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AN OPTIMAL DIVIDEND POLICY WITH DELAYED CAPITAL INJECTIONS

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Abstract

This work focuses on finding optimal dividend payment and capital injection policies to maximize the present value of the difference between the cumulative dividend payment and the possible capital injections with delays. Starting from the classical Cramér–Lundberg process, using the dynamic programming approach, the value function obeys a quasi-variational inequality. With delays in capital injections, the company will be exposed to the risk of financial ruin during the delay period. In addition, the optimal dividend payment and capital injection strategy should balance the expected cost of the possible capital injections and the time value of the delay period. In this paper, the closed-form solution of the value function and the corresponding optimal policies are obtained. Some limiting cases are also discussed. A numerical example is presented to illustrate properties of the solution. Some economic insights are also given.

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1. Introduction

Designing dividend payment policies has long been an important issue in financial and actuarial sciences. The dividend payment decision is crucial because it not only represents an important signal about a firm's future growth opportunities and profitability but also may influence the investment and financing decisions of the firm. A practitioner will manage the bank capital and dividend payment against asset risks so that the bank can satisfy its minimum capital requirement. For insurance companies, because of the nature of their products, insurers tend to accumulate relatively large amounts of cash, cash equivalents, and investments in order to pay future claims and to avoid financial ruin. It is thus desirable to study the dividend decisions of insurance companies because the payment of dividends to shareholders may reduce an insurer's ability to survive adverse investment and underwriting experiences. Recently,

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the financial crisis has led to controversial discussion on the dividend policy of the European insurance industry [13].

Stochastic optimal control problems regarding dividend strategies for an insurance corporation have drawn increasing attention since the introduction of the optimal dividend payment model proposed by De Finetti [3]. Recently, there have been increasing efforts to use advanced methods of stochastic control to study the optimal dividend policy [1, 10, 17]. As an extension of previous work, the dividend is assumed to be paid out with the constraint that a transaction cost must be paid. Optimal dividend problems with transaction costs with a compound Poisson process were studied by Thonhauser and Albrecher [15], and with a Brownian risk model by He and Liang [8]. On the other hand, to maximize the expected total discounted dividend payments, the company will bankrupt almost surely if the dividend payment is paid out as a barrier strategy. In practice, Dickson and Waters [4] suggested that capital injections can be taken into account to maintain the business when cash flow is insufficient. Furthermore, a penalty will be paid at the time of ruin, which can be considered as the transaction cost (proportional and fixed) of capital injection [11, 14, 16]. Recently, there have been resurgent efforts devoted to the study of time delay on stochastic models. Puera and Keppo [12] considered the problem of a bank's optimal strategy of recapitalization with a fixed delay period. Bayraktar and Egami [2] proposed a direct solution method for delayed impulse control problems of one-dimensional diffusions and solved an optimal labour force problem with firing delay. Egami and Young [5] studied the optimal reinsurance strategy under fixed cost and delay. Considering such a delay in the capital injection makes our formulation more general and realistic. Whenever the company is on the verge of ruin, it can raise sufficient funds to survive. A natural payoff function is to maximize the difference between the expected total discounted dividend payment and the capital injections with costs under the optimal controls.

With the classical capital injection policies, the company could run the business in the absence of risk of ruin. However, empirical studies indicate that traditional surplus models with capital injections fail to capture the impact of regulatory processes of capital raising transactions. To better reflect reality, we have to consider the factor that the transactions of capital injections need a certain amount of time to be carried out after the decision of injecting extra capital is made. The time needed can be modelled by using delays. In the real world, the capital injection can never happen instantaneously. Time delays cannot be ignored and are unavoidable. Time delays occur naturally in insurance decision-making problems such as improving the capital reserve to a positive capital buffer level by capital injections. Many companies face regulatory delays (for example, preparatory and administrative work), which need to be taken into account when the companies make decisions under uncertainty of insolvency during the delays. In the presence of delay, the corporations will be exposed to a strictly positive probability of insolvency during the waiting period. In addition, dividend payment is not allowed during this cash insufficient period. Unlike the models where capital injections can be implemented instantaneously to remove the ruin completely, the positive probability of liquidation risk in our model makes the decision of capital injections with delays more realistic but more complicated.

As far as the significance of the contributions is concerned, this paper reveals clearly the differences between the strategies with and without delays. The problem of finding the optimal strategy under the condition of delayed capital injections involves a stochastic delay system with impulse controls. There is a fundamental difference between the models with and without delays. In the absence of delays, capital injections will only be implemented when surplus hits zero. The size of the capital injections is always a constant to increase the surplus to a positive capital buffer level [16]. One of the novel contributions of the current paper is that, taking into account the time delays, the impulse controls of capital injections depend on the surplus and can be very large. Together with the unrestricted dividend payment policy, using a quasi-variational inequality approach, we demonstrate that these state-dependent capital injections lead to the formulation of a free boundary problem. Under general assumptions, the analytical solution of the free boundary problem and the optimal state-dependent "threshold" strategies are obtained in this paper.

In addition, our new findings indicate that systems without delays in the traditional sense are a special case considered in our paper, namely $\Delta = 0$. Note that for the case of capital injection without delays, financial ruin can be completely avoided, whereas when delays are allowed and when surplus approaches zero, the optimal capital injection value converges to zero. This demonstrates that even when capital injections are available, the insurance company will be unlikely to avoid financial ruin due to delay when the surplus is sufficiently low. This has not been considered in the literature, to the best of our knowledge. Furthermore, the capital injection barriers are not monotone with respect to the delay period and exhibit hump-shaped curves. Furthermore, we show that the dividend payment barrier is very sensitive to the delay when the delay period is relatively short, but is more stable with relatively long delays. It also demonstrates that for large delays, a capital injection strategy is not optimal. We have found the most valuable moment of capital injections. To maximize the performance, it is shown that the injected capital is most valuable to the insurance company when the surplus reaches a neighbourhood of the capital injection barrier with a reasonable distance below it.

The new model we constructed involves the consideration of an important factor, the delay factor in the capital injection process. Through our numerical experiments, it can be clearly seen that the delay factor leads to state-dependent optimal strategies, which provide insights for the insurance company in their decision-making process and risk analysis. Not only are such results theoretically sound, but they are crucial in insurance practice. Capital injections and dividend payment policies with transaction costs are introduced as impulse and singular stochastic controls. The imposed time delay on the capital injections makes the problem more complicated. By adopting the diffusion approximation technique and using a diffusion model, we obtain a quasi-variational inequality (QVI) in this paper. The closed-form solution of the QVI is obtained under certain general assumptions. The value function is verified to be a concave function and defined separately in three regions, which are the capital injection region, continuation region, and dividend payment region. The capital injection barrier and dividend payment barrier are also given. Finally, the optimal capital injection and dividend payment strategies are obtained.

The paper is organized as follows. A formulation of the optimal capital injection strategies and dividend policies is presented in Section 2. Section 3 deals with the construction of the value function and dividend payment strategy and the verification of the solution of the value function. Some limiting cases on different values of Δ are considered in Section 4. A numerical example is provided in Section 5 to illustrate the effect of the parameters on properties of the solution. Further remarks are given in Section 6. Appendix A gives some technical results.

