{o(\nu)}_\theta$. Let $\beta = max\{o(\nu), \theta\}$; it is clear that β is as wanted.

Case $\nu \in I^{o(\nu)}_\delta \backslash I_\delta$. Recall $I_\delta = I^\delta_\delta$; clearly $\delta < o(\nu)$. There is $\nu_0 \in I_\delta$, such that $\nu \in Gen(\nu_0)$, with the road to ν equal to $\{\nu_i\}_{i \leq n}$, and $\nu_1 \notin I^\delta$. Since $\nu_0 \in I^\delta_\delta$ and δ is a limit, $o(\nu_0) < \delta$ and there is $\theta < \delta$ such that $\nu_0 \in I^{o(\nu_0)}_\theta$. Let $\beta = max\{o(\nu_0), \theta\}$.

CLAIM 2.19.1. *β is as wanted.*

PROOF. Let $\sigma \in I^\delta_\delta$ be such that $\sigma > \nu$. Since $\nu_1 \notin I^\delta$, $o(\nu_1) = \gamma + 1 > \delta$, and ν_0 and ν have the same type of basic formulas over $I^\gamma \backslash \{\nu_0\}$. In particular ν_0 and ν have the same type of basic formulas over $I^\delta \backslash \{\nu_0\}$, so $\sigma > \nu_0 > \nu$. Since $\nu_0 \in I^\beta_\beta$, $\sigma' = \nu_0$ is as wanted. ⊣

Case $\nu \notin I^{o(\nu)}_\delta$. Let $\theta = max\{o(\nu), \delta\}$; thus $\nu \in I^\theta$ (notice that we are talking about the order I^θ and not the element I_θ of the κ-representation $\langle I_\alpha \mid \alpha < \kappa \rangle$) and by Lemma 2.18 there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^\theta_\delta [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \, (\sigma > \sigma' > \nu)].$$

CLAIM 2.19.2. *β is as wanted.*

PROOF. Let $\sigma \in I^\delta_\delta$ be such that $\sigma > \nu$. Since $\delta \leq \theta$, $\sigma \in I^\theta_\delta$. Therefore, there is $\sigma' \in I^0_\beta$ such that $\sigma > \sigma' > \nu$. The claim follows from $I^0_\beta \subseteq I^\beta_\beta = I_\beta$. ⊣

⊣

FACT 2.20 (Hyttinen–Tuuri [8, Lemma 8.12]). *Let A be a linear order of size κ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ a κ-representation. Then the following are equivalent*:

(1) *A is (κ, bs, bs)-nice.*
(2) *There is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$, for all $x \in A$ there is $\beta < \delta$ such that one of the following holds*:
   - $\forall \sigma \in A_\delta [\sigma \geq x \Rightarrow \exists \sigma' \in A_\beta \, (\sigma \geq \sigma' \geq x)]$.
   - $\forall \sigma \in A_\delta [\sigma \leq x \Rightarrow \exists \sigma' \in A_\beta \, (\sigma \leq \sigma' \leq x)]$.

The previous fact is stated as it is in [8]. Due to Lemma 2.19, the second bullet point of item (2) is not needed for our purposes. The following corollary follows from Lemma 2.19.

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COROLLARY 2.21. *I is (κ, bs, bs)-nice.*

Notice that if κ is inaccessible, I is $(< \kappa, bs)$-stable. This can be generalize to successors κ.

LEMMA 2.22. *Suppose $\kappa = \lambda^+$. I^0 is $(< \kappa, bs)$-stable.*

PROOF. Recall the linear order $\mathcal{R}$ from Definition 2.8. From the general assumption on κ, we know that $\lambda^\omega = \lambda$.

For all $A \subseteq I^0$ define $Pr(A)$ as the set $\{f_1 \mid f \in A\}$. Let $A \subseteq I^0$ be such that $|A| < \kappa$. Since $|\mathbb{Q}| = \omega$, $|\{tp_{bs}(a, A, I^0) \mid a \in I^0\}| \leq |\{tp_{bs}(a, Pr(A), \mathcal{R}) \mid a \in \mathcal{R}\} \times 2^\omega|$. By Fact 2.9 and since $\lambda^\omega = \lambda$, $|\{tp_{bs}(a, A, I^0) \mid a \in I\}| < \kappa$. ⊣

LEMMA 2.23. *Suppose $\kappa = \lambda^+$. I is $(< \kappa, bs)$-stable.*

PROOF. Let us fix $A \subset I$ such that $|A| < \kappa$. From Fact 2.17, for all $a \in I$ and $v \in I^0$ such that $a \in Gen(v)$ the following holds:

$$b \models tp_{bs}(a, A, I) \Leftrightarrow b \models tp_{bs}(v, A \backslash Gen(v), I) \cup tp_{bs}(a, A \cap Gen(v), Gen(v)).$$

Thus for all $a \in I$ and $v \in I^0$ with $a \in Gen(v)$, the type of a is determined by $tp_{bs}(v, A \backslash Gen(v), I)$ and $tp_{bs}(a, A \cap Gen(v), Gen(v))$. Let $A' \subseteq I^0$ be such that the following hold:

- For all $x \in A$ there is $y \in A'$, $x \in Gen(y)$.
- For all $y \in A'$ there is $x \in A$, $x \in Gen(y)$.

Clearly $|A'| \leq |A|$, and by Fact 2.17, for all $v \in I^0$, $tp_{bs}(v, A \backslash Gen(v), I)$ is determined by $tp_{bs}(v, A' \backslash \{v\}, I^0)$. So for all $a \in I$ and $v \in I^0$ with $a \in Gen(v)$, $tp_{bs}(a, A, I)$ is determined by $tp_{bs}(v, A' \backslash \{v\}, I^0)$ and $tp_{bs}(a, A \cap Gen(v), Gen(v))$. Therefore $|\{tp_{bs}(a, A, I) \mid a \in I\}|$ is bounded by

$$|\{tp_{bs}(v, A', I^0) \mid v \in I^0\}| \times Sup(\{\alpha_v \mid v \in I^0\}),$$

where

$$\alpha_v = |\{tp_{bs}(a, A \cap Gen(v), Gen(v)) \mid a \in Gen(v)\}|.$$

CLAIM 2.23.1. *For all $v \in I^0$, $Gen(v)$ with the induced order is $(< \kappa, bs)$-stable.*

PROOF. Recall the order $(Gen, <_{Gen})$. By the non-inductive construction of I, it is enough to show that $(Gen, <_{Gen})$ is $(<\kappa, bs)$-stable.

Let $D \subseteq Gen$ be such that $|D| < \kappa$, and let

$$\beta = sup\{f(n) + 1 \mid f \in D \ \& \ n < \omega\}.$$

Since for all $f \in A$, f is constant to 0 starting at some m, $\beta < \kappa$. On the other hand, for all $f, g \in Gen$, f and g eventually become constants to 0, and the order $f <_{Gen} g$ (or $g <_{Gen} f$) is determined by the values of $f(i)$ and $g(i)$, where i is the least ordinal such that $f(i) \neq g(i)$. Therefore, for all $f \in Gen$, $tp_{bs}(f, D, Gen)$ is entirely determined by the coordinates n of f in which $f(n)$ is smaller than $\beta + 1$. Since $\lambda^\omega = \lambda$, and $\beta < \kappa$, $|\{tp_{bs}(f, D, Gen) \mid f \in Gen\}| \leq |\beta^{<\omega}| \leq \lambda < \kappa$. ⊣

From the previous claim, we conclude that for all $v \in I^0$, $\alpha_v < \kappa$. Since $\kappa = \lambda^+$, $Sup(\{\alpha_v \mid v \in I^0\}) \leq \lambda$. From Lemma 2.22 we know that $|\{tp_{bs}(v, A', I^0) \mid v \in$

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$I^0\}| < \kappa$, so $|\{tp_{bs}(v, A', I^0) \mid v \in I^0\}| \leq \lambda$. We conclude $|\{tp_{bs}(a, A, I) \mid a \in I\}| < \kappa$. ⊣

THEOREM 2.24. *There is a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.*

PROOF. From Corollary 2.21 and Lemma 2.23, we only need to show that I is κ-colorable. For all $v \in I$ let us define $Succ_I(v)$ as follows:

$$Succ_I(v) = \{\sigma \in I \mid \sigma = v^{o(\sigma)}\}.$$

We use the same notation of ordered trees because I can be seen as an ordered tree. Notice that for all $v \in I$, $|Succ_I(v)| = \kappa$ and either $o(v) = 0$, or there is a unique $v' \in I$ such that $v = (v')^{o(v)}$ (i.e. $v \in Succ_I(v')$).

