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ON NUMERICAL RANGES OF GENERALIZED DERIVATIONS AND RELATED PROPERTIES

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Abstract

This paper is concerned with the numerical range and some related properties of the operator $\Delta|\mathcal{S}$: $T \to AT - TB\,(T \in \mathcal{S})$, where A, B are (bounded linear) operators on the normed linear spaces $\mathbf{X}$ and $\mathbf{Y}$, respectively, and $\mathcal{S}$ is a linear subspace of the space $\mathcal{L}(\mathbf{Y},\mathbf{X})$ of all operators from $\mathbf{Y}$ to $\mathbf{X}$. $\mathcal{S}$ is assumed to contain all finite operators, to be invariant under Δ, and to be suitably normed (not necessarily with the operator norm). Then the algebra numerical range of $\Delta|\mathcal{S}$ is equal to the difference of the algebra numerical ranges of A and B. When $\mathbf{X} = \mathbf{Y}$ and $\mathcal{S} = \mathcal{L}(\mathbf{X})$, Δ is Hermitian (resp. normal) in $\mathcal{L}(\mathcal{L}(\mathbf{X}))$ if and only if $A - \lambda$ and $B - \lambda$ are Hermitian (resp. normal) in $\mathcal{L}(\mathbf{X})$ for some scalar λ; if $\mathbf{X} := \mathbf{H}$ is a Hilbert space and if $\mathcal{S}$ is a C^*-algebra or a minimal norm ideal in $\mathcal{L}(\mathbf{H})$, then any Hermitian (resp. normal) operator on S is of the form $\Delta|\mathcal{S}$ for some Hermitian (resp. normal) operators A and B. $AT = TB$ implies $A^*T = TB^*$, provided that A and B^* are hyponormal operators on the Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, and T is a Hilbert-Schmidt operator from $\mathbf{H}_2$ to $\mathbf{H}_1$.

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0. Introduction

Let $\mathbf{X}$ be a complex normed linear space and let $\mathcal{L}(\mathbf{X})$ be the algebra of all (bounded linear) operators on $\mathbf{X}$. The spatial numerical range of an operator A on $\mathbf{X}$ is given by $W(A; \mathcal{L}(\mathbf{X})) := \{f(Ax);\ (x, f) \in \pi(\mathbf{X})\}$, where $\pi(\mathbf{X})$ denotes the set of all pairs $(x, f) \in \mathbf{X} \times \mathbf{X}'$ such that $\|x\| = \|f\| = f(x) = 1$. The algebra numerical range of A in $\mathcal{L}(\mathbf{X})$ is given by $V(A;\ \mathcal{L}(\mathbf{X})) := \{F(A);\ (I, F) \in \pi(\mathcal{L}(\mathbf{X}))\}$, where I is the identity operator. It is known that $V(A; \mathcal{L}(\mathbf{X}))$ is compact

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and is the closed convex hull of $W(A; \mathcal{L}(\mathbf{X}))$. When $\mathbf{X} = \mathbf{H}$ is a Hilbert space with inner product $\langle\ ,\ \rangle$, $W(A; \mathcal{L}(\mathbf{X}))$ reduces to the classical numerical range $W(A) := \{\langle Ax, x\rangle; x \in \mathbf{H}, \|x\| = 1\}$ (see [4] for details).

If B is an operator on another normed linear space Y, then the generalized derivation $\Delta = \Delta_{A,B}$, defined by $\Delta(T) := AT - TB$, is an operator on the space $\mathcal{L}(\mathbf{Y}, \mathbf{X})$ of all operators from $\mathbf{Y}$ to $\mathbf{X}$. In this paper, we consider the numerical ranges of restrictions of Δ to certain invariant subspaces $\mathcal{S}$ and some consequences. First, the algebra numerical range of $\Delta | \mathcal{S}$ is shown to be the difference of $V(A; \mathcal{L}(\mathbf{X}))$ and $V(B; \mathcal{L}(\mathbf{Y}))$, provided that $\mathcal{S}$ contains all finite-rank operators and is suitably normed. Then it is applied to determine when Δ or $\Delta | \mathcal{S}$ is Hermitian or normal, and to derive a Fuglede-Putnam theorem for hyponormal operators. The results will extend some theorems of Kyle [8], Sourour [13] and Berberian [3], respectively.