2. Formulation

We work with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\{\mathcal{F}_t\}$ (or simply $\mathcal{F}_t$) is a filtration satisfying the usual condition. That is, $\mathcal{F}_t$ is a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$ and $\mathcal{F}_0$ contains all null sets. In the traditional set-up of risk theory, one assumes that the surplus $X(t)$ of an insurance company in the absence of dividend payments and capital injections satisfies the classical Cramér–Lundberg process,

$$X(t) = x + ct - S(t), \quad t \ge 0,$$

where x is the initial surplus and c is the rate of premium. Let ϱ_j be the inter-arrival time of the jth claim, $\nu_n = \sum_{j=1}^{n} \varrho_j$, and let

$$N(t) = \max\{n \in \mathbb{N} : \nu_n \le t\}$$

be the number of claims up to time t, which is a Poisson counting process. Let

$$S(t) = \sum_{i=1}^{N(t)} Y_i$$

be a compound Poisson process representing claims with arrival rate λ, where $\{Y_i\}$ is a sequence of independent and identically distributed random variables known as the magnitudes of claim sizes and Y_1 has the distribution $\Pi(\cdot)$. Assume that $f(y)$, $y \ge 0$, is the probability density and let μ denote the expectation of Y. Then the Poisson measure $N(\cdot)$ has intensity $\lambda dt \times \Pi(dy)$ where $\Pi(dy) = f(y)dy$.

A dividend strategy $Z(\cdot)$ is an $\mathcal{F}_t$-adapted process $\{Z(t) : t \ge 0\}$ corresponding to the accumulated amount of dividends paid up to time t, such that $Z(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous with left limits. Throughout the paper, we use the convention that $Z(0^-) = 0$. The jump size of Z at time $t \ge 0$ is denoted by

$$\Delta Z(t) := Z(t) - Z(t^-),$$

and

$$Z^c(t) := Z(t) - \sum_{0\le s\le t} \Delta Z(s)$$

denotes the continuous part of $Z(t)$.

We treat the model that allows capital injections to avoid bankruptcy. The capital injection process $L(t) = \sum_{n=1}^{\infty} I_{\{\tau_n \le t\}}\zeta_n$ is described by a sequence of increasing stopping times $\{\tau_n : n = 1, 2, \ldots\}$ and a sequence of random variables $\{\zeta_n : n = 1, 2, \ldots\}$, which represent the times and the sizes of capital injections. The capital injections are associated with a time delay of length Δ. A control policy u is described by

$$u = \{Z; L\} = \{Z; \tau_1, \ldots, \tau_n, \ldots; \zeta_1, \ldots, \zeta_n, \ldots\}.$$

Assume that the evolution of $X(t)$, subject to capital injections and dividend payments, follows a one-dimensional process on an unbounded domain $G = [0, \infty)$. We impose $X(t) = 0$ for all $t > \tau_0$, where $\tau_0 = \inf\{t \ge 0 : X(t) < 0\}$ represents the time of ruin. The controlled asset process is given by

$$X(t) = x + ct - S(t) - Z(t) + \sum_n I_{\{\tau_n+\Delta\le t\}}\zeta_n$$

for all $t < \tau_0$.

Let $\mathrm{E}[Y_i] = \mu$ and $\mathrm{E}[Y_i^2] = \sigma^2$. By adopting diffusion approximation techniques, $X(t)$ can be approximated by

$$X(t) = x + (c - \lambda\mu)t + \sigma\lambda^{1/2}W(t) - Z(t) + \sum_n I_{\{\tau_n+\Delta\le t\}}\zeta_n,$$

where $W(t)$ is a standard Brownian motion. In our set-up, there is no safety loading. Then this formulation is consistent with the safety loading factor $\eta \to 0$ as used by Asmussen and Taksar [1].

REMARK 2.1. While claims do not arrive continuously, the continuous rate of arriving claims has been always a good approximation. The central limit theorem is applied to characterize the fluctuations of the claims, and the discrete-valued claim process is replaced by a diffusion process with a similar distribution of the infinitesimal increments. The classical reserve processes will converge weakly to some Brownian motion with drift. Of course any approximation has its limitations; however, the diffusion approximation has been successfully employed since the time of its inception.

We assume that shareholders need to pay $K + \zeta$ to meet the capital injection of ζ where $K > 0$ is the fixed transaction cost. We omit the fixed transaction cost in the dividend payout process. Denote by $r > 0$ the discounting factor. For an arbitrary admissible pair $u = (Z, L)$, the performance functional is

$$J(x, u) = E_x\Big[\int_0^{\tau_0} e^{-rt}\, dZ - \sum_n e^{-r(\tau_n+\Delta)}(K + \zeta_n) I_{\{\tau_n+\Delta<\tau_0\}}\Big]. \tag{2.1}$$

The pair $u = (Z, L)$ is said to be *admissible* if Z and L satisfy the following requirements:

(i) $Z(t)$ and $L(t)$ are nonnegative for any $t \geq 0$;
(ii) Z is càdlàg (that is, it is right continuous and has left limits), nondecreasing and adapted to $\{\mathcal{F}_t\}$;
(iii) τ_n is a sequence of stopping times with respect to $\{\mathcal{F}_t\}$ and $0 \leq \tau_1 < \cdots < \tau_n < \cdots$ almost surely;
(iv) ζ_n is measurable with respect $\mathcal{F}_{\tau_n}$;
(v) $\mathbb{P}(\lim_{n\to\infty} \tau_n < T) = 0$ for all $T > 0$; and
(vi) $J(x, u) < \infty$ for any x and admissible pair $u = (Z, L)$, where J is the functional defined in (2.1).

In addition, we assume that the admissible control u satisfies:

(a) $\tau_{n+1} - \tau_n \geq \Delta$ for all $n \geq 1$;
(b) $dZ(t) = 0$ for all $t \in [\tau_n, \tau_n + \Delta]$, $n \geq 1$.

Condition (a) tells us that a new capital injection should not be placed during the waiting period of the previous capital injection. Condition (b) states that dividends will not be paid during the waiting periods of the capital injections.

Suppose that $\mathcal{A}$ is the collection of all admissible pairs. Define the value function as

$$V(x) := \sup_{u\in\mathcal{A}} J(x, u). \tag{2.2}$$

For all $V(\cdot) \in C^2(\mathbb{R})$, define an operator $\mathcal{L}$ by

$$\mathcal{L}V(x) = (c - \lambda\mu)V_x(x) + \tfrac{1}{2}\lambda\sigma^2 V_{xx}(x),$$

where V_x and $V_{xx}(x)$ denote the first and second derivatives with respect to x, respectively. For X_Δ satisfying $dX(t) = (c - \lambda\mu)\,dt + \sigma\lambda^{1/2}\,dW(t)$ with $X(0) = x$, define a capital injection operator $\mathcal{M}$ by

$$\mathcal{M}V(x) = \mathrm{E}_x\Big[e^{-r\Delta} \sup_{s\geq 0}\{V(X_\Delta + s) - s - K\} I_{\{\tau_0>\Delta\}}\Big], \tag{2.3}$$

in which the integrability is assumed to ensure that $\mathcal{M}V(x)$ is well defined; in the following development, we find a suitable function ensuring this. If the value function V defined in (2.2) is sufficiently smooth, by applying the dynamic programming principle [6], we formally derive the following QVI:

$$\max\{\mathcal{L}V(x) - rV(x), 1 - V_x(x), \mathcal{M}V(x) - V(x)\} = 0. \tag{2.4}$$

The derivation of (2.4) is given in Appendix A.

Similar to Yao et al. [16], we divide the set of the surplus into three regions:

(i) continuation region,

$$C := \{\mathcal{L}V(x) - rV(x) = 0, 1 < V_x(x), \mathcal{M}V(x) < V(x)\};$$

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(ii) dividend payout region,

$$\mathcal{D} := \{\mathcal{L}V(x) - rV(x) < 0, 1 = V_x(x), \mathcal{M}V(x) < V(x)\};$$

(iii) capital injection region,

$$\mathcal{I} := \{\mathcal{L}V(x) - rV(x) < 0, 1 < V_x(x), \mathcal{M}V(x) = V(x)\}.$$

Boundary conditions. The capital injection is taken into account when there is not enough solvency capital to maintain the business. To make the company run continuously, the capital injections will definitely occur at the moments when $x = 0$. Intuitively, in the absence of the capital injection delays, on the boundary of the capital injection region, the value function obeys

$$\mathcal{M}'V(x) = \sup_{s\geq 0}\{V(x+s) - s - K\}.$$

Then the optimal payoff or the value function $V(x)$ will not be zero with instantaneous capital injections, which could always guarantee the stability of the company's capital structure [9]. However, with the capital injection delays, the company will violate the capital adequacy if the capital injection is held while the surplus hits zero. Thus, taking into account the time delay of the capital injections, the value of the $V(x)$ on the boundary can be obtained as

$$V(0) = 0. \tag{2.5}$$

In addition, capital injections also occur whenever the surplus is sufficiently low. The impulse control of a capital injection depends on the surplus state and leads to a free boundary of the capital injection region.