Let us fix $G : \kappa \to \kappa \times \kappa$ a bijection, and G_1, G_2 be the functions such that $G(\alpha) = (G_1(\alpha), G_2(\alpha))$. For all $v \in I$ let us fix a bijection $g_v : Succ_I(v) \to \kappa$. Let us define $F : I \to \kappa$ by

$$F(v) = \begin{cases} 0, & \text{if } o(v) = 0, \\ G_1(g_{v'}(v)), & \text{where } (v')^{o(v)} = v. \end{cases}$$

CLAIM 2.24.1. *F is a κ-coloration of I.*

PROOF. Let $B \subseteq I$, $|B| < \kappa$, $b \in I \setminus B$, and $p = tp_{bs}(b, B, I)$. Since $|B| < \kappa$, there is $\gamma < \kappa$ such that $B \subset I^\gamma$. Let $\theta = \max\{o(b), \gamma\}$, so for all $v \in \{a \in Succ_I(b) \mid o(a) > \theta\}$, b and v have the same type of basic formulas over $I^\theta \setminus \{b\}$. In particular for all $v \in \{a \in Succ_I(b) \mid o(a) > \theta\}$, $v \models p$. By the way F was defined, we conclude that for any $\alpha < \kappa$, $|\{a \in Succ_I(b) \mid o(a) > \theta \ \&\ F(a) = \alpha\}| = \kappa$. Which implies that for any $\alpha < \kappa$, $|\{a \in Succ_I(b) \mid a \models p \ \&\ F(a) = \alpha\}| = \kappa$. ⊣

⊣

§3. Ordered coloured trees.

3.1. Coloured trees. We will use the κ-colorable linear order I to construct trees with $\omega + 1$ levels, $A^f(I)$, for every $f \in \kappa^\kappa$ with the property $A^f(I) \cong A^g(I)$ if and only if $f =^\kappa_\omega g$. These tress will be a mix of coloured tree and ordered trees. For clarity and to avoid misunderstandings, in this section we will denote trees by $(T, \prec)$. Later on we will see that $\prec$ is the initial segment relation of the trees that we construct. The coloured trees that we will use in this section, are essentially the same trees used by Hyttinen and Weinstein (né Kulikov) in [5] and by Hyttinen and Moreno in [7].

Let t be a tree; for every $x \in t$ we denote by $ht(x)$ the height of x, the order type of $\{y \in t | y \prec x\}$. Define $(t)_\alpha = \{x \in t | ht(x) = \alpha\}$ and $(t)_{<\alpha} = \cup_{\beta<\alpha}(t)_\beta$, denote by $x \restriction \alpha$ the unique $y \in t$ such that $y \in (t)_\alpha$ and $y \prec x$. If $x, y \in t$ and $\{z \in t | z \prec x\} = \{z \in t | z \prec y\}$, then we say that x and y are $\sim$-related, $x \sim y$, and we denote by $[x]$ the equivalence class of x for $\sim$.

An α, β-tree is a tree t with the following properties:

- $|[x]| < \alpha$ for every $x \in t$.
- All the branches have order type less than β in t.
- t has a unique root.
- If $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

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DEFINITION 3.1. Let λ be a cardinal smaller than κ, and β a cardinal smaller than or equal to κ. A coloured tree with β colors is a pair (t, c), where t is a κ^+, $(\lambda+2)$-tree and c is a map $c : (t)_\lambda \to \beta$ (the color function).

Two coloured trees (t, c) and (t', c') are isomorphic, if there is a tree isomorphism $f : t \to t'$ such that for every $x \in (t)_\lambda$, $c(x) = c'(f(x))$. We will denote by $\cong_{ct}$ the isomorphism of coloured trees

We will only consider trees in which every element with height less than λ, has infinitely many immediate successors, and every maximal branch has order type $\lambda+1$. Notice that the intersection of two distinct branches has order type less than λ. We can see every coloured tree as a downward closed subset of $\kappa^{\leq\lambda}$. In this section all the coloured trees have $\lambda = \omega$.

An ordered coloured tree with β colors, $\beta \leq \kappa$, is a tree $T \in K^\omega_{tr}$ with a color function $c : (t)_\omega \to \beta$.

We will follow the construction used in [5, 7].

Let us start from coloured trees which are subsets of $(\omega \times \kappa^4)^{\leq\omega}$, and let us make some preparation before the actual construction. Order the set $\omega \times \kappa \times \kappa \times \kappa \times \kappa$ lexicographically, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) > (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ if for some $1 \leq k \leq 5$, $\alpha_k > \theta_k$ and for every $i < k$, $\alpha_i = \theta_i$. Order the set $(\omega \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\omega}$ as a tree by initial segments.

For all $f \in \beta^\kappa$, define the tree (R_f, r_f) as: R_f be the set of all strictly increasing functions from some $n \leq \omega$ to κ, and r_f be the color function such that for each η with domain ω, $r_f(\eta) = f(sup(rng(\eta)))$.

For every pair of ordinals α and θ, $\alpha < \theta < \kappa$, and $i < \omega$, define

$$R(\alpha, \theta, i) = \bigcup_{i<j\leq\omega} \{\eta : [i, j) \to [\alpha, \theta) \mid \eta \text{ strictly increasing}\}.$$

DEFINITION 3.2. If $\alpha < \theta < \kappa$ and $\alpha, \theta, \gamma \neq 0$, let $\{Z^{\alpha,\theta}_\gamma | \gamma < \kappa\}$ be an enumeration of all downward closed subtrees of $R(\alpha, \theta, i)$ for all i, in such a way that each possible coloured tree appears cofinally often in the enumeration. Let $Z^{0,0}_0$ be the tree (R_f, r_f).

This enumeration is possible because there are at most $|\bigcup_{i<\omega} \mathcal{P}(R(\alpha, \theta, i))| \leq \omega \times \kappa = \kappa$ downward closed coloured subtrees. Since for all $\theta < \kappa$, $|R(\alpha, \theta, i)| < \kappa$ there are at most $\kappa \times \kappa^{<\kappa} = \kappa$ coloured trees.

DEFINITION 3.3. Let $\beta \leq \kappa$ be a cardinal. Define for every $f \in \beta^\kappa$ the coloured tree (J_f, c_f) with β colors, by the following construction. Let $J_f = (J_f, c_f)$ as the tree of all $\eta : s \to \omega \times \kappa^4$, where $s \leq \omega$, ordered by endextension, and such that the following conditions hold for all $i, j < s$: Denote by η_i, $1 < i < 5$, the functions from s to κ that satisfies

$$\eta(n) = (\eta_1(n), \eta_2(n), \eta_3(n), \eta_4(n), \eta_5(n)).$$

(1) $\eta\restriction_n \in J_f$ for all $n < s$.
(2) η is strictly increasing with respect to the lexicographical order on $\omega \times \kappa^4$.
(3) $\eta_1(i) \leq \eta_1(i+1) \leq \eta_1(i) + 1$.
(4) $\eta_1(i) = 0$ implies $\eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.
(5) $\eta_1(i) < \eta_1(i+1)$ implies $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$.