1. The numerical range

We will assume that $\mathcal{S}$ is a linear subspace of $\mathcal{L}(\mathbf{Y}, \mathbf{X})$ equipped with a norm $|||\cdot|||$ (possibly different from the operator norm $\|\cdot\|$) such that the following conditions are satisfied:

(1) $A\mathcal{S} \subset \mathcal{S}$ and $\mathcal{S}B \subset \mathcal{S}$; (2) If $D \in \mathcal{L}(\mathbf{X})$, $T \in \mathcal{S}$, $E \in \mathcal{L}(\mathbf{Y})$ and $DTE \in \mathcal{S}$, then $|||DTE||| \leqslant \|D\|\,|||T|||\,\|E\|$; (3) $\|T\| \leqslant |||T|||$ for all T in $\mathcal{S}$, and the equality holds whenever T has rank one; (4) $\mathcal{S}$ contains all finite rank operators from $\mathbf{Y}$ to $\mathbf{X}$.

It follows from (1) that $\mathcal{S}$ is an invariant subspace of $\Delta = \Delta_{A,B}$, and from (2) that the restriction $\Delta | \mathcal{S}$ of Δ is a bounded linear operator on $(\mathcal{S}, |||\cdot|||)$. We consider the numerical range of $\Delta | \mathcal{S}$.

THEOREM 1.1. *For operators $A \in \mathcal{L}(\mathbf{X})$ and $B \in \mathcal{L}(\mathbf{Y})$ let $(\mathcal{S}, |||\cdot|||)$ be the normed linear space as mentioned above. Then*

$$(*) \qquad V(\Delta | \mathcal{S}; \mathcal{L}(\mathcal{S})) = V(A; \mathcal{L}(\mathbf{X})) - V(B; \mathcal{L}(\mathbf{Y})).$$

PROOF. We first prove that the left side is contained in the right side. So, let λ be an arbitrary element of $V(\Delta | \mathcal{S}; \mathcal{L}(\mathcal{S}))$. Then $\lambda = f(\Delta | \mathcal{S})$ for some $f \in (\mathcal{L}(\mathcal{S}))'$ such that $\|f\| = f(I_{\mathcal{L}(\mathcal{S})}) = 1$, where $I_{\mathcal{L}(\mathcal{S})}$ denotes the identity operator in $\mathcal{L}(\mathcal{S})$. It is clear that the set $\mathfrak{M}$ (resp. $\mathfrak{N}$) of all $D \in \mathcal{L}(\mathbf{X})$ (resp. $E \in \mathcal{L}(\mathbf{Y})$) such that $D\mathcal{S} \subset \mathcal{S}$ (resp. $\mathcal{S}E \subset \mathcal{S}$) is a linear subspace of $\mathcal{L}(\mathbf{X})$ (resp. $\mathcal{L}(\mathbf{Y})$) containing $I_{\mathcal{L}(\mathbf{X})}$ and A (resp. $I_{\mathcal{L}(\mathbf{Y})}$ and B). Define a linear functional F on $\mathfrak{M}$ by $F(D) = f(L_D | \mathcal{S})$ $(D \in \mathfrak{M})$ and a linear functional G on $\mathfrak{N}$ by $G(E) = f(R_E | \mathcal{S})$ $(E \in \mathfrak{N})$, where L_D and R_E stand for the left multiplication by D and the right multiplication by