We consider the dividend payment strategy with delayed capital injections. Combining (2.4) and (2.5), the following QVI with boundary condition is obtained:

$$\max\{\mathcal{L}V(x) - rV(x), 1 - V_x(x), \mathcal{M}V(x) - V(x)\} = 0, \quad V(0) = 0. \tag{2.6}$$

REMARK 2.2. The value function $V(x)$ is not necessarily smooth. In fact, the second derivative of the value function is not always continuous. In the absence of a classical solution of the QVI, one alternative definition for a solution to (2.6) is that of a viscosity solution. However, we can interpret the differential generator in terms of left or right derivatives [12].

3. Value function, dividend strategy and verification

3.1. Value function and dividend strategy To solve the QVI, we guess the form of a solution and verify the validation of the constructed solution in a general case. The solution can be formulated based on the strategies in each of the optimal regions. Referring to Jin et al. [9], we consider the dividend payment strategy with capital injections as a band strategy. The decision maker takes no action until the surplus

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reaches the lower barrier, where an impulse control of capital injection is taken. The dividend is paid out immediately when the surplus reaches the upper barrier. That is, suppose there exist two thresholds b_1 and b_2 separating the three regions, where $0 < b_1 < b_2 < \infty$. The form of the solution is as follows:

(1) for $x \in [0, b_1)$, it is optimal to inject the capitals;
(2) for $x \in [b_1, b_2)$, it is optimal neither to order the new capitals nor to pay dividends;
(3) for $x \in [b_2, \infty)$, it is optimal to pay all the extra surplus as dividends.

In view of (2.6), we look for a function that satisfies each component of the QVI in the corresponding regions defined above. According to Remark 2.2, this function is continuously differentiable at the impulse control barrier b_1 and twice continuously differentiable at the singular control barrier b_2.

We now construct the solutions in the continuation region when the equality holds. Write $f(x)$ for the candidate solution in the continuation region. The equality in the continuation region then becomes

$$\tfrac{1}{2}\lambda\sigma^2 V_{xx}(x) + (c - \lambda\mu)V_x(x) - rV(x) = 0. \tag{3.1}$$

The solution to (3.1) is

$$f(x) = m_1 e^{d_+ x} + m_2 e^{d_- x}, \tag{3.2}$$

where

$$d_\pm = \frac{-(c - \lambda\mu) \pm \sqrt{(c - \lambda\mu)^2 + 2r\lambda\sigma^2}}{\lambda\sigma^2}.$$

In addition, the equality in the dividend payout region is represented as

$$1 = V_x(x). \tag{3.3}$$

Then the solution in the dividend payout region, denoted by $g(x)$, is given by

$$g(x) = x + a.$$

Based on the rule of the solution forms, it is shown that b_2 is the threshold to separate the continuation region and dividend payout region. Thus, the solution should satisfy both (3.1) and (3.3) at b_2. On the other hand, the twice continuous differentiability of $g(x)$ at b_2 requires that $f_x(b_2) = 1$ and $f_{xx}(b_2) = 0$. Imposing these boundary conditions on (3.2) yields

$$f(x; b_2) = a_1 e^{-d_+(b_2 - x)} + a_2 e^{-d_-(b_2 - x)}, \tag{3.4}$$

where

$$a_1 = \frac{d_-}{d_+(d_- - d_+)} > 0, \quad a_2 = \frac{d_+}{d_-(d_+ - d_-)} < 0.$$

Furthermore, substituting $f_x(b_2) = 1$ and $f_{xx}(b_2) = 0$ into (3.1) yields

$$f(b_2; b_2) = \frac{c - \lambda\mu}{r}. \tag{3.5}$$

Moreover, subject to the boundary condition at b_2, $g(x)$ becomes

$$g(x; b_2) = x + \frac{c - \lambda\mu}{r} - b_2. \tag{3.6}$$

Finally, we need only construct the solution in the capital injection region. Assume that a concave function denoted by $h(x; b_2)$ satisfies $V_x(x; b_2)|_{x=b_2} = 1$ and (3.5). Because of the concavity of the function $h(x; b_2)$, it is shown that the supremum is achieved when $s = b_2 - X_\Delta$. Thus, (2.3) is simplified to

$$\begin{aligned} h(x; b_2) &= e^{-r\Delta}\mathrm{E}_x[\{V(b_2; b_2) - b_2 + X_\Delta - K\}I_{\{\tau_0>\Delta\}}] \\ &= e^{-r\Delta}\mathrm{E}_x\Big[\Big\{X_\Delta + \frac{c - \lambda\mu}{r} - b_2 - K\Big\}I_{\{\tau_0>\Delta\}}\Big]. \end{aligned} \tag{3.7}$$

Let $\gamma = (c - \lambda\mu)/r - b_2 - K$. This parameter γ can be interpreted as the benefit of capital injections. Expression (3.7) can be further simplified as

$$h(x; b_2) = e^{-r\Delta}\mathrm{E}_x[\{X_\Delta + \gamma\}I_{\{\tau_0>\Delta\}}] = e^{-r\Delta}\mathrm{E}_{x+\gamma}[X_\Delta I_{\{\tau_\gamma>\Delta\}}], \tag{3.8}$$

where $\tau_\gamma = \inf\{t \geq 0 : X(t) = \gamma\}$. Define a Markov transition probability density function $p(\Delta, x + \gamma, y)$ with $y \in [0, \infty)$, which is the density of the absorbed process $X_{\Delta\wedge\tau_\gamma}$ that starts at $x + \gamma$. Then

$$\begin{aligned} p(\Delta, x + \gamma, y) &= \varphi(y, (c - \lambda\mu)\Delta + x + \gamma, \sigma\lambda^{1/2}\sqrt{\Delta}) \\ &\quad - \exp\Big(-\frac{2(c - \lambda\mu)x}{\sigma^2}\Big)\varphi(y, (c - \lambda\mu)\Delta - x + \gamma, \sigma\lambda^{1/2}\sqrt{\Delta}), \end{aligned}$$

where $\varphi(y, \hat{\mu}, \hat{\sigma})$ denotes the density of a normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$, that is,

$$\varphi(y, \hat{\mu}, \hat{\sigma}) = \frac{1}{\hat{\sigma}\sqrt{2\pi}}\exp\Big(-\frac{(y - \hat{\mu})^2}{2\hat{\sigma}^2}\Big).$$

Hence, (3.8) yields

$$h(x; b_2) = e^{-r\Delta}\mathrm{E}_{x+\gamma}[X_{\Delta\wedge\tau_\gamma}I_{\{\tau_\gamma>\Delta\}}] = e^{-r\Delta}\int_\gamma^\infty yp(\Delta, x + \gamma, y)\,dy.$$

Finally, we obtain the constructed function as follows:

$$\begin{aligned} h(x; b_2) = e^{-r\Delta}\Big\{&\Big[x + (c - \lambda\mu)\Delta + \frac{c - \lambda\mu}{r} - b_2 - K\Big]\Phi\Big(\frac{x + (c - \lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big) \\ &+ \sigma\lambda^{1/2}\sqrt{\Delta}\varphi\Big(\frac{x + (c - \lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big) - \exp\Big(-\frac{2(c - \lambda\mu)x}{\lambda\sigma^2}\Big) \\ &\times\Big[\Big[-x + (c - \lambda\mu)\Delta + \frac{c - \lambda\mu}{r} - b_2 - K\Big]\Phi\Big(\frac{-x + (c - \lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big) \\ &+ \sigma\lambda^{1/2}\sqrt{\Delta}\varphi\Big(\frac{-x + (c - \lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big)\Big]\Big\}, \end{aligned} \tag{3.9}$$

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where $\Phi(\cdot)$ and $\varphi(\cdot)$ are the cumulative standard normal distribution and its density function, respectively. It is clear that $h(0; b_2) = 0$.