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(6) $\eta_1(i) = \eta_1(i+1)$ implies $\eta_k(i) = \eta_k(i+1)$ for $k \in \{2,3,4\}$.
(7) If for some $k < \omega$, $[i,j) = \eta_1^{-1}\{k\}$, then

$$\eta_5{\restriction}_{[i,j)} \in Z^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}.$$

Note that (7) implies $Z^{\eta_2(i),\eta_3(i)}_{\eta_4(i)} \subset R(\alpha,\theta,i)$.
(8) If $s = \omega$, then either
  (a) there exists a natural number m such that $\eta_1(m-1) < \eta_1(m)$, for every $k \geq m\, \eta_1(k) = \eta_1(k+1)$, and the color of η is determined by $Z^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$:

$$c_f(\eta) = c(\eta_5{\restriction}_{[m,\omega)}),$$

  where c is the coloring function of $Z^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$,
  or
  (b) there is no such m and then $c_f(\eta) = f(sup(rng(\eta_5)))$.

Notice that for every $f \in \beta^\kappa$ and $\delta < \kappa$ with $cf(\delta) = \omega$, there is $\eta \in J_f$ such that $rng(\eta_1) = \omega$ and η_5 is cofinal to δ. This η can be constructed by taking $\langle \xi(i) \mid i < \omega \rangle$ a cofinal sequence to δ, let $\eta_1 = id$; let η_2, η_3, and η_4 be such that for every $i < \omega$, $\xi{\restriction}\{i\} \in Z^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}$. Finally let $\eta_5{\restriction}\{i\} = \xi{\restriction}\{i\}$. It is clear that $\eta \in J_f$, $rng(\eta_1) = \omega$, and η_5 is cofinal to δ. In particular this η satisfies $c_f(\eta) = f(\delta)$.

FACT 3.4 (Hyttinen–Kulikov [5, Lemma 2.5]; Hyttinen–Moreno [7, Lemma 4.7]). *Suppose κ is such that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$. For every $f, g \in \beta^\kappa$ the following holds*:

$$f \ =^\beta_\omega \ g \Leftrightarrow J_f \cong_{ct} J_g,$$

where $\cong_{ct}$ is the isomorphism of coloured trees.

The previous fact is an important step in [5, 7] to construct a reductions from $=^2_\omega$ to the isomorphism relation of different stable unsuperstable theories. We will use the coloured trees J_f to construct ordered coloured trees. Before we start with the construction of the ordered coloured trees, let us prove an important property of the coloured trees.

LEMMA 3.5. *For every $f \in \beta^\kappa$, $\theta < \beta$, and $\eta \in (J_f)_{<\omega}$, there is $\xi \in (J_f)_\omega$ such that $\eta \prec \xi$ and $c_f(\xi) = \theta$.*

PROOF. Let $f \in \beta^\kappa$, such that $\eta \in (J_g)_{<\omega}$, and $n = dom(\eta)$.
Let us construct ξ, $\eta \prec \xi$ and $c_f(\xi) = \theta$.

- $\xi{\restriction}n = \eta$.
- If $n \leq m < \omega$,
  - $\xi_1(m) = \xi_1(n-1) + 1$.
  - $\xi_2(m) = \xi_3(n-1) + \xi_4(n-1)$.
  - $\xi_3(m) = \xi_2(n) + \omega$.
  - Let γ and ζ be such that $dom(\zeta) = [n,\omega)$, $\zeta \in Z^{\xi_2(n),\xi_3(n)}_\gamma$ with $c(\zeta) = \theta$. Such γ and ζ exist by Definition 3.2.
  - $\xi_4(m) = \gamma$.
  - $\xi_5{\restriction}_{[n,\omega)} = \zeta$.

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By the way we defined ξ, we know that $\xi \in J_f$ and $\eta \prec \xi$. By the item (8)(a) on the construction of J_f, we know that $c_f(\xi) = c(\xi_5{\restriction}_{[n,\omega)}) = \theta$. ⊣

Notice that for any $f, g \in \beta^\kappa$, J_f and J_g are isomorphic as trees but not as coloured trees. This is because f is only used to define the color function of J_f.

3.2. Construction of ordered coloured trees. For each $f \in \beta^\kappa$ we will use the coloured trees J_f to construct ordered coloured trees, which will be the base for the construction of the models in Section 4.

Let us define the following subtrees:

$$J_f^\alpha = \{\eta \in J_f \mid \exists \theta < \alpha\ (rng(\eta) \subset \omega \times \theta^4)\}.$$

Notice that $J_f^0 = \{\emptyset\}$ and $dom(\emptyset) = 0$. Let us denote by $\mathrm{acc}(\kappa) = \{\alpha < \kappa \mid \alpha = 0$ or α is a limit ordinal$\}$. For all $\alpha \in \mathrm{acc}(\kappa)$ and $\eta \in J_f^\alpha$ with $dom(\eta) = m < \omega$ define

$$W_\eta^\alpha = \{\zeta \mid dom(\zeta) = [m, s), m \leq s \leq \omega, \eta^\frown\zeta \in J_f^{\alpha+\omega}, \eta^\frown(\zeta \restriction \{m\}) \notin J_f^\alpha\}.$$

Notice that by the way J_f was constructed, for every $\eta \in J_f$ with finite domain and $\alpha < \kappa$, the set

$$\{(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in (\omega \times \kappa^4) \backslash (\omega \times \alpha^4) \mid \eta^\frown(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in J_f^{\alpha+\omega}\}$$

is either empty or has size ω. Let σ_η^α be an enumeration of this set, when this set is not empty.

Let us denote by $\mathcal{T} = (\kappa \times \omega \times \mathrm{acc}(\kappa) \times \omega \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\omega}$. For every $\xi \in \mathcal{T}$ there are functions $\{\xi_i \in \kappa^{\leq\omega} \mid 0 < i \leq 8\}$ such that for all $i \leq 8$, $dom(\xi_i) = dom(\xi)$ and for all $n \in dom(\xi)$, $\xi(n) = (\xi_1(n), \xi_2(n), \xi_3(n), \xi_4(n), \xi_5(n), \xi_6(n), \xi_7(n), \xi_8(n))$. For every $\xi \in \mathcal{T}$ let us denote $(\xi_4, \xi_5, \xi_6, \xi_7, \xi_8)$ by $\overline{\xi}$.

Definition 3.6. For all $\alpha \in \mathrm{acc}(\kappa)$ and $\eta \in \mathcal{T}$ with $\overline{\eta} \in J_f$, $dom(\eta) = m < \omega$ define Γ_η^α as follows:

If $\overline{\eta} \in J_f^\alpha$, then Γ_η^α is the set of elements ξ of $\mathcal{T}$ such that:

(1) $\xi{\restriction}m = \eta$,
(2) $\overline{\xi}{\restriction}dom(\xi)\backslash m \in W_\eta^\alpha$,
(3) ξ_3 is constant on $dom(\xi)\backslash m$,
(4) $\xi_3(m) = \alpha$,
(5) for all $n \in dom(\xi)\backslash m$, let $\xi_2(n)$ be the unique $r < \omega$ such that $\sigma_\zeta^\alpha(r) = \overline{\xi}(n)$, where $\zeta = \overline{\xi}{\restriction}n$.

If $\overline{\eta} \notin J_f^\alpha$, then $\Gamma_\eta^\alpha = \emptyset$.

Notice that $\xi_2(n)$ and $\xi_3(n)$ can be calculated from $\overline{\xi}(n)$ and η.