E, respectively. Now the Hahn-Banach theorem guarantees the existence of $\hat{F}$ in $(\mathfrak{L}(\mathbf{X}))'$ and $\hat{G}$ in $(\mathfrak{L}(\mathbf{Y}))'$ such that $\hat{F}|\mathfrak{M} = F$, $\|\hat{F}\| = \|F\|$, $\hat{G}|\mathfrak{N} = G$ and $\|\hat{G}\| = \|G\|$. Since $\hat{F}(I_{\mathfrak{L}(\mathbf{X})}) = F(I_{\mathfrak{L}(\mathbf{X})}) = f(I_{\mathfrak{L}(\mathbb{S})}) = 1$ and since $|F(D)| \leqslant \|f\| \|L_D|\mathbb{S}\| = \sup\{|||DT|||;\ T \in \mathbb{S},\ |||T||| = 1\} \leqslant \|D\|$ for all D in $\mathfrak{M}$, we have that $\|\hat{F}\| = \|F\| \leqslant 1 = \hat{F}(I_{\mathfrak{L}(\mathbf{X})}) \leqslant \|\hat{F}\|$, that is, $(I_{\mathfrak{L}(\mathbf{X})}, \hat{F})$ belongs to $\pi(\mathfrak{L}(\mathbf{X}))$. Similarly, we have $(I_{\mathfrak{L}(\mathbf{Y})}, \hat{G}) \in \pi(\mathfrak{L}(\mathbf{Y}))$. Hence

$$\begin{aligned}\lambda = f(\Delta|\mathbb{S}) &= f(L_A|\mathbb{S}) - f(R_B|\mathbb{S}) \\ &= F(A) - G(B) = \hat{F}(A) - \hat{G}(B) \\ &\in V(A; \mathfrak{L}(\mathbf{X})) - V(B; \mathfrak{L}(\mathbf{Y})).\end{aligned}$$

To prove the other inclusion, it suffices to show that

$$V(\Delta|\mathbb{S}; \mathfrak{L}(\mathbb{S})) \supset W(A; \mathfrak{L}(\mathbf{X})) - W(B; \mathfrak{L}(\mathbf{Y}))$$

since the closed convex hull of the set on the right side is $V(A;\ \mathfrak{L}(\mathbf{X})) - V(B;\ \mathfrak{L}(\mathbf{Y}))$, by an elementary proof. So, let $\alpha = g(Ax)$ with $(x, g) \in \pi(\mathbf{X})$ and let $\beta = h(By)$ with $(y, h) \in \pi(\mathbf{Y})$. Using the usual notation $x \otimes h$ for the rank-one operator: $z \to h(z)x$ $(z \in \mathbf{Y})$, we define the linear functional P on $\mathfrak{L}(\mathbb{S})$ by

$$P(\Omega) := g([\Omega(x \otimes h)]y) \quad (\Omega \in \mathfrak{L}(\mathbb{S})).$$

Clearly we have $P(I_{\mathfrak{L}(\mathbb{S})}) = 1$ and, by (3) and (4),

$$\begin{aligned}|P(\Omega)| &\leqslant \|g\| \|\Omega(x \otimes h)\| \|y\| \leqslant |||\Omega(x \otimes h)||| \leqslant \|\Omega\|\, |||x \otimes h||| \\ &= \|\Omega\| \|x \otimes h\| = \|\Omega\|,\end{aligned}$$

that is, $(I_{\mathfrak{L}(\mathbb{S})}, P) \in \pi(\mathfrak{L}(\mathbb{S}))$. Hence $V(\Delta|\mathbb{S};\ \mathfrak{L}(\mathbb{S}))$ contains the number

$$\begin{aligned}P(\Delta|\mathbb{S}) &= g(A(x \otimes h)y - (x \otimes h)By) \\ &= g(Ax)h(y) - g(x)h(By) = \alpha - \beta.\end{aligned}$$

The proof is complete.

REMARK. Conditions (3) and (4) are used only in proving the direction "$\supset$", therefore the inclusion "$\subset$" will hold for any subspace satisfying (1) and (2). That (4) is essential for the direction "$\supset$" is easily seen from the example where $\mathbf{X} = \mathbf{Y} = C^2$, $A = B = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$, $\mathbb{S} = \operatorname{span}\{A\}$ and $|||\cdot||| = \|\cdot\|$.