On the other hand, the boundary condition of $h(x; b_2)$ on the boundary of the capital injection region b_1 can be formulated as

$$h(b_1; b_2) = f(b_1; b_2), \tag{3.10}$$

$$\frac{\partial h(x; b_2)}{\partial x}\bigg|_{x=b_1} = \frac{\partial f(x; b_2)}{\partial x}\bigg|_{x=b_1}. \tag{3.11}$$

The explicit expressions for the region boundaries b_1 and b_2 are not easy to obtain because of the nonlinearity of the QVI. However, the existence of b_1 and b_2 is verified in the next section under certain conditions. Combining (3.4), (3.6) and (3.9), given the existence of b_1 and b_2, the value function $V(x)$ can be written as

$$V(x) = \begin{cases} h(x; b_2) & \text{if } 0 \le x < b_1 \\ f(x; b_2) & \text{if } b_1 \le x < b_2 \\ g(x; b_2) & \text{if } b_2 \le x < \infty. \end{cases} \tag{3.12}$$

3.2. Verification theorem Here we verify the existence of the boundaries of the continuation region b_1 and b_2. Under general conditions, sufficient conditions for the existence of b_1 and b_2 are given. Moreover, the value function $V(x)$ defined in (3.12) is verified as the solution to (2.6). To prove the theorem, we need to establish a series of technical lemmas. These are given in Appendix A.

THEOREM 3.1. *Assume that a solution to* (3.10)–(3.11) *as defined in Lemma* A.5 *exists and* $V(x)$ *is defined by* (3.12). *Then* $V(x)$ *is a concave solution to* (2.6).

PROOF. We prove the concavity of $V(x)$ in the three regions. In the dividend payout region, $V_{xx}(x) = 0$. In the capital injection region, $h(x; b_2)$ is concave following from Lemma A.2. In the continuation region, differentiating $f(x; b_2)$ three times, it is shown that $f_{xxx}(x; b_2) > 0$ on $x \in [b_1, b_2)$. Combining with the value of the second-order derivative on the boundary, $f_{xx}(b_2; b_2) = 0$, we have $f_{xx}(b_2; b_2) < 0$ for all $x \in [b_1, b_2)$. Hence, $V(x)$ is concave in the continuation region $[b_1, b_2)$. Thus, $V(x)$ is concave.

The proof that $V(x)$ satisfies (2.6) consists of four steps.

Step 1. $V(0) = h(0; b_2) = 0$.

Step 2. For $x \in [b_2, \infty)$, $V_x(x) = 1$ by construction. For $x \in [0, b_2)$, $V_x(x) > 1$ following from the concavity of $V(x)$.

Step 3. $V(x) = h(x; b_2) = \mathcal{M}V(x)$ for $x \in [0, b_1)$ by construction. For $x \in [b_1, b_2)$, $V(x) = f(x; b_2) \ge h(x; b_2) = \mathcal{M}(x)$ following from Lemma A.5. For $x \in [b_2, \infty)$, $V(x) = g(x; b_2) > \mathcal{M}(x)$. Hence, $V(x) \ge \mathcal{M}(x)$ globally.

Step 4. For $0 \le x < b_1$, $V(x) = h(x; b_2)$. Denote $\tau_\varepsilon^* = \tau_\varepsilon \wedge \varepsilon$ for some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \in (0, b_1)$ and $\tau_\varepsilon = \inf\{t \ge 0 : X(t) \notin (X(0) - \varepsilon, X(0) + \varepsilon)\}$. Following from Dynkin's formula,

$$\mathrm{E}_x[e^{-r\tau_\varepsilon^*} h(X(\tau_\varepsilon^*); b_2)] = h(x; b_2) + \mathrm{E}_x\bigg[\int_0^{\tau_\varepsilon^*} (\mathcal{L} - r)h(X(s); b_2)\, ds\bigg],$$

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where $\mathrm{E}_x[e^{-r\tau^*_\varepsilon}h(X(\tau^*_\varepsilon); b_2)]$ is the present value of the optimal payoff or the value function where a new capital injection occurs after τ^*_ε. Because of the rule of optimization that immediate ordering of capital is the optimal policy in the capital injection region,

$$\mathrm{E}_x[e^{-r\tau^*_\varepsilon}h(X(\tau^*_\varepsilon); b_2)] \le h(x; b_2).$$

Thus, in the capital injection region $x \in [0, b_1)$, we obtain

$$\mathrm{E}_x\Big[\int_0^{\tau^*_\varepsilon}(\mathcal{L} - r)V(X(s))\,ds\Big] \le 0.$$

Taking the limit yields

$$(\mathcal{L} - r)V(x) = \lim_{\varepsilon\to 0}\frac{1}{\mathrm{E}_x[\tau^*_\varepsilon]}\mathrm{E}_x\Big[\int_0^{\tau^*_\varepsilon}(\mathcal{L} - r)V(X(s))\,ds\Big] \le 0,$$

where $x \in [0, b_1)$. For $x \in [b_1, b_2)$, $(\mathcal{L} - r)V(x) = 0$ by construction. For $x \in [b_2, \infty)$, $V(x) = (c - \lambda\mu)/r + x - b_2$. Then

$$\begin{aligned}(\mathcal{L} - r)V(x) &= (c - \lambda\mu)V_x(x) + \tfrac{1}{2}\lambda\sigma^2 V_{xx}(x) - rV(x)\\ &= c - \lambda\mu - r\Big(\frac{c - \lambda\mu}{r} + x - b_2\Big) = r(b_2 - x) \le 0.\end{aligned}$$

Thus, $(\mathcal{L} - r)V(x) \le 0$ globally.

To summarize, the value function $V(x)$ in (3.12) satisfies the QVI (2.6). □

4. Discussion of different cases of Δ

4.1. $\Delta = 0$ In the absence of any capital delays, the optimal dividend payment strategy is a barrier strategy, where the extra surplus is paid out as dividends beyond a certain barrier level. To maximize performance, the capital injection time is postponed and capital could fall to arbitrarily close to zero. Thus zero is an absorbing boundary. The value function is concave and monotone increasing. The initial value $V(0)$ is nonzero because capital injections can always guarantee the continuity of business even with zero initial surplus. Furthermore, starting as a curve, the value function increases linearly after the barrier level, which means that extra surplus will all be paid out as dividends after reaching a certain barrier.

Taking $\Delta = 0$ in (3.9), the function $h(x; b_2)$ is simplified to

$$h(x; b_2) = x + \frac{c - \lambda\mu}{r} - K - b_2,$$

where $x > 0$. Hence, the initial value of the value function $V(0)$ under the condition $\Delta = 0$ can be written as

$$V(0) = \max\Big(\lim_{x\to 0^+} h(x; b_2|_{\Delta=0}), 0\Big) = \max\Big(\frac{c - \lambda\mu}{r} - K - b_2, 0\Big). \tag{4.1}$$

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Since the capital injection region is reduced to one point, we only have one barrier in this limiting case. This barrier separates the continuation region and dividend payout region. The value function and the corresponding barrier are given in the following proposition.