For $\eta \in \mathcal{T}$ with $\overline{\eta} \in J_f$, $dom(\eta) = m < \omega$ define

$$\Gamma(\eta) = \bigcup_{\alpha \in \mathrm{acc}(\kappa)} \Gamma_\eta^\alpha.$$

Finally we can define A^f by induction. Let $T_f(0) = \{\emptyset\}$ and for all $n < \omega$,

$$T_f(n+1) = T_f(n) \cup \bigcup_{\eta \in T_f(n)\ dom(\eta) = n} \Gamma(\eta),$$

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and for $n = \omega$,

$$T_f(\omega) = \bigcup_{n<\omega} T_f(n).$$

For $0 < i \leq 8$ let us denote by $s_i(\eta) = sup\{\eta_i(n) \mid n < \omega\}$ and $s_\omega(\eta) = sup\{s_i(\eta) \mid i \leq 8\}$, finally

$$A^f = T_f(\omega) \cup \{\eta \in \mathcal{T} \mid dom(\eta) = \omega, \forall m < \omega(\eta \restriction m \in T_f(\omega))\}.$$

Define the color function d_f by

$$d_f(\eta) = \begin{cases} c_f(\overline{\eta}), & \text{if } s_1(\eta) < s_\omega(\eta), \\ f(s_1(\eta)), & \text{if } s_1(\eta) = s_\omega(\eta). \end{cases} \tag{2}$$

It is clear that A^f is closed under initial segments; indeed, the relations $\prec$, $(P_n)_{n\leq\omega}$, and h of Definition 2.1 have a canonical interpretation in A^f.

Now we finish the construction of A^f by using the κ-colorable linear order I. We only have to define $< \restriction Suc_{A^f}(\eta)$ for all $\eta \in A^f$ with finite domain. Properly speaking, A^f will not be an ordered coloured tree as in Definition 2.1, but it will be isomorphic to an ordered coloured tree as in Definition 2.1.

Let us proceed to define $< \restriction Suc_{A^f}(\eta)$. Let $F : I \to \kappa$ be a κ-coloration of I.

For any $\eta \in A^f$ with domain $m < \omega$, we will define the order $< \restriction Suc_{A^f}(\eta)$ such that it is isomorphic to I and satisfies the following:

$(*)$ *For any set $B \subset Suc_{A^f}(\eta)$ of size less than κ, $p(x)$ a type of basic formulas over B in the variable x, and any tuple $(\theta_2, \theta_3) \in \omega \times \text{acc}(\kappa)$ with $\theta_3 \geq \eta_3(m-1)$, if $p(x)$ is realized in $Suc_{A^f}(\eta)$, then there are κ many $\gamma < \kappa$ such that $\eta^\frown(\gamma, \theta_2, \theta_3, \sigma_{\overline{\eta}}^{\theta_3}(\theta_2)) \models p(x)$.*

By the construction of A^f, an isomorphism between $\{(\theta_1, \theta_2, \theta_3) \in \kappa \times \omega \times \text{acc}(\kappa) \mid \theta_3 \geq \eta_3(m-1)\}$ and I, induces an order in $Suc_{A^f}(\eta)$.

Definition 3.7. Recall the coloration F of I in Theorem 2.24. For all $\theta, \alpha < \kappa$, let fix bijections $\tilde{G}_\theta : \{(\theta_2, \theta_3) \in \omega \times \text{acc}(\kappa) \mid \theta_3 \geq \theta\} \to \kappa$ and $\tilde{H}_\alpha : F^{-1}[\alpha] \to \kappa$. Notice that these functions exist because F is a κ-coloration of I and there are κ tuples (θ_2, θ_3) of this form.

Let us define $\tilde{\mathcal{G}}_\theta : \{(\theta_1, \theta_2, \theta_3) \in \kappa \times \omega \times \text{acc}(\kappa) \mid \theta_3 \geq \theta\} \to I$, by $\tilde{\mathcal{G}}_\theta((\theta_1, \theta_2, \theta_3)) = a$ where a and α are the unique elements that satisfy:

- $\tilde{G}_\theta((\theta_2, \theta_3)) = \alpha$;
- $\tilde{H}_\alpha(a) = \theta_1$.

For any $\eta \in A^f$ with domain $m < \omega$ and $\eta_3(m-1) = \theta$, the isomorphism $\tilde{\mathcal{G}}_\theta$ induces an order in $Suc_{A^f}(\eta)$. Let us define $< \restriction Suc_{A^f}(\eta)$ as the induced order given by $\tilde{\mathcal{G}}_\theta$.

Fact 3.8. *Suppose $\eta \in A^f$ has domain $m < \omega$ and $\eta_3(m-1) = \theta$. Then $< \restriction Suc_{A^f}(\eta)$ satisfies $(*)$.*

Proof. Let $b \in Suc_{A^f}(\eta)$, $(\theta_2, \theta_2) \in \omega \times \text{acc}(\kappa)$ such that $\theta_3 \geq \eta_3(m-1) = \theta$, and $B \subseteq Suc_{A^f}(\eta)$ have size less than κ. Let us denote by q the type

$$tp_{bs}(\tilde{\mathcal{G}}_\theta(b_1, b_2, b_3), \tilde{\mathcal{G}}_\theta(B \cap (\kappa \times \omega \times \text{acc}(\kappa))), I).$$

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By the construction of $\tilde{G}_\theta$, since F is a κ-coloration of I,

$$|\{a \in I \mid a \models q \ \& \ F(a) = \tilde{G}_\theta(\theta_2, \theta_3)\}| = \kappa.$$

Therefore, for all a such that $a \models q$ and $F(a) = \tilde{G}_\theta(\theta_2, \theta_3)$,

$$\eta^\frown(\tilde{H}_{\tilde{G}_\theta((\theta_2,\theta_3))}(a), \theta_2, \theta_3, \sigma^{\theta_3}_{\overline{\eta}}(\theta_2)) \models p. \qquad \dashv$$

It is clear that $(A^f, \prec, (P_n)_{n\leq\omega}, <, h)$ is isomorphic to a subtree of $I^{\leq\omega}$ in the sense of Definition 2.1.

REMARK 3.9. Notice that for any $\eta \in A^f$, $< \restriction Suc_{A^f}(\eta)$ is isomorphic to I. Therefore, for any $\zeta, \eta \in A^f$, $< \restriction Suc_{A^f}(\zeta)$ and $< \restriction Suc_{A^f}(\eta)$ are isomorphic. Even more, the construction of $< \restriction Suc_{A^f}(\eta)$ only depends on $\eta_3(m-1)$, where $m < \omega$ is the domain of η.

Notice that the only property we used from I to construct the ordered coloured trees was that it is a κ-colorable linear order. Therefore the construction can be done with any κ-colorable linear order.

THEOREM 3.10. *Suppose $f, g \in \beta^\kappa$, then $f =^\beta_\omega g$ if and only if $A^f \cong A^g$ (as ordered coloured trees).*

PROOF. For every $f \in \beta^\kappa$ let us define the κ-representation $\mathbb{A}^f = \langle A^f_\alpha \mid \alpha < \kappa\rangle$ of A^f,

$$A^f_\alpha = \{\eta \in A^f \mid rng(\eta) \subseteq \theta \times \omega \times \theta \times \omega \times \theta^4 \text{ for some } \theta < \alpha\}.$$

Let f and g be such that $f =^\beta_\omega g$. Thus, there is G a coloured tree isomorphism between J_f and J_g. Let $C \subseteq \kappa$ be a club such that $\{\alpha \in C \mid cf(\alpha) = \omega\} \subseteq \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$. We will show that there are sequences $\{\alpha_i\}_{i<\kappa}$ and $\{F_i\}_{i<\kappa}$ with the following properties:

- $\{\alpha_i\}_{i<\kappa}$ is a club.
- If i is a successor, then there is $\theta \in C$ such that $\alpha_{i-1} < \theta < \alpha_i$.
- If $i = \gamma + n$ and n is odd, F_i is a partial isomorphism between A^f and A^g, and $A^f_{\alpha_i} \subseteq dom(F_i)$.
- If $i = \gamma + n$ and n is even, F_i is a partial isomorphism between A^f and A^g, and $A^g_{\alpha_i} \subseteq rng(F_i)$.
- If i is limit, then $F_i : A^f_{\alpha_i} \to A^g_{\alpha_i}$.
- If $i < j$, then $F_i \subseteq F_j$.
- For all $\eta \in dom(F_i)$, $G(\overline{\eta}) = \overline{F_i(\eta)}$.