The following are some examples of qualified subspaces $(\mathbb{S}, |||\cdot|||)$:

(a) Components in $\mathfrak{L}(\mathbf{Y}, \mathbf{X})$ of all the operator ideals on Banach spaces, as studied in Chapter 1 of [9], such as the classes of finite operators, approximate operators, compact operators, weakly compact operators, completely continuous

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operators, unconditionally summing operators, separable operators, Kato operators and Pełczynski operators.

(b) Components in $\mathfrak{L}(\mathbf{Y},\mathbf{X})$ of all the normed operator ideals on Banach spaces, as studied in Chapter 6 of [9], such as nuclear operators, integral operators, absolutely summing operators and Hilbert operators. (Compare the assumptions on $(\mathfrak{S}, |||\cdot|||)$ with Definitions 6.1.1, 6.2.2 and Propositions 6.1.4, 6.1.5 of [9].)

(c) The Schatten p-class $C_p(\mathbf{H}_2,\mathbf{H}_1)$ $(1 \leqslant p < \infty)$ or approximable operators from a Hilbert space $\mathbf{H}_2$ to another Hilbert space $\mathbf{H}_1$, that is, operators T such that $\|T\|_p := [\mathrm{trace}(T^*T)^{p/2}]^{1/p} < \infty$, (see [9, page 216]). In the case where $p = 1$ these are the operators of trace class (nuclear operators), and $p = 2$ yields the Hilbert space of Hilbert-Schmidt operators (see [2, Chapter 12]).

COROLLARY 1.2. *For any* $A \in \mathfrak{L}(\mathbf{X})$ *and* $B \in \mathfrak{L}(\mathbf{Y})$,

$$V(\Delta; \mathfrak{L}(\mathfrak{L}(\mathbf{Y},\mathbf{X}))) = V(A; \mathfrak{L}(\mathbf{X})) - V(B; \mathfrak{L}(\mathbf{Y})).$$

This contains Kyle's result [8] (for the case $\mathbf{X} = \mathbf{Y}$) as a special case.

COROLLARY 1.3. *Let A and B be any operators on Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, and let $C_p(\mathbf{H}_2,\mathbf{H}_1)$ be normed with $\|\cdot\|$ or $\|\cdot\|_p$. Then*

$$V\big(\Delta \,|\, C_p(\mathbf{H}_2,\mathbf{H}_1); \mathfrak{L}\big(C_p(\mathbf{H}_2,\mathbf{H}_1)\big)\big) = W(A)^{-} - W(B)^{-}.$$

Thus Corollary 1.3 becomes a numerical range analogue of Fialkow's [5] formula for spectra: $\sigma(\Delta \,|\, C_p(\mathbf{H})) = \sigma(A) - \sigma(B)$.

We end this section by deriving from Theorem 1.1 the following known property, which will be of use in Section 2.

COROLLARY 1.4. *If $AT = TB$ holds for all rank-one operators T in $\mathfrak{L}(\mathbf{Y},\mathbf{X})$, then $A = \lambda I_{\mathfrak{L}(\mathbf{X})}$ and $B = \lambda I_{\mathfrak{L}(\mathbf{Y})}$ for some scalar λ.*

PROOF. Take $\mathfrak{S}$ to be the space of all finite rank operators. Then $\Delta \,|\, \mathfrak{S} = 0$ and so $V(A;\, \mathfrak{L}(\mathbf{X})) - V(B;\, \mathfrak{L}(\mathbf{Y})) = V(\Delta \,|\, \mathfrak{L}(\Sigma)) = \{0\}$, or equivalently, $V(A;\, \mathfrak{L}(\mathbf{X})) = V(B;\, \mathfrak{L}(\mathbf{Y})) = \{\lambda\}$ for some scalar λ. It follows that $V(A - \lambda I_{\mathfrak{L}(\mathbf{X})};\, \mathfrak{L}(\mathbf{X})) = \{0\}$ and

$$\|A - \lambda I_{\mathfrak{L}(\mathbf{X})}\| \leqslant e \max\big\{|\mu| \,;\, \mu \in V\big(A - \lambda I_{\mathfrak{L}(\mathbf{X})}; \mathfrak{L}(\mathbf{X})\big)\big\} = 0$$

(see [4, page 34]). Hence $A = \lambda I_{\mathfrak{L}(\mathbf{X})}$, and similarly $B = \lambda I_{\mathfrak{L}(\mathbf{Y})}$.