PROPOSITION 4.1. *If* $(c - \lambda\mu)/r - K - b_0 > 0$, *where* b_0 *is given in* (A.6), *then the value function satisfies*

$$V(x) = \begin{cases} a_1 e^{-d_+(\bar{b}-x)} + a_2 e^{-d_-(\bar{b}-x)} & \text{if } x < \bar{b} \\ (c - \lambda\mu)/r + x - \bar{b} & \text{if } x \geq \bar{b}, \end{cases}$$

where $\bar{b} < b_0$ *is the unique positive solution for* b *in the equation*

$$\tilde{f}(0, b) = \frac{c - \lambda\mu}{r} - K - b.$$

Otherwise, $\bar{b} = b_0$.

PROOF. Assume that the maximum in (4.1) is achieved by the first term. Following from (3.4), the boundary of the dividend payout region $\bar{b}$ is determined by

$$\tilde{f}(0, b) = a_1 e^{-d_+ b} + a_2 e^{-d_- b} = \frac{c - \lambda\mu}{r} - K - b. \tag{4.2}$$

Since $\tilde{f}_b(0, b) < 0$ and $\tilde{f}(0, 0) = (c - \lambda\mu)/r > (c - \lambda\mu)/r - K - b > \tilde{f}(0, \infty) = 0$, (4.2) is solved by a unique $\bar{b} > 0$. In addition, $\bar{b}$ satisfies

$$\tilde{f}(0, \bar{b}) = \frac{c - \lambda\mu}{r} - K - \bar{b} > 0. \tag{4.3}$$

On the other hand, $\tilde{f}(0, b_0) = 0$ by (A.6). Thus, the negative derivative of $\tilde{f}(0, b)$ with respect to b implies $\bar{b} < b_0$. To guarantee the validity of (4.3), a sufficient condition for $(c - \lambda\mu)/r - K - b_0 > 0$ must be satisfied. That is, $\tilde{f}(0, b)$ intersects $(c - \lambda\mu)/r - K - b$ with positive value at $\bar{b}$ under the assumption. □

4.2. Capital injection is not optimal for large Δ When the capital injection delay Δ or the transaction cost K are sufficiently large, capital injection is no longer optimal. Thus, the problem is reduced to analysing a dividend payment strategy without capital injection [7]. Hence the capital injection region does not exist in this case. Then the corresponding value function becomes a special form of (3.12) with $b_1 = 0$ and $b_2 = b_0$, given by

$$V(x) = \begin{cases} a_1 e^{-d_+(b_0-x)} + a_2 e^{-d_-(b_0-x)} & \text{if } x < b_0 \\ (c - \lambda\mu)/r + x - b_0 & \text{if } x \geq b_0, \end{cases} \tag{4.4}$$

where a_1, a_2 and $d_\pm$ are defined as above.

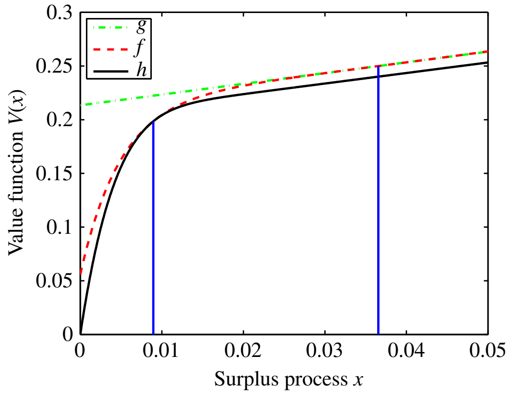

FIGURE 1. Optimal value function versus surplus process.

5. Numerical example

This section is devoted to a numerical example. Under the assumptions, the value function is constructed. We also analyse the effects of claim frequency, delay on the capital injection, and capital injection cost on optimal capital injection and dividend policies. We assume that $S(t)$ is a compound Poisson process. The claim severity distribution follows $\mu = 0.01$ and $\sigma = 0.01$. The rate of premium c is assumed to be 0.02. Furthermore, $\{\nu_{n+1} - \nu_n\}$ is a sequence of exponentially distributed random variables with $\lambda = 1$. The discounting factor is $r = 0.04$.

5.1. Value function Let the delay of capital injections be $\Delta = 0.5$, and the capital injection cost $K = 0.01$. Then the value function $V(x)$ against the surplus process is depicted in Figure 1. It can be shown that the value function is monotonically increasing and concave as described in (3.12). The capital injection barrier $b_1 = 0.9\%$ and the dividend payout barrier $b_2 = 3.66\%$ separate the three regions.

The value function $V(x)$ coincides with h in the capital injection region $[0, b_1)$, where it is optimal to order the capital injection immediately. $V(x)$ matches f in the continuation region $[b_1, b_2)$, in which neither capital injection nor dividend payment is optimal. We also see that h and f are smoothly connected with the first derivatives being equal, but different second derivatives. Since h is more concave than f at b_1, the second derivative is discontinuous at b_1. When the surplus hits b_2, extra surplus is paid out immediately to maximize the payoff function. Thus $V(x)$ follows g in the dividend payout region $[b_2, \infty)$. Moreover, f and g are also smoothly connected with equal first and second derivatives, but not third derivatives, where $f_{xxx}(b_2) = -2r/\sigma^2 = -800$ and $g_{xxx}(b_2) = 0$.

5.2. Optimal dividend and capital injection strategies Figure 2 shows the effects of claim frequency λ on optimal dividend and capital injection strategies. Let

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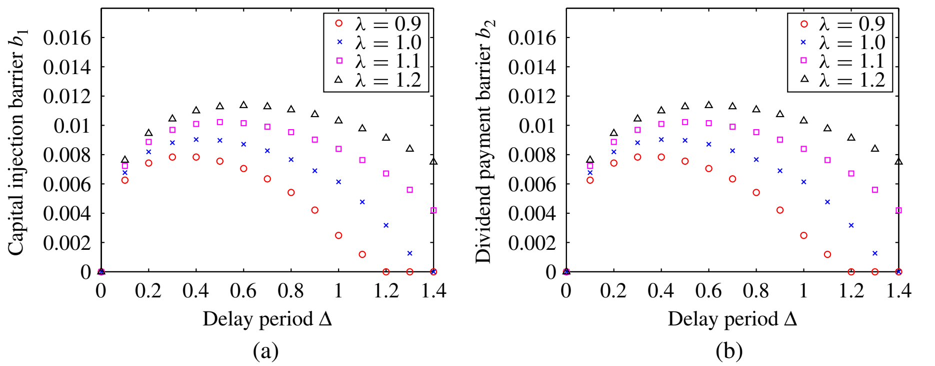

FIGURE 2. (a) Capital injection barrier b_1 versus delay period Δ. (b) Dividend payment barrier b_2 versus delay period Δ.

the parameters c, K, r, μ and σ be fixed, with different claim frequencies. The corresponding barriers b_1 and b_2 are obtained against the delay period Δ.

Figure 2(a) shows that the capital injection barriers are increasing with respect to the claim frequency λ. The more claims, the more risk of liquidation the company will be exposed to during the delay period, and thus the decision makers are more willing to order capital injections. Furthermore, the capital injection barrier converges to zero as the delay Δ converges to zero, which is consistent with the result of Dao et al. [16]. When $\Delta = 0$, due to the time value of money, the capital injection should be postponed until the surplus reaches arbitrarily close to zero to maximize performance.

Moreover, the capital injection barriers are not monotone with respect to the delay period. These humped curves show that the capital injection barrier is positively related to increasing delay period when the delay is relatively short. When the delay is relatively long, the response to the increasing delay period is negative in the capital injection barrier. This is because the capital injection barrier is influenced by the delay in two ways. On one hand, since dividend payment is prohibited during the delay period, it is optimal to take a “wait and see” approach instead of ordering capital injections immediately to avoid the potential loss of missing dividends in the future, leading to a decrease of the capital barrier. On the other hand, higher risk of financial ruin with respect to a longer delay period yields a higher capital injection barrier. When the delay is relatively long, the former reason is dominant; when the delay is relatively short, the latter reason is dominant.