We will proceed by induction over i, for the case $i = 0$, let $\alpha_0 = 0$ and $F_0(\emptyset) = \emptyset$. Suppose $i = \gamma + n$ with n even is such that F_i is a partial isomorphism, $A^g_{\alpha_i} \subseteq rng(F_i)$ for all $j < i$, $F_j \subseteq F_i$, and $G(\overline{\eta}) = \overline{F_i(\eta)}$ for all $\eta \in dom(F_i)$.

Let us choose α_{i+1} to be a successor ordinal such that $\alpha_i < \theta < \alpha_{i+1}$ holds for some $\theta \in C$ and enumerate $A^f_{\alpha_i}$ by $\{\eta_j \mid j < \Omega\}$ for some $\Omega < \kappa$. Denote by B_j the set $\{x \in A^f_{\alpha_{i+1}} \backslash dom(F_i) \mid \eta_j \prec x\}$.

By the induction hypothesis, we know that for all $j < \Omega$, $x \in B_j$, $\overline{F_i(\eta_j)} \prec G(\overline{x})$. By Remark 3.9, for all $\eta \in A^f$ and $\xi \in A^g$, $< \restriction Suc_{A^f}(\eta)$ and $< \restriction Suc_{A^g}(\xi)$ are

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isomorphic. Thus, since $|A^f_{\alpha_i}|, |B_0| < \kappa$, by $(*)$ there is an embedding F^0_i from $(A^f_{\alpha_i} \cup B_0, \prec, <)$ to $(A^g, \prec, <)$ that extends F_i and for all $\eta \in dom(F^0_i)$, $\overline{F^0_i(\eta)} = G(\overline{\eta})$.

For the case B_j for $j > 0$, let us suppose that $t < \Omega$ is such that the following hold:

- There is a sequence of embeddings $\{F^j_i \mid j < t\}$, where F^j_i is an embedding from $(A^f_{\alpha_i} \cup \bigcup_{l \leq j} B_l, \prec, <)$ into A^g.
- $F^l_i \subseteq F^j_i$ holds for all $l < j < t$.
- For all $\eta \in dom(F^j_i)$, $\overline{F^j_i(\eta)} = G(\overline{\eta})$.

Since $|A^f_{\alpha_i} \cup \bigcup_{j<t} B_j|, |B_t| < \kappa$, by $(*)$ there is an embedding F^t_i from $(A^f_{\alpha_i} \cup \bigcup_{j \leq t} B_j, \prec, <)$ to $(A^g, \prec, <)$ that extends $\bigcup_{j<t} F^j_i$ and for all $\eta \in dom(F^t_i)$, $\overline{F^t_i(\eta)} = G(\overline{\eta})$.

Finally $F_{i+1} = \bigcup_{j<\Omega} F^j_i$ is as wanted.

The case $i = \gamma + n$ with n odd is similar. For i limit, we define $\alpha_i = \bigcup_{j<i} \alpha_j$ and $F_{\alpha_i} = \bigcup_{j<i} F_j$.

It is clear that $F = \bigcup_{j<\kappa} F_j$ witnesses that A^f and A^g are isomorphic as ordered trees. Let us show that $d_f(\eta) = d_g(F(\eta))$, and suppose $\eta \in A^f$ is a leaf. Let l be the least ordinal such that $\eta \in A^f_{\alpha_l}$. If there is $n < \omega$ such that for all $j < l$, $\eta{\restriction}n \notin A^f_{\alpha_j}$, then by the way F was constructed, $d_f(\eta) = d_g(F(\eta))$. On the other hand, if for all $n < \omega$ there is $j < l$ such that $\eta{\restriction}n \in A^f_{\alpha_j}$, then there is an ω-cofinal ordinal i such that $s_\omega(\eta) = \alpha_i$ and $i + 1 = l$.

By the construction of A^f (recall equation (2)) we know that

$$d_f(\eta) = \begin{cases} c_f(\overline{\eta}), & \text{if } s_1(\eta) < s_\omega(\eta), \\ f(s_1(\eta)), & \text{if } s_1(\eta) = s_\omega(\eta). \end{cases}$$

Since $s_\omega(\eta) = \alpha_i$, either $d_f(\eta) = f(s_1(\eta))$ (if $s_1(\eta) = \alpha_i$) or $d_f(\eta) = c_f(\overline{\eta})$ (if $s_1(\eta) < \alpha_i$).

Therefore, if $s_1(\eta) = \alpha_i$, then $d_f(\eta) = f(\alpha_i)$.

Let us calculate $d_f(\eta)$, when $s_1(\eta) < s_\omega(\eta)$. Notice that $\overline{\eta} \in J_j$, so there is $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)$ such that $\overline{\eta} = \zeta \in J_j$.

From Definition 3.3 items (5) and (7), since $\zeta \in (J_f)_\omega \backslash J^{\alpha_i}_f$ and for all $n < \omega$, $\zeta{\restriction}n \in J^{\alpha_i}_f$ holds,

$$sup(rng(\zeta_4)) \leq sup(rng(\zeta_2)) = sup(rng(\zeta_3)) = sup(rng(\zeta_5)).$$

Since $\overline{\eta} = \zeta$,

$$s_7(\eta) \leq s_8(\eta) = s_6(\eta) = s_8(\eta) = sup(rng(\zeta_5).$$

It is easy to see that $s_2(\eta), s_3(\eta), s_4(\eta) \leq s_5(\eta)$.

We conclude that $s_\omega(\eta) = s_8(\eta) = sup(rng(\zeta_5))$ and $\alpha_i = sup(rng(\zeta_5))$. From Definition 3.3(8),

$$c_f(\overline{\eta}) = c_f(\zeta) = f(sup(rng(\zeta_5))) = f(\alpha_i).$$

Therefore $d_f(\eta) = f(\alpha_i)$ in both cases ($s_1(\eta) = s_\omega(\eta)$ and $s_1(\eta) < s_\omega(\eta)$).

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By the same argument and using the definition of F, we can conclude that $d_g(F(\eta)) = g(\alpha_i)$. Finally since i is a limit ordinal with cofinality ω, α_i is an ω-limit of C. Thus $d_f(\eta) = f(\alpha_i) = g(\alpha_i) = d_g(F(\eta))$ and F is a coloured tree isomorphism.

Now let us prove that if A^f and A^g are isomorphic ordered coloured trees, then $f =^{\beta}_{\omega} g$.