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2. Hermitian and normal derivations

An operator A on a normed linear space $\mathbf{X}$ is said to be Hermitian if its numerical range is contained in the real line and it is normal if $A = H + iK$ for some commuting Hermitian operators H and K. In this section we try to answer partly the question about when the operator $\Delta | \mathbb{S}$ is Hermitian or normal.

First, from the formula (*) comes immediately the following

COROLLARY 2.1. *Let* $(\mathbb{S}, |||\cdot|||)$ *be as assumed in Theorem* 1.1. *Then* $\Delta | \mathbb{S}$ *is Hermitian in* $\mathcal{L}(\mathcal{L}(\mathbf{Y},)$ *if and only if* $A - \lambda I \in \mathcal{L}(\mathbf{X})$ *and* $B - \lambda I \in \mathcal{L}(\mathbf{Y})$ *are Hermitian for some scalar* λ.

Kyle [8] has proved that when $\mathbf{X} = \mathbf{Y}$ is a Banach space and when $A = B$, Δ is normal if and only if A is normal in $\mathcal{L}(\mathbf{X})$. We shall extend this result under various situations. The statement for the most general situation is as follows.

THEOREM 2.2. *Let* $A \in \mathcal{L}(\mathbf{X})$ *and* $B \in \mathcal{L}(\mathbf{Y})$ *be of the forms* $A = H + iK$ *and* $B = M + iN$, *where* H, K, M *and* N *are Hermitian operators. Suppose that* $(\mathbb{S}, |||\cdot|||)$ *satisfies* $H\mathbb{S} \subset \mathbb{S}$, $K\mathbb{S} \subset \mathbb{S}$, $\mathbb{S}M \subset \mathbb{S}$, $\mathbb{S}N \subset \mathbb{S}$ *and conditions* (2), (3), (4). *Then* $\Delta | \Sigma$ *is normal if and only if both* A *and* B *are normal.*

PROOF. $\Delta_{A,B}$ can be written as $\Delta_{H,M} + i\Delta_{K,N}$, where $\Delta_{H,M}$ and $\Delta_{K,N}$ and their restrictions to $\mathbb{S}$ are Hermitian, by Corollary 2.1. Now, from the easily verified identity

$$\Delta_{H,M}\Delta_{K,N} - \Delta_{K,N}\Delta_{H,M} = \Delta_{HK-KH,MN-NM}$$

we see that $\Delta | \mathbb{S}$ is normal if and only if $(HK - KH)T = T(MN - NM)$ for all T in $\mathbb{S}$. Since $\mathbb{S}$ contains all finite rank operators, the latter condition is, by Corollary 1.4, equivalent to that $HK - KH = \lambda I_{\mathcal{L}(\mathbf{X})}$ and $MN - NM = \lambda I_{\mathcal{L}(\mathbf{Y})}$ for some scalar λ. But this is possible only when $\lambda = 0$, that is, A and B are normal (see [10, page 332]).

It follows that for $\Delta | \mathbb{S}$ to be normal it is sufficient that A and B are normal. That this is also necessary in case $\mathbf{X}$ and $\mathbf{Y}$ are Hilbert spaces is already contained in the above theorem.