In Figure 2(b), the dividend payment barriers are decreasing with respect to the claim frequency λ. It can be interpreted that the dividend payment should be postponed to avoid surplus shortage if more frequent claims come. As the delay period increases, the dividend payment barriers are monotonically increasing because of prohibition of dividend payment during the delay period. Moreover, higher risk of financial ruin due to the increasing delay period leads to a higher reserve of the surplus to guarantee financial safety, thus yielding an increasing dividend payment barrier.

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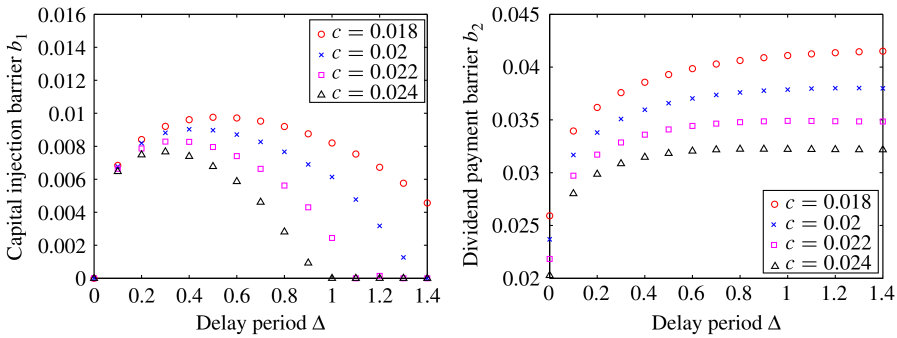
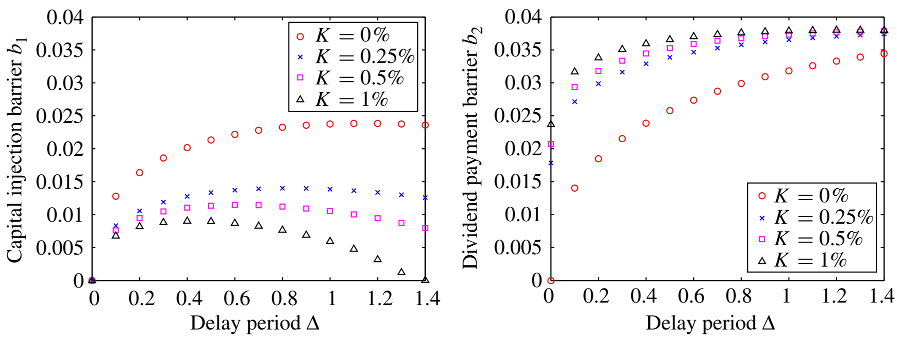


(a) Capital injection barrier b_1 versus delay period Δ

(b) Dividend payment barrier b_2 versus delay period Δ

FIGURE 3. Effects of the rate of premium c on optimal dividend and capital injection strategies with $\lambda = 1$, $K = 0.01$, $r = 0.04$, $\mu = 0.01$, $\sigma = 0.01$.



(a) Capital injection barrier b_1 versus delay period Δ

(b) Dividend payment barrier b_2 versus delay period Δ

FIGURE 4. Effects of the transaction cost K on optimal dividend and capital injection strategies with $c = 0.02$, $\lambda = 1$, $r = 0.04$, $\mu = 0.01$, $\sigma = 0.01$.

Figure 3 analyses the effects of the rate of premium c on optimal dividend and capital injection strategies. We see that both the capital injection barrier and dividend payment barrier are decreasing with respect to c. It is shown that the capital injection is postponed with lower risk of financial ruin because of the higher rate of premium income, suggesting a decrease of the capital injection barrier. In addition, the dividend payment is triggered at a lower surplus status with higher rate of premium. The impact of the delay on the capital injection and dividend payment barriers under various rates of premium has the same trend and interpretation as for Figure 2.

Figure 4 shows the impact of the capital injection cost K on optimal dividend and capital injection strategies. In Figure 4(a), unlike the effects of λ and c, the capital injection barriers are decreasing in the capital injection cost K. With higher capital

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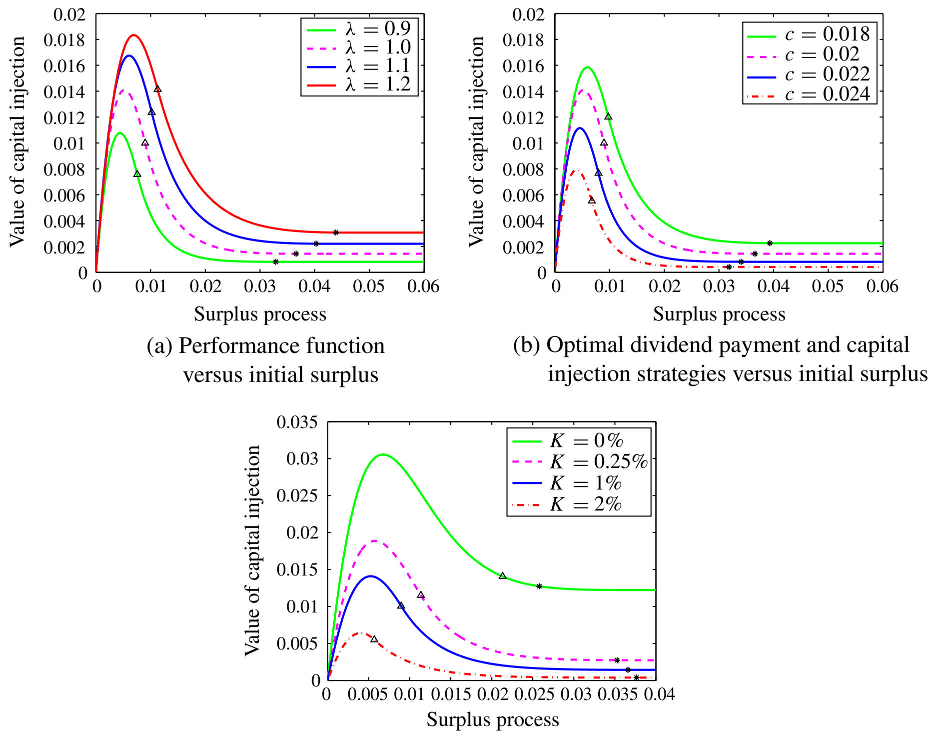

(a) Performance function versus initial surplus

(b) Optimal dividend payment and capital injection strategies versus initial surplus

(c) Optimal capital injection versus initial surplus

FIGURE 5. Exponential claim size distribution with $r = 0.04$, $\mu = 0.01$, $\sigma = 0.01$.

injection cost, decision makers are less willing to order extra capital injections, which yields a lower barrier of the capital injection region. In Figure 4(b), the dividend payment barriers are increasing with respect to the capital injection cost K, since an increasing capital injection cost increases the surplus reserved needed to guarantee financial safety, thus postponing the dividend payment.

We also find that the dividend barrier is nondecreasing with respect to K. However, it is shown that the dividend payment barrier is very sensitive to the delay when the delay period is relatively short, while for the case of a relatively long delay period, the dividend barrier is more stable. In particular, when $K = 0$ and $\Delta = 0$, it is shown that the dividend barrier is zero. In this case, the so-called perfect market, the surplus is paid out immediately as dividend to maximize performance.

5.3. Value of capital injection The opportunity of capital injection is an option for the insurance company, and thus will not reduce the value of the company. The value of the capital injection is determined as the difference of (3.12) and (4.4), where no capital injections are exercised. In Figure 5, the values of the optimal capital injections are

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depicted under the effects of parameters. The corresponding capital injection barriers and dividend payment barriers are also marked as triangles and stars, respectively.