Let us start by defining the following function $H_f \in \beta^\kappa$. For every $\alpha \in \kappa$ with cofinality ω, define $B_\alpha = \{\eta \in A^f \backslash A^f_\alpha \mid dom(\eta) = \omega \wedge \forall n < \omega\, (\eta \restriction n \in A^f_\alpha)\}$. Notice that by the construction of A^f and the definition of A^f_α, for all $\eta \in B_\alpha$ we have $d_f(\eta) = f(s_\omega(\eta)) = f(\alpha)$. Therefore, the value of $f(\alpha)$ can be obtained from B_α and d_f, and we can define the function $H_f \in \beta^\kappa$ as:

$$H_f(\alpha) = \begin{cases} f(\alpha), & \text{if } cf(\alpha) = \omega, \\ 0, & \text{otherwise.} \end{cases}$$

This function can be obtained from the κ-representation $\{A^f_\alpha\}_{\alpha<\kappa}$ and d_f. It is clear that $f =^{\beta}_{\omega} H_f$.

Claim 3.10.1. *If A^f and A^g are isomorphic ordered coloured trees, then $H_f =^{\beta}_{\omega} H_g$.*

Proof. Let F be an ordered coloured tree isomorphism. It is easy to see that $\{F[A^f_\alpha]\}_{\alpha<\kappa}$ is a κ-representation. Define $C = \{\alpha < \kappa \mid F[A^f_\alpha] = A^g_\alpha\}$. Since F is an isomorphism, for all $\alpha \in C$, $H_f(\alpha) = H_g(\alpha)$. Therefore it is enough to show that C is ω-closed and unbounded. By the definition of κ-representation, if $(\alpha_n)_{n<\omega}$ is a sequence of elements of C cofinal to γ, then $A^g_\gamma = \bigcup_{n<\omega} A^g_{\alpha_n} = \bigcup_{n<\omega} F[A^f_{\alpha_n}] = F[A^f_\gamma]$. We conclude that C is ω-closed.

Let us finish by showing that C is unbounded. Fix an ordinal $\alpha < \kappa$, and let us construct a sequence $(\alpha_n)_{n\leq\omega}$ such that $\alpha_\omega \in C$ and $\alpha_\omega > \alpha$. Define $\alpha_0 = \alpha$. For every odd n, define α_{n+1} to be the least ordinal bigger than α_n such that $F[A^f_{\alpha_n}] \subseteq A^g_{\alpha+1}$. For every even n, define α_{n+1} to be the least ordinal bigger than α_n such that $A^g_{\alpha_n} \subseteq F[A^f_{\alpha+1}]$. Define $\alpha_\omega = \bigcup_{n<\omega} \alpha_n$. Clearly $\bigcup_{i<\omega} F[A^f_{\alpha_{2i}}] = \bigcup_{i<\omega} A^g_{\alpha_{2i+1}}$. We conclude that $\alpha_\omega \in C$. ⊣

⊣

Remark 3.11. Same as in the construction of the coloured trees J_f, the function $f \in \beta^\kappa$ is only used to define the color function in the construction of A^f. So if $f, g \in \beta^\kappa$ and α are such that $f \restriction \alpha = g \restriction \alpha$, then $J^\alpha_f = J^\alpha_g$. As a consequence $f \restriction \alpha = g \restriction \alpha$ implies that $A^f_\alpha = A^g_\alpha$.

Notice that the only property of $< \restriction Suc_{A^f}(\eta)$ that we used in the previous theorem was $(*)$. Therefore, the previous theorem can be generalized to the following corollary.

Corollary 3.12. *Suppose l is a κ-colorable linear order and $\beta \leq \kappa$. Then for any $f \in \beta^\kappa$, there is an ordered coloured tree $A^f(l)$ that satisfies: For all $f, g \in \beta^\kappa$,*

$$f =^{\beta}_{\omega} g \Leftrightarrow A^f(l) \cong A^g(l).$$

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§4. The models.

4.1. Generalized Ehrenfeucht–Mostowski models. In this section we will use the generalized Ehrenfeucht–Mostowski models (see [12, Chapter VII.2] or [8, Section 8]) to construct the models of unsuperstable theories, and we will use the previous constructed ordered coloured trees (from I) as the skeleton of the construction.

DEFINITION 4.1 (Generalized Ehrenfeucht–Mostowski models). We say that a function Φ is proper for K^{γ}_{tr}, if there is a vocabulary $\mathcal{L}^1$ and for each $A \in K^{\gamma}_{tr}$, there is a model $\mathcal{M}_1$ and tuples a_s, $s \in A$, of elements of $\mathcal{M}_1$ such that the following two hold:

- Every element of $\mathcal{M}_1$ is an interpretation of some $\mu(a_s)$, where μ is an $\mathcal{L}^1$-term.
- $tp_{at}(a_s, \emptyset, \mathcal{M}_1) = \Phi(tp_{at}(s, \emptyset, A))$.

Notice that for each A, the previous conditions determine $\mathcal{M}_1$ up to isomorphism. We may assume $\mathcal{M}_1, a_s, s \in A$, are unique for each A. We denote $\mathcal{M}_1$ by $EM^1(A, \Phi)$. We call $EM^1(A, \Phi)$ an *Ehrenfeucht–Mostowski model*.

Suppose T is a countable complete theory in a countable vocabulary $\mathcal{L}$, $\mathcal{L}^1$ a Skolemization of $\mathcal{L}$, and T^1 the Skolemization of T by $\mathcal{L}^1$. If there is Φ a proper function for K^{γ}_{tr}, then for every $A \in K^{\gamma}_{tr}$, we will denote by $\mathrm{EM}(A, \Phi)$ the $\mathcal{L}$-reduction of $EM^1(A, \Phi)$. The following result ensures the existence of a proper function Φ for unsuperstable theories T and $\gamma = \omega$.

FACT 4.2 (Shelah [11, Theorem 1.3], proof in [12, Chapter VII.3]). *Suppose $\mathcal{L} \subseteq \mathcal{L}^1$ are vocabularies, T is a complete first order theory in $\mathcal{L}$, and T^1 is a complete theory in $\mathcal{L}^1$ extending T and with Skolem-functions. Suppose T is unsuperstable and $\{\phi_n(x, y_n) \mid n < \omega\}$ witnesses this. Then there is a proper function Φ such that for all $A \in K^{\omega}_{tr}$, $EM^1(A, \Phi)$ is a model of T^1, and for $s \in P^A_n$, $t \in P^A_{\omega}$, $EM^1(A, \Phi) \models \phi_n(a_t, a_s)$ if and only if $A \models s \prec t$.*

The models that we will construct are of the form $EM(A, \Phi)$.

4.2. Reduction of the isomorphism relation. Before we deal with the construction of the models and the reduction, we need to do some preparations.

DEFINITION 4.3. For any $A \in K^{\omega}_{tr}$ with size κ and $\mathbb{A}$ a κ-representation of A, we define $S(\mathbb{A})$ as the set ordinal $\delta < \kappa$ that satisfies:

- δ is a limit ordinal,
- $\exists \eta \in P^A_{\omega}, \{\eta{\restriction}n \mid n < \omega\} \subseteq A_\delta \ \wedge \ \forall \alpha < \delta(\{\eta{\restriction}n \mid n < \omega\} \not\subseteq A_\alpha)$.

FACT 4.4 (Shelah [11, Fact 2.3], Hyttinen–Tuuri [8, Lemma 8.6]). *S is a CUB-invariant function.*

This fact allows us to define $S(A)$ for $A \in K^{\omega}_{tr}$ as $\left[S(\mathbb{A})\right]^2_{=_{CUB}}$ for any $\mathbb{A}$ κ-representation of A.

Notice that for a function $f \in \kappa^\kappa$ and $\mathbb{A} = \langle A^f_\alpha \mid \alpha < \kappa \rangle$, the κ-representation from Theorem 3.10, $S(A^f)$ is the set of ω-cofinal ordinals δ for which there is $\eta \in (A^f)_\omega \backslash A^f_\delta$, such that for all $n < \omega$, $\eta{\restriction}n \in A^f_\delta$. Thus, $S(A^f)$ does not depend on the color function. This can be fixed by restricting ourselves to the generalized Cantor space 2^κ and making a small modification to the trees A^f.