COROLLARY 2.3. *Let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be Hilbert spaces. For* $A \in \mathcal{L}(\mathbf{H}_1)$ *and* $B \in \mathcal{L}(\mathbf{H}_2)$ *let* $\mathbb{S} \subset \mathcal{L}(\mathbf{H}_2, \mathbf{H}_1)$ *be a subspace satisfying conditions* (1)–(4) (*for example,* $C_p(\mathbf{H}_2, \mathbf{H}_1)$ *with norm* $\|\cdot\|_p$ *or operator norm* $\|\cdot\|$). *Then* $\Delta | \mathbb{S}$ *is normal if and only if both* A *and* B *are normal.*

REMARK. When A and B are normal operators on a Hilbert space $\mathbf{H}$, Δ and $\Delta \mid C_2(\mathbf{H})$ become normal operators on the Banach space $\mathfrak{L}(\mathbf{H})$ and the Hilbert space $(C_2(\mathbf{H}), \|\cdot\|_2)$, respectively. It follows (see [6, Theorem A] or [1]) that the null space $N(\Delta)$ is orthogonal to the range $R(\Delta)$ of Δ. Hence we have

$$R(\Delta)^- \oplus N(\Delta) \supset \left(R(\Delta \mid C_2(\mathbf{H}))^= \oplus N(\Delta \mid C_2(\mathbf{H}))\right)^- = C_2(\mathbf{H})^-,$$

where the superscripts "$-$" and "$=$" denote the closure relative to $\|\cdot\|$ and $\|\cdot\|_2$, respectively. Thus $R(\Delta)^- \oplus N(\Delta)$ contains all compact operators while it is in general strictly less than $\mathfrak{L}(\mathbf{H})$ [1].

Since a general operator on a normed linear space is not necessarily of the form $H + IK$, with H and K Hermitian, it is not known from Theorem 2.2 whether a normal $\Delta_{A,B} \mid \mathfrak{S}$ ($A \in \mathfrak{L}(\mathbf{X})$, $B \in \mathfrak{L}(\mathbf{Y})$) must be made of two normal A and B. But, at least when $\mathbf{X}$ is equal to $\mathbf{Y}$ and when $\mathfrak{S} = \mathfrak{L}(\mathbf{X})$, this is true, as is shown by the following extension of Kyle's result.

THEOREM 2.4. *Let A and B be operators on a normed linear space X. Then $\Delta_{A,B}$ is normal in $\mathfrak{L}(\mathfrak{L}(\mathbf{X}))$ if and only if both A and B are normal in $\mathfrak{L}(\mathbf{X})$.*

This will follow from Theorem 2.2 (with $\mathfrak{S} = \mathfrak{L}(\mathbf{X})$) and the next

LEMMA 2.5. *$\Delta_{A,B} = \Phi + i\Psi$ for some Hermitian operators Φ and Ψ in $\mathfrak{L}(\mathfrak{L}(\mathbf{X}))$ if and only if $A = H + iK$ and $B = M + iN$ for some Hermitian operators H, K, M and N in $\mathfrak{L}(\mathbf{X})$.*

PROOF. Suppose $\Delta = \Phi + i\Psi$ where Φ and Ψ are Hermitian. Fix a pair (x_0, f) in $\pi(\mathbf{X})$ and define operators H_1, K_1, M_1 and N_1 by $H_1 x := (\Phi(x \otimes f))x_0$, $K_1 x := (\Psi(x \otimes f))x_0$ $(x \in X)$, $M_1 := H_1 - \Phi(I)$ and $N_1 := K_1 - \Psi(I)$, respectively, where I is the identity operator on $\mathbf{X}$.

We first show that these operators are Hermitian. To show that H_1 is Hermitian, we will prove that $g(H_1 x)$ is real for any pair (x, g) in $\pi(\mathbf{X})$. Indeed, for a fixed (x, g) in $\pi(\mathbf{X})$ there corresponds the linear functional $G\colon T \to g(Tx_0)$ $(T \in \mathfrak{L}(\mathbf{X}))$ on $\mathfrak{L}(\mathbf{X})$ which satisfies: $\|G\| = G(x \otimes f) = \|x \otimes f\| = 1$, that is, $(x \otimes f, G) \in \pi(\Lambda(\mathbf{X}))$. This implies that

$$g(H_1 x) = g((\Phi(x \otimes f))x_0) = G(\Phi(x \otimes f)) \in W(\Phi; \mathfrak{L}(\mathfrak{L}(\mathbf{X}))) \subset R.$$