Figure 5 shows that the optimal capital values all follow hump-shaped curves with respect to surplus. The value of the optimal capital injection is increasing with respect to λ and decreasing with respect to c and K. This result is reasonable since λ and c have the opposite effect on the financial status, and moreover, higher claim frequency demands a larger value of capital injection to guarantee financial stability. For the capital injection cost K, it is shown that the value of optimal capital injection is reduced with higher capital raising cost.

When surplus approaches zero, the value of the optimal capital injection also converges to zero. This can be interpreted as the insurance company being unlikely to avoid financial ruin due to the delay when the surplus is sufficiently low, while for sufficiently large surplus, the company is financially stable and unlikely to suffer ruin. Hence, the value of the capital injection is rather flat and stable. In addition, the optimal capital injection is highest when the surplus is near and below the capital injection barrier. Surprisingly, the capital injection is most valuable to the insurance company when the surplus reaches a neighbourhood of the capital injection barrier with a reasonable distance below it.

6. Further remarks

We study optimal dividend and capital injection strategies with constant time delays in capital injection processes. Closed-form solutions of the value function and optimal strategies are obtained. To provide guidance for decision makers in the insurance industry, numerical experiments are presented and corresponding economic insights are given. More general time delay models that can be a function of time or even a random process may be considered in future. These deserve further thought and consideration. Nevertheless, in the more general models, it will be very difficult to obtain a closed-form solution, although numerical algorithms can be constructed. In any case, this paper provides some insight for more complex models.

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Appendix A. Technical results

A.1. Derivation of (2.4) Proving the following four inequalities is equivalent to deriving (2.4):

$$\mathcal{L}V(x) - rV(x) \le 0, \tag{A.1}$$

$$1 - V_x(x) \le 0, \tag{A.2}$$

$$\mathcal{M}V(x) - V(x) \le 0, \tag{A.3}$$

$$(\mathcal{L}V(x) - rV(x))(1 - V_x(x))(\mathcal{M}V(x) - V(x)) = 0. \tag{A.4}$$

Since $V(x)$ is optimal, $V(x) \geq \mathrm{E}_x[e^{-r\delta}V(X(\delta))]$ for any $\delta > 0$. Then

$$0 \geq \mathrm{E}_x \frac{V(X(\delta)) - V(x)}{\delta} + \frac{e^{-r\delta} - 1}{\delta} \mathrm{E}_x V(X(\delta)).$$

By virtue of Itô's lemma and letting $\delta \to 0$, (A.1) holds. Consider an admissible strategy u_0 with $J(y, u_0) \geq V(y) - \varepsilon$ for any $\varepsilon > 0$. For any $x \geq y$, we define a new strategy u_1 which pays $x - y$ as dividend immediately and follows u_0. Then for any $\varepsilon > 0$,

$$V(x) \geq x - y + J(y, u_0) \geq x - y + V(y) - \varepsilon.$$

Since ε is arbitrary, $V(x) \geq x - y + V(y)$. Thus, (A.2) holds as $V(x)$ is assumed to be sufficiently smooth. Inequality (A.3) holds since $V(x)$ is the performance function with optimal strategies at x and is always larger than the performance function $\mathcal{M}V(x)$ that is associated with the best immediate capital injection strategies. In addition, equality holds when it is optimal to inject capitals. Equation (A.4) holds since one of the three inequalities (A.1)–(A.3) must hold in an optimal strategy. That is, in the optimal strategy, we should either take no action, pay dividend or inject capital.

A.2. Lemmas

Lemma A.1. *Let $q(x, t) = \mathrm{P}[\tau_0 \leq t \mid X(0) = x]$. Then*

$$\mathrm{E}_x[X_\Delta I_{\{\tau_0 > \Delta\}}] = x + (c - \lambda\mu)\Delta - (c - \lambda\mu) \int_0^\Delta q(x, t)]\, dt.$$

Proof. The equality is obtained as follows:

$$\begin{aligned}
\mathrm{E}_x[X_\Delta I_{\{\tau_0 > \Delta\}}] &= \mathrm{E}_x[X_\Delta] - \mathrm{E}_x[X_\Delta I_{\{\tau_0 \leq \Delta\}}] \\
&= \mathrm{E}_x[X_\Delta] - \int_0^\Delta \mathrm{E}[X_\Delta \mid X(t) = 0] \frac{\partial}{\partial t} q(x, t)\, dt \\
&= x + (c - \lambda\mu)\Delta - \int_0^\Delta (c - \lambda\mu)(\Delta - t) \frac{\partial}{\partial t} q(x, t)\, dt \\
&= x + (c - \lambda\mu)\Delta(1 - q(x, \Delta)) + (c - \lambda\mu) \int_0^\Delta t \frac{\partial}{\partial t} q(x, t)\, dt \\
&= x + (c - \lambda\mu)\Delta - (c - \lambda\mu) \int_0^\Delta q(x, t)\, dt.
\end{aligned}$$

□

Lemma A.2. *If $\gamma \geq 0$ then, for all $x > 0$,*

$$\frac{\partial h(x; b_2)}{\partial x} > 0 \quad \textit{and} \quad \frac{\partial^2 h(x; b_2)}{\partial x^2} < 0.$$

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PROOF. In view of (3.8),

$$
\begin{aligned}
h(x;b_2) &= e^{-r\Delta}\mathrm{E}_x[\{X_\Delta+\gamma\}I_{\{\tau_0>\Delta\}}]\\
&= e^{-r\Delta}[\gamma\mathrm{E}_x[I_{\{\tau_0>\Delta\}}]+\mathrm{E}_x[X_\Delta I_{\{\tau_0>\Delta\}}]]\\
&= e^{-r\Delta}[\gamma(1-q(x,\Delta))+\mathrm{E}_x[X_\Delta I_{\{\tau_0>\Delta\}}]]\\
&= e^{-r\Delta}\Big[\gamma(1-q(x,\Delta))+x+(c-\lambda\mu)\Delta-(c-\lambda\mu)\int_0^\Delta q(x,t)\,dt\Big].
\end{aligned}
$$

The first- and second-order derivatives of $h(x;b_2)$ are

$$
\begin{aligned}
\frac{\partial h(x;b_2)}{\partial x} &= e^{-r\Delta}\Big[-\gamma\frac{\partial q(x,\Delta)}{\partial x}+1-(c-\lambda\mu)\int_0^\Delta\frac{\partial q(x,\Delta)}{\partial x}\,dt\Big],\\
\frac{\partial^2 h(x;b_2)}{\partial x^2} &= e^{-r\Delta}\Big[-\gamma\frac{\partial^2 q(x,\Delta)}{\partial x^2}-(c-\lambda\mu)\int_0^\Delta\frac{\partial^2 q(x,\Delta)}{\partial x^2}\,dt\Big].
\end{aligned}
$$

Since $q(x,t)=\mathrm{P}[\tau_0\le t\mid X(0)=x]$ satisfies the Kolmogorov backward equation and its derivatives satisfy

$$
\frac{\partial q(x,t)}{\partial x}<0,\quad \frac{\partial^2 q(x,t)}{\partial x^2}>0,\quad \frac{\partial q(x,t)}{\partial t}>0,
$$

we obtain

$$
\frac{\partial h(x;b_2)}{\partial x}>0 \quad\text{and}\quad \frac{\partial^2 h(x;b_2)}{\partial x^2}<0
$$

when $\gamma>0$. □

Lemma A.2 shows the concavity of the value function in the capital injection region, which means that the new capital issues can be optimized when $\gamma>0$. Now we consider a positive barrier of the continuation region. Define the function $\tilde f(x,b)$ as

$$
\tilde f(x,b)=a_1e^{-d_+(b-x)}+a_2e^{-d_-(b-x)}, \tag{A.5}
$$

where $d_\pm$ are defined as in (3.2). Consider a positive barrier b_0 which satisfies $\tilde f(0,b_0)=0$. Then we deduce that

$$
b_0=\frac{2}{d_+-d_-}\ln\Big(-\frac{d_-}{d_+}\Big). \tag{A.6}
$$

LEMMA A.3. *Let b_2 and b_0 be the barriers defined above. Then*

$$
\max(b_2,b_0)<(c-\lambda\mu)/r.
$$

PROOF. Differentiating (A.5) with respect to b, we obtain

$$
\tilde f_b(x,b)=-f_x(x;b)<-1 \quad \text{for } x\in[0,b_2).
$$

Moreover,

$$
\tilde f(0,b_0)=0 \quad\text{and}\quad \tilde f(0,0)=f(b_2;b_2)=\frac{c-\lambda\mu}{r}.
$$