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DEFINITION 4.5. Let I be the $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order from Section 2. For every $f \in 2^\kappa$, let A^f be the tree constructed in Section 3. Define the tree $A_f \subseteq A^f$ by: $x \in A_f$ if and only if x is not a leaf of A^f or x is a leaf such that $d_f(x) = 1$. Denote by $\mathcal{A}^f$ the model $\mathrm{EM}(A_f, \Phi)$.

Notice that for all $\eta \in A_f$ such that $\eta \notin P_\omega^{A_f}$, $Suc_{A_f}(\eta)$ is infinite. On the other hand by Lemma 3.5, there is $\xi \in P_\omega^{A_f}$ such that $\eta \prec \xi$. Therefore, since I is (κ, bs, bs)-nice, by Fact 3.8 the trees A_f are locally (κ, bs, bs)-nice. Notice that since the branches of the trees A_f have length at most $\omega + 1$ and I is $(<\kappa, bs)$-stable, then the trees A_f are $(<\kappa, bs)$-stable.

By the way the models $\mathrm{EM}(A, \Phi)$ were defined, we know that if $A, A' \in K^\omega_{tr}$ are isomorphic, then $\mathrm{EM}(A, \Phi)$ and $\mathrm{EM}(A', \Phi)$ are isomorphic. Thus if A_f and A_g are isomorphic, then $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.

Notice that since we are working under the assumption that κ is an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$, $\kappa > |\mathcal{L}^1|$.

From Theorem 3.10 we know that for all $f, g \in 2^\kappa$,

$$f \ =^2_\omega \ g \ \Leftrightarrow \ A^f \cong A^g.$$

By using Fact 4.4 we can obtain a similar characterization of $=^2_\omega$, with the operator S. The following lemma states this characterization and relies essentially on Fact 4.4.

LEMMA 4.6. *For every $f, g \in 2^\kappa$,*

$$f \ =^2_\omega \ g \text{ if and only if } S(A_f) = S(A_g).$$

PROOF. By Fact 4.4, S is CUB-invariant; therefore, it is enough to find a κ-representation $\mathbb{A}_f$ of A_f for every $f \in 2^\kappa$, such that for all $f, g \in 2^\kappa$, $f \ =^2_\omega \ g$ if and only $\mathbb{A}_f \ =^2_{CUB} \ \mathbb{A}_g$.

Similar as in the proof of Theorem 3.10, for all $f \in 2^\kappa$ let us define the κ-representation $\mathbb{A}_f = \langle A_{f,\alpha} \mid \alpha < \kappa \rangle$ by

$$A_{f,\alpha} = \{\eta \in A_f \mid rng(\eta) \subseteq \theta \times \omega \times \theta \times \omega \times \theta^4 \text{ for some } \theta < \alpha\}.$$

By definition,

$$S(\mathbb{A}_f) = \{\delta < \kappa \mid \exists \eta \in P_\omega^{A_f}, \{\eta \restriction n \mid n < \omega\} \subseteq (A_{f,\delta} \ \& \ \forall \alpha < \delta(\{\eta \restriction n \mid n < \omega\} \not\subseteq A_{f,\alpha})\}.$$

CLAIM 4.6.1. *$\delta \in S(\mathbb{A}_f)$ if and only if $cf(\delta) = \omega$ and there is $\eta \in P_\omega^{A_f}$ with $max(\{sup(rng(\eta_i)) \mid i \leq 8\}) = \delta$.*

PROOF. The direction from right to left follows from Definition 4.3. The other direction follows from the definition of $S(\mathbb{A}_f)$ and $A_{f,\alpha}$. ⊣

By the way A_f was constructed, $\eta \in P_\omega^{A_f}$ if and only if $\eta \in P_\omega^{A^f}$ and $d_f(\eta) = 1$. By the previous claim we know that if $\delta \in S(\mathbb{A}_f)$ and $\eta \in P_\omega^{A_f}$ witnesses it, then $\eta \in P_\omega^{A^f}$ and $1 = d_f(\eta)$. In the same way as in the proof of Theorem 3.10, we can conclude that $d_f(\eta) = f(max\{s_1(\eta), s_8(\eta)\})$, so

$$1 = f(max\{sup(rng(\eta_1)), sup(rng(\eta_8))\}).$$

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Recall from the proof of Theorem 3.10 that

$$max(\{sup(rng(\eta_i)) \mid i \leq 8\}) = max\{sup(rng(\eta_1)), sup(rng(\eta_8))\}.$$

We conclude that $1 = f(\delta)$.

Therefore we can rewrite $S(\mathbb{A}_f)$ as

$$S(\mathbb{A}_f) = \{\delta < \kappa \mid cf(\delta) = \omega \wedge f(\delta) = 1\}.$$

It follows that $S(\mathbb{A}_f) =^2_{CUB} S(\mathbb{A}_g)$ holds if and only if $f =^2_\omega g$. ⊣

Now we proceed to prove that the models $\mathcal{A}^f$ are as wanted, i.e., $f =^2_\omega g$ if and only if $\mathcal{A}^f \cong_T \mathcal{A}^g$.

Fact 4.7 (Shelah [11, Theorem 2.4]). *Suppose T is a countable complete unsuperstable theory in a countable vocabulary. If κ is a regular uncountable cardinal, $A_1, A_2 \in K^\omega_{tr}$ have size κ, A_1, A_2 are locally (κ, bs, bs)-nice and $(< \kappa, bs)$-stable, $EM(A_1, \Phi)$ is isomorphic to $EM(A_2, \Phi)$, then $S(A_1) = S(A_2)$.*

Lemma 4.8. *If T is a countable complete unsuperstable theory over a countable vocabulary, then for all $f, g \in 2^\kappa$, $f =^2_\omega g$ if and only if $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.*

Proof. From left to right. Suppose $f, g \in 2^\kappa$ are such that $f =^2_\omega g$. By Theorem 3.10 and Definition 4.5 we know that $f =^2_\omega g$ if and only if $A_f \cong A_g$. Finally $A_f \cong A_g$ implies that $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.

From right to left. Suppose $f, g \in 2^\kappa$ are such that $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic. By Definition 4.5 and Fact 4.7, $S(A_f) = S(A_g)$. From Lemma 4.6 we conclude $f =^2_\omega g$. ⊣

Theorem 4.9. *If T is a countable complete unsuperstable theory over a countable vocabulary, $\mathcal{L}$, then $=^2_\omega \hookrightarrow_c \cong_T$.*

Proof. Let us construct a continuous function $G : 2^\kappa \to 2^\kappa$ with $\mathcal{A}_{G(f)} \cong EM(A_f, \Phi)$.

By Remark 3.11, Definition 4.5, and the definition of $A_{f,\alpha}$,

$$f \restriction \alpha = g \restriction \alpha \Leftrightarrow A_{f,\alpha} = A_{g,\alpha}.$$

Let us denote by $SH(X)$ the Skolem-hull of X, i.e., $\{\mu(a) \mid a \in X, \mu \text{ an } \mathcal{L}^1\text{-term}\}$. For all α, $A \in K^\omega_{tr}$, and a κ-representation $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ of A, let us denote by $\tilde{A}_\alpha$ the set $\{a_s \mid s \in A_\alpha\}$ (recall the construction of $EM^1(A, \Phi)$ in Definition 4.1). Since for all $\alpha < \kappa$,

$$A_{f,\alpha} = A_{g,\alpha} \Leftrightarrow SH(\tilde{A}_{f,\alpha}) = SH(\tilde{A}_{g,\alpha}).$$

Thus

$$f \restriction \alpha = g \restriction \alpha \Leftrightarrow SH(\tilde{A}_{f,\alpha}) \restriction \mathcal{L} = SH(\tilde{A}_{g,\alpha}) \restriction \mathcal{L}.$$

For every $f \in 2^\kappa$ there is a bijection $E_f : dom(EM(A_f, \Phi)) \to \kappa$, such that for every $f, g \in 2^\kappa$ and $\alpha < \kappa$ it holds that: If $f \restriction \alpha = g \restriction \alpha$, then

$$E_f \restriction dom(SH(\tilde{A}_{f,\alpha}) \restriction \mathcal{L}) = E_g \restriction dom(SH(\tilde{A}_{g,\alpha}) \restriction \mathcal{L})$$

(see [10]).