By a similar way one can show that K_1 is also Hermitian. To claim that $\Phi(I)$ is Hermitian we observe that $F(\Phi(I))$ belongs to $W(\Phi, \mathfrak{L}(\mathfrak{L}(\mathbf{X})))$ for every (I, F) in $\pi(\mathfrak{L}(\mathbf{X}))$, or equivalently, $V(\Phi(I); \mathfrak{L}(\mathbf{X})) \subset (\Phi; \mathfrak{L}(\mathfrak{L}(\mathbf{X}))) \subset R$. Similarly, $\Psi(I)$ is Hermitian.

Now we have, for $D \in \mathfrak{L}(\mathbf{X})$ and $x \in \mathbf{X}$,

$$\left(\Delta_{H_1,M_1}(D)\right)x = (H_1D - DM_1)x = H_1Dx - DH_1x + D\Phi(I)x$$
$$= (\Phi(Dx \otimes f))x_0 - D(\Phi(x \otimes f))x_0 + D\Phi(I)x,$$

and similarly

$$\left(\Delta_{K_1,N_1}(D)\right)x = (\Psi(Dx \otimes f))x_0 - D(\Psi(x \otimes f))x_0 + D\Psi(I)x.$$

Thus

$$\begin{aligned}
&\left((\Delta_{H_1,M_1} + i\Delta_{K_1,N_1})D\right)x \\
&\quad = (\Delta_{A,B}(Dx \otimes f))x_0 - D(\Delta_{A,B}(x \otimes f))x_0 + D(A - B)x \\
&\quad = A(Dx \otimes f)x_0 - (Dx \otimes f)Bx_0 \\
&\quad\quad - D(A(x \otimes f) - (x \otimes f)B)x_0 + D(A - B)x \\
&\quad = ADx - DAx + D(A - B)x \\
&\quad = \Delta_{A,B}(D)x.
\end{aligned}$$

That is, $(A - H_1 - iK_1)D = D(B - M_1 - iN_1)$ holds for every D in $\mathfrak{L}(\mathbf{X})$. It follows from Corollary 1.4 that $A = H_1 + iK_1 + \lambda I$ and $B = M_1 + iN_1 + \lambda I$ for some scalar λ. Now we can take $H = H_1 + (\operatorname{Re}\lambda)I$, $K = K_1 + (\operatorname{Im}\lambda)I$, $M = M_1 + (\operatorname{Re}\lambda)I$ and $N = N_1 + (\operatorname{Im}\lambda)I$ as the desired Hermitian operators.

So far, the question about when $\Delta_{A,B}|\mathfrak{S}$ is Hermitian in $\mathfrak{L}(\mathfrak{S})$ ($\mathfrak{S} \subset \mathfrak{L}(\mathbf{Y},\mathbf{X})$) has been answered by Corollary 2.1, and the question about when $\Delta|\mathfrak{S}$ is normal has been answered by Theorem 2.2 for special operators A and B on normed linear spaces $\mathbf{X}$ and $\mathbf{Y}$, by Corollary 2.3 for the case where $\mathbf{X}$ and $\mathbf{Y}$ are Hilbert spaces, and by Theorem 2.4 for the case where $\mathbf{X}$ is the same normed linear space as $\mathbf{Y}$ and $\mathfrak{S}$ is $\mathfrak{L}(\mathbf{X})$. But the latter question for the more general case where $\mathbf{X} \neq \mathbf{Y}$ or where $\mathbf{X} = \mathbf{Y}$ and $\mathfrak{S} \neq \mathfrak{L}(\mathbf{X})$ remain unanswered. It is unknown whether there exist nonnormal operators A, B such that $\Delta_{A,B}|\mathfrak{S}$ is normal.

