SPECTRAL THEORY FOR A CLASS OF NON-NORMAL OPERATORS II

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1. Introduction. In a previous paper **(2)** we have developed a spectral theory and a unitary equivalence theory for a certain class of non-normal operators. We dealt primarily with operators which were called J_2 operators. At present we are interested in studying the uniformly closed rings generated by such operators.

The notation will be the same as in **(2)**.

2. Classification of J_2 operators. In view of the uniqueness of the projection valued measure on $Z \cup Q$ for a given operator B, operators may be classified by the nature of the collection of null sets in the measure.

(2.1) A pure J_2 operator is a J_2 operator whose associated projection valued measure satisfied $E(Z) = 0$. This is the same as saying that the normal kernel is 0.

(2.2) A boundedly pure J_2 operator is a pure J_2 operator which has the additional property that there exists an $\alpha > 0$ such that the measure on Q is concentrated on the set (λ, μ, a) where $a \geqslant \alpha$.

(2.3) A separated J_2 operator is a J_2 operator whose associated projection valued measure restricted to Q is concentrated on the set (λ, μ, a) where $a \geqslant \alpha$ for a given positive α. (Note that there is no restriction on the measure restricted to Z.)

Even though the concept of a separated operator might seem artificial, it will be seen that it comes up naturally in the study of uniformly closed rings. The separated case turns out to be simple whereas strange difficulties appear in the non-separated case. It will be convenient to annex the points $(\lambda, \mu, 0)$ with $\lambda \geqslant \mu$ to Q to obtain the space Q_∞ in dealing with non-separated operators.

3. The abstract spectrum. We are given that $B = \int \lambda \, dE(\lambda)$ where E is a projection valued measure on $Z \cup Q$. The abstract spectrum of A will consist of points in $Z \cup Q_\infty$.

(3.1) p is in the discrete spectrum $\leftrightarrow E(p) > 0$.

(3.2) p is in the spectrum $\leftrightarrow$ every open set containing p has positive measure. (Open sets in $Z \cup Q_\infty$ are defined in the natural manner, that is, unions of open sets in Z and open sets in Q_∞ regarded as a subspace of $Z \oplus Z \oplus Z$.)

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(3.3) p is in the continuous spectrum $\leftrightarrow p$ is in the spectrum but not in the discrete spectrum.

It is easily seen that for normal operators the definitions are equivalent to the usual definitions.

LEMMA 3.1.
(a) *B is a pure J_2 operator $\leftrightarrow$ spectrum of $B \subset Q_\infty$.*
(b) *B is a separated J_2 operator $\leftrightarrow$ spectrum of $B \subset Q \cup Z$.*
(c) *B is a boundedly pure J_2 operator $\leftrightarrow$ spectrum of $B \subset Q$.*

The proofs are trivial. Note that the measure is concentrated on a compact set (in fact the set $\{\lambda\} \cup \{\lambda, \mu, a\}$ where $|\lambda|, |\mu|, a \leqslant ||A||$).

LEMMA 3.2. *p is in the spectrum $\leftrightarrow$ every set of a complete base of neighbourhoods of p has positive measure.*

LEMMA 3.3. *The spectrum is compact.*

Proof. Clearly the spectrum is bounded. Suppose $p \notin K$ where K is the spectrum. Then there exists an open set U containing p of zero measure. By definition $U \cap K$ is empty. This shows that the spectrum is closed.

LEMMA 3.4. *The complement of the spectrum has measure zero.*

Proof. It suffices to remark that $Z \cup Q_\infty$ has a countable base for open sets.

LEMMA 3.5. *The spectrum K is the unique minimal closed set the complement of which has measure zero.*

Proof. Suppose C is a closed set with the above property. Then $K' \cup C'$ is an open set of measure zero. Hence $K' \cup C' \subset K'$. Thus $K \subset C$ and this proves the minimality of K.

COROLLARY 1. All non-empty open subsets of K in the relative topology induced by K have positive measure.

COROLLARY 2. If L is a null set in K, $\overline{K - L} = K$.

For completion we state two further lemmas the proofs of which are relatively simple.

LEMMA 3.6. *A compact set K is the spectrum of some operator if and only if $\overline{K \cap Q} = K \cap Q_\infty$.*

LEMMA 3.7. *λ_0 is in the ordinary spectrum of B if and only if the abstract spectrum of B contains either λ_0, a point of the form (λ_0, λ, a) or a point of the form (λ, λ_0, a).*

4. Uniformly closed rings. The spectrum will play an important part in the study of the uniformly closed ring $C(A, A^*)$ generated by A and A^*. It will be convenient to break up the study of these rings into cases.

Case 1. *A is boundedly pure.* It can be seen from **(2)** that the weakly closed ring $R(A, A^*)$ generated by A and A^* corresponds to the set of all L_∞ second order matrix functions on Q. By Lemma 3.4 the spectrum of A may be used as the domain space. The task ahead of us is to see what happens to $C(A, A^*)$ in this correspondence.