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Let π be the bijection in Definition 1.4, and define the function G by

$$G(f)(\alpha) = \begin{cases} 1, & \text{if } \alpha = \pi(m, a_1, a_2, \dots, a_n) \text{ and} \\ & EM(A_f, \Phi) \models Q_m(E_f^{-1}(a_1), E_f^{-1}(a_2), \dots, E_f^{-1}(a_n)), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\mathcal{A}_{G(f)} \cong EM(A_f, \Phi)$.

To show that $G : 2^\kappa \to 2^\kappa$ is continuous, let $[\zeta \restriction \alpha]$ be a basic open set and $\xi \in G^{-1}[[\zeta \restriction \alpha]]$. There is $\beta < \kappa$ such that for all $\gamma < \alpha$, if $\gamma = \pi(m, a_1, \dots, a_n)$, then $E_\xi^{-1}(a_i) \in dom(SH(\tilde{A}_{\xi,\beta}) \restriction \mathcal{L})$ holds for all $i \leq n$. Since for all $\eta \in [\xi \restriction \beta]$ it holds that $SH(\tilde{A}_{\eta,\beta}) \restriction \mathcal{L} = SH(\tilde{A}_{\xi,\beta}) \restriction \mathcal{L}$, for any $\gamma < \alpha$ that satisfies $\gamma = \pi(m, a_1, \dots, a_n)$

$$EM(A_\eta, \Phi) \models Q_m(E_\eta^{-1}(a_1), E_\eta^{-1}(a_2), \dots, E_\eta^{-1}(a_n))$$

if and only if

$$EM(A_\xi, \Phi) \models Q_m(E_\xi^{-1}(a_1), E_\xi^{-1}(a_2), \dots, E_\xi^{-1}(a_n)).$$

We conclude that G is continuous. ⊣

4.3. Corollaries. In this section we will prove Theorem A and Theorem B. For any stationary set $X \subseteq \kappa$, let us denote by $\diamondsuit_X$ the following principle:

There is a sequence $\{D_\alpha \subset \alpha \mid \alpha \in X\}$ *such that for all* $B \subseteq \kappa$, *the set* $\{\alpha \in X \mid D_\alpha = B \cap \alpha\}$ *is stationary.*

Let us denote by $\diamondsuit_\omega$ the diamond principle $\diamondsuit_X$ when $X = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$.

FACT 4.10 (Hyttinen–Kulikov–Moreno,[6, Lemma 2]). *Assume T is a countable complete classifiable theory over a countable vocabulary. If $\diamondsuit_\omega$ holds, then $\cong_T \hookrightarrow_c =^2_\omega$.*

FACT 4.11 (Friedman–Hyttinen–Kulikov [4, Theorem 77]). *If a first-order countable complete theory over a countable vocabulary T is classifiable, then $=^2_\omega \not\hookrightarrow_c \cong_T$.*

COROLLARY 4.12. *Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T_1 is a countable complete classifiable theory, and T_2 is a countable complete unsuperstable theory, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ and $\cong_{T_2} \not\hookrightarrow_c \cong_{T_1}$.*

PROOF. Since $\lambda^\omega = \lambda$, $cf(\lambda) > \omega$. By [13] we know that if $\kappa = \lambda^+ = 2^\lambda$ and $cf(\lambda) > \omega$, then $\diamondsuit_\omega$ holds. The proof follows from Theorem 4.9, Fact 4.10, and Fact 4.11. ⊣

We will finish this section with a corollary about Σ^1_1-completeness. Before we state the corollary we need to recall some definitions from [2], in particular the definition of $\mathrm{DI}^*_S(\Pi^1_2)$. For more on $\mathrm{DI}^*_S(\Pi^1_2)$ see [2].

A Π^1_2-sentence ϕ is a formula of the form $\forall X \exists Y \varphi$ where φ is a first-order sentence over a relational language $\mathcal{L}$ as follows:

- $\mathcal{L}$ has a predicate symbol ε of arity 2.
- $\mathcal{L}$ has a predicate symbol $\mathbb{X}$ of arity $m(\mathbb{X})$.
- $\mathcal{L}$ has a predicate symbol $\mathbb{Y}$ of arity $m(\mathbb{Y})$.
- $\mathcal{L}$ has infinitely many predicate symbols $(\mathbb{B}_n)_{n\in\omega}$, each $\mathbb{B}_n$ is of arity $m(\mathbb{B}_n)$.

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DEFINITION 4.13. For sets N and x, we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.

Suppose that a set N sees an ordinal α, and that $\phi = \forall X \exists Y \varphi$ is a Π^1_2-sentence, where φ is a first-order sentence in the above-mentioned language $\mathcal{L}$. For every sequence $(B_n)_{n\in\omega}$ such that, for all $n \in \omega$, $B_n \subseteq \alpha^{m(\mathbb{B}_n)}$, we write

$$\langle \alpha, \in, (B_n)_{n\in\omega} \rangle \models_N \phi$$

to express that the two hold:

(1) $(B_n)_{n\in\omega} \in N$.
(2) $\langle N, \in \rangle \models (\forall X \subseteq \alpha^{m(\mathbb{X})})(\exists Y \subseteq \alpha^{m(\mathbb{Y})})[\langle \alpha, \in, X, Y, (B_n)_{n\in\omega} \rangle \models \varphi]$, where:
   - $\in$ is the interpretation of ε;
   - X is the interpretation of $\mathbb{X}$;
   - Y is the interpretation of $\mathbb{Y}$, and
   - for all $n \in \omega$, B_n is the interpretation of $\mathbb{B}_n$.

DEFINITION 4.14. Let κ be a regular and uncountable cardinal, and $S \subseteq \kappa$ stationary.

$\mathrm{Dl}^*_S(\Pi^1_2)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

(1) For every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α.
(2) For every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$.
(3) Whenever $\langle \kappa, \in, (B_n)_{n\in\omega} \rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, (B_n \cap (\alpha^{m(\mathbb{B}_n)}))_{n\in\omega} \rangle \models_{N_\alpha} \phi$.

FACT 4.15 (Fernandes–Moreno–Rinot [2, Theorem C]). *If* $\mathrm{Dl}^*_S(\Pi^1_2)$ *holds for* $S = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$, *then* $=^2_\omega$ *is* Σ^1_1*-complete.*

COROLLARY 4.16. *If* $\mathrm{Dl}^*_S(\Pi^1_2)$ *holds for* $S = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$, *and* T *is a countable complete unsuperstable theory, then* $\cong_T$ *is* Σ^1_1*-complete.*

PROOF. It follows from Fact 4.15 and Theorem 4.9. ⊣

FACT 4.17 (Fernandes–Moreno–Rinot [3, Lemma 4.10 and Proposition 4.14]). *There exists a* $< \kappa$*-closed* κ^+*-cc forcing extension in which* $\mathrm{Dl}^*_S(\Pi^1_2)$ *holds.*

COROLLARY 4.18. *There exists a* $< \kappa$*-closed* κ^+*-cc forcing extension in which for all countable complete unsuperstable theory* T, $\cong_T$ *is* Σ^1_1*-complete.*

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