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Since the absolute value of the derivative of $\tilde{f}$ with respect to b is greater than 1, $b_0 < (c - \lambda\mu)/r$. On the other hand, $b_2 < (c - \lambda\mu)/r - K < (c - \lambda\mu)/r$ following from $\gamma > 0$. Thus, $\max(b_2, b_0) < (c - \lambda\mu)/r$. □

LEMMA A.4. *For all $b_2 \in (0, b_0]$, $h(x; b_2) < g(x; b_2) = (c - \lambda\mu)/r + x - b_2$.*

PROOF. Referring to (3.7), by using Lemmas A.1 and A.2,

$$\begin{aligned}
h(x; b_2) &= e^{-r\Delta}\mathrm{E}_x\Big[\Big(X_\Delta + \frac{c-\lambda\mu}{r} - b_2 - K\Big)I_{\{\tau_0>\Delta\}}\Big] \\
&= e^{-r\Delta}\mathrm{E}_x[\{X_\Delta\}I_{\{\tau_0>\Delta\}}] + \mathrm{E}_x\Big[\Big(\frac{c-\lambda\mu}{r} - b_2 - K\Big)I_{\{\tau_0>\Delta\}}\Big] \\
&= e^{-r\Delta}\Big[x + (c-\lambda\mu)\Delta \\
&\qquad - (c-\lambda\mu)\int_0^\Delta q(x,t)\,dt + \Big(\frac{c-\lambda\mu}{r} - b_2 - K\Big)(1 - q(x,\Delta))\Big] \\
&\le e^{-r\Delta}\Big[\frac{c-\lambda\mu}{r} + x + (c-\lambda\mu)\Delta - b_2 \\
&\qquad - (c-\lambda\mu)\int_0^\Delta q(x,t)\,dt - \Big(\frac{c-\lambda\mu}{r} - b_2\Big)q(x,\Delta)\Big] \\
&< e^{-r\Delta}\Big[\frac{c-\lambda\mu}{r} + x + (c-\lambda\mu)\Delta - b_2\Big] \\
&< \frac{c-\lambda\mu}{r} + x - b_2 \\
&= g(x; b_2).
\end{aligned}$$

Thus, the inequality is verified. □

Define a two-component function $\tilde{h}(x, b)$, where $(x, b) \in \mathbb{R}_+ \times \mathbb{R}_+$, as

$$\begin{aligned}
\tilde{h}(x, b) = e^{-r\Delta}\Big\{&\Big[x + (c-\lambda\mu)\Delta + \frac{c-\lambda\mu}{r} - b - K\Big]\Phi\Big(\frac{x + (c-\lambda\mu)\Delta}{\sigma\lambda^{\frac{1}{2}}\sqrt{\Delta}}\Big) \\
&+ \sigma\lambda^{1/2}\sqrt{\Delta}\varphi\Big(\frac{x + (c-\lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big) - \exp\Big(-\frac{2(c-\lambda\mu)x}{\lambda\sigma^2}\Big) \\
&\times \Big[\Big[-x + (c-\lambda\mu)\Delta + \frac{c-\lambda\mu}{r} - b - K\Big]\Phi\Big(\frac{-x + (c-\lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big) \\
&+ \sigma\lambda^{1/2}\sqrt{\Delta}\varphi\Big(\frac{-x + (c-\lambda\mu)\Delta}{\sigma\lambda^{1/2}\sqrt{\Delta}}\Big)\Big]\Big\},
\end{aligned} \tag{A.7}$$

where $\tilde{h}(x, b_2) = h(x; b_2)$.

LEMMA A.5. *If $\partial\tilde{h}(x, b_0)/\partial x|_{x=0} > \partial\tilde{f}(x, b_0)/\partial x|_{x=0}$ and $\gamma \ge 0$ then there exists a pair of solutions (b_1, b_2) to (3.10)–(3.11) satisfying $0 < b_1 < b_2 < b_0$ such that $\tilde{h}(x, b_2) \le \tilde{f}(x, b_2)$ for all $0 \le x \le b_2$.*

PROOF. *Step 1.* In view of (A.7), we have $\tilde{h}(0, b_0) = 0$. Also, $\tilde{f}(x, b_0) = 0$ from the definition. The condition $\partial\tilde{h}(x, b_0)/\partial x|_{x=0} > \partial\tilde{f}(x, b_0)/\partial x|_{x=0}$ implies that there exists some point $x_1 \in (0, b_0)$ such that $\tilde{h}(x_1, b_0) > \tilde{f}(x_1, b_0)$. On the other hand, from Lemma A.4, substituting $b_2 = b_0$ in $\tilde{h}(x, b_2)$, we have $\tilde{h}(b_0, b_0) < (c - \lambda\mu)/r = \tilde{f}(b_0, b_0)$. This shows that $\tilde{h}(x, b_0)$ must cross $\tilde{f}(x, b_0)$ from above at some point x_2 in the interval $(0, b_0)$.

Step 2. For all $b_2 \in (0, b_0)$, since $\tilde{f}_b(x, b) < -1$, we have $\tilde{h}(0, b_2) = 0 = \tilde{f}(0, b_0) < \tilde{f}(0, b_2)$. In addition, $\tilde{h}(b_2, b_2) < \tilde{f}(b_2, b_2) = (c - \lambda\mu)/r$ by Lemma A.4. From (A.5) and (A.7),

$$\lim_{b_2 \to 0} \tilde{h}(b_2, b_2) = 0 < \frac{c - \lambda\mu}{r} = \lim_{b_2 \to 0} \tilde{f}(0, b_2).$$

By Lemma A.2, $\gamma \geq 0$ implies that $h(x; b_2)$ is increasing and concave, and so is $\tilde{h}(x, b_2)$ with respect to x; $\tilde{f}(x, b_2)$ is also increasing in x. Thus, combining with the previous inequality, we can always find a positive b_2 in the interval $(0, b_0)$ such that $\tilde{h}(x, b_2) < \tilde{f}(x, b_2)$ for $0 \leq x \leq b_2$.

Step 3. In view of Steps 1–2, following from the continuity of $\tilde{h}(x, b)$ and $\tilde{f}(x, b)$ with respect to x and b, there exists b_2 in the interval $(0, b_0)$ such that $\tilde{h}(b_1, b_2) = \tilde{f}(b_1, b_2)$ for some $0 < b_1 < b_2$, where $\tilde{h}(x, b_2) \leq \tilde{f}(x, b_2)$ for all $0 \leq x \leq b_2$. For the (b_1, b_2) we have chosen, the continuous differentiability of $\tilde{h}(x, b_2)$ and $\tilde{f}(x, b_2)$ with respect to x yields $\partial\tilde{h}(x, b_2)/\partial x|_{x=b_1} = \partial\tilde{f}(x, b_2)/\partial x|_{x=b_1}$. The equality is established because the two continuously differentiable lines have the same derivative if they coincide but do not cross at some point. In view of the definition of the two functions $\tilde{f}(b_1, b_2)$ and $\tilde{h}(b_1, b_2)$, we find that (3.10)–(3.11) hold. □

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