On the other hand, when $\mathbf{X} = \mathbf{Y} = \mathbf{H}$ is a Hilbert space, one can deduce a stronger result than Corollaries 2.1 and 2.3. Indeed, a result of Sinclair [12, page 213] states that a Hermitian operator on a C^*-algebra (with identity) is the sum of a left multiplication by a Hermitian element in the algebra and a *-derivation, and a result of Kadison [7] and Sakai [11] asserts that every derivation of a C^*-algebra acting on H is spatial (that is, of the form $\Delta_{A,A}$, with A and element in the weak operator closure of the algebra). These facts together with Corollary 2.1 imply that an operator on a C^*-algebra $\mathfrak{S}$ in $\mathfrak{L}(\mathbf{H})$ is Hermitian if and only if it is of the form $\Delta_{H,M}|\mathfrak{S}$ for some Hermitian operators H and M (in the weak operator closure of $\mathfrak{S}$). Recently, Sourour [13] has proved the same assertion for the case where $\mathfrak{S}$ is a minimal norm ideal (including the $C_p(\mathbf{H})$ ideals, $p \neq 2$). Thus every normal operator on such $\mathcal{S}$ has to be of the form $\Delta_{A,B}|\mathcal{S}$, with $A = H + iK$ and $B = M + iN$ for some Hermitian H, K, M and N. Using Theorem 2.2 we obtain the following

THEOREM 2.6. *Let $\mathcal{S}$ be a minimal norm ideal or a C^*-algebra in $\mathcal{L}(\mathbf{H})$, which contains all finite rank operators. Then an operator U on $\mathcal{S}$ is Hermitian (resp. normal) in $\mathcal{L}(\mathcal{S})$ if and only if $U = \Delta_{A,B}|\mathcal{S}$ for some Hermitian (resp. normal) operators A and B.*

3. Berberian's theorem

As another application of Theorem 1.1, we shall derive an extension of the Fuglede-Putnam theorem to hyponormal operators A and B^* on Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. It is also a slight extension of a theorem of Berberian [3] who proved in a different way the special case where $\mathbf{H}_1 = \mathbf{H}_2$.

A natural and consistent definition for a hyponormal operator A on a normed linear space $\mathbf{X}$ is that it can be written as $A = H + iK$ for some Hermitian operators H and K such that $A^*A - AA^* = 2i(HK - KH)$ is positive (that is, has nonnegative numerical range). Suppose $B^* = M - iN$ is a hyponormal operator on another normed linear space $\mathbf{Y}$. Then the operator $\Delta_{A,B}|\mathcal{S}$ is also hyponormal. Indeed, from the easily verified identity:

$$\begin{aligned}\Delta^*\Delta - \Delta\Delta^* &= 2i(\Delta_{H,M}\Delta_{K,N} - \Delta_{K,N}\Delta_{H,M}) = 2i\Delta_{HK-KH,MN-NM}\\ &= \Delta_{A^*A-AA^*,B^*B-BB^*}\end{aligned}$$

we see that the numerical range of $(\Delta|\mathcal{S})^*(\Delta|\mathcal{S}) - (\Delta|\mathcal{S})(\Delta|\mathcal{S})^*$, as the sum of the numerical ranges of the two positive operators $A^*A - AA^*$ and $BB^* - B^*B$ is nonnegative.

In particular, if $\mathbf{X} = \mathbf{H}_1$ and $\mathbf{Y} = \mathbf{H}_2$ are two Hilbert spaces, then $\Delta | C_2(\mathbf{H}_2, \mathbf{H}_1)$ is a hyponormal operator on the Hilbert space $(C_2(\mathbf{H}_2, \mathbf{H}_1), \|\cdot\|_2)$. Hence we have $\|\Delta T\|_2 \geqslant \|\Delta^* T\|_2$ for all T in $C_2(\mathbf{H}_2, \mathbf{H}_1)$. Since $(\Delta_{A,B})^* = \Delta_{A^*,B^*}$, we have proved the following

THEOREM 3.1. *Let A and B^* be hyponormal operators on the Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. If T is a Hilbert-Schmidt operator from $\mathbf{H}_2$ to $\mathbf{H}_1$ such that $AT = TB$, then $A^*T = TB^*$.*

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