Suppose $B \in C(A, A^*)$. Then B can be uniformly approximated by polynomials in A and A^*. Let B_n be a sequence of polynomials approaching B uniformly. Then $B_n(t)$ approaches $B(t)$ uniformly except on a null set, hence by Corollary 2 to Lemma 3.5, $B_n(t)$ approaches $B(t)$ uniformly on a dense subset of K. Since $B_n(t)$ is continuous for all n, $B(t)$ may be redefined on the null set if necessary so as to become a continuous function on K. The new $B(t)$ still corresponds to B.

Strictly speaking B corresponds to an equivalence class of matrix functions on K. However, by Corollary 1 to Lemma 3.5, there is at most one continuous function in any equivalence class. Thus we have a well-defined mapping from operators in $C(A,A^*)$ into the set of continuous second order matrix functions on K. It is easily verified that this mapping is norm-preserving and an algebraic isomorphism into. Note that some caution is required in the proof since the map on $C(A, A^*)$ is not just the restriction of the map on $R(A, A^*)$. For example, to verify that

$$||B(t)||_\infty = \max ||B(t)||,$$

the continuity of $B(t)$ must be used as well as the fact that on K, open sets have positive measure. Since K is compact it follows from **(3)** that the mapping is onto. Thus we have proved

THEOREM 1. *The uniformly closed * ring generated by a boundedly pure J_2 operator A is algebraically isomorphic and isometric to the algebra of all continuous second order matrix valued functions on the spectrum of A.*

By * ring is meant a ring which is closed under the adjoint operation.

This technique gives an alternative way of proving the corresponding well-known theorem for normal operators.

Case 2. *A is separated.* It is easy to generalize Theorem 1 to this case. K is the union of a compact set in Q with a compact set in Z. Every $B \in C(A, A^*)$ can be decomposed in such a way that $B(t)$ is a continuous second order matrix valued function on t restricted to Q, and a continuous complex valued function on t restricted to Z. Again using **(3)** we obtain

THEOREM 2. *The uniformly closed * ring generated by a separated operator A is algebraically isomorphic and isometric to the algebra of all functions on the spectrum of A which are continuous second order matrix valued on $K \cap Q$ and continuous complex valued on $K \cap Z$.*

Case 3. *A is not separated.* This case is more difficult than the separated case because not all continuous functions on the spectrum are obtained. For

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example, if λ, μ, and $(\lambda, \mu, 0)$ are in the spectrum, and if f is a uniform limit of polynomials, then

$$f(\lambda, \mu, 0) = \begin{pmatrix} f(\lambda) & 0 \\ 0 & f(\mu) \end{pmatrix}.$$

Thus the continuous functions that correspond to operators in $C(A, A^*)$ satisfy certain conditions.

At any rate, the previous technique remains valid up to the use of **(3)**. Thus we have a one-one norm-preserving mapping of $C(A, A^*)$ into the set of all continuous functions on the spectrum. (It is understood that the values are matrices of order two on Q_∞ and complex numbers on Z.) It is clear that all functions g in the image have the following property when restricted to $(Q_\infty - Q) \cup Z$: There exists a continuous function h on the complex numbers z such that $g(\lambda) = h(\lambda)$ for all λ in the spectrum and

$$g(\lambda, \mu, 0) = \begin{pmatrix} h(\lambda) & 0 \\ 0 & h(\mu) \end{pmatrix}$$

for all $(\lambda, \mu, 0)$ in the spectrum. It will be shown that this condition is also sufficient for a continuous function to be in the image.

Let $g(t)$ be any continuous function satisfying the above condition. Choose a polynomial function $f(t)$ which satisfies $|g(t) - f(t)| < \frac{1}{8}\epsilon$ for all t which either are in the part of the spectrum which is in z or have the property that $(\lambda, t, 0)$ or $(t, \lambda, 0)$ is in the spectrum for some λ. It follows immediately that

$$||g(t) - f(t)|| < \tfrac{1}{4}\epsilon,\ t \in (Q_\infty - Q) \cup Z.$$

By uniform continuity we can even guarantee that $||g(t) - f(t)|| < \frac{1}{2}\epsilon$ for all t in the spectrum except possibly those of the form (λ, μ, a) for $a \geqslant \alpha$ for some positive α. Now consider the set of all (λ, μ, a) in the spectrum which satisfy $a \geqslant \frac{1}{2}\alpha$. There exists a polynomial $h(t)$ which satisfied $||g(t) - h(t)|| < \frac{1}{2}\epsilon$ for all such a. By the same technique as in **(3)** it may be shown that there exists a positive definite self-adjoint matrix valued function $e(t)$ satisfying $||e(t)|| \leqslant 1$ for all t which is a uniform limit of a sequence of polynomial functions, and satisfying $e(t) = 1$ for all (λ, μ, a) such that $a \geqslant \alpha$ and $e(t) = 0$ for all $t \in Z$ and all (λ, μ, a) such that $a \leqslant \frac{1}{2}\alpha$. Note that even though separation of points fails to hold in the spectrum, the proof goes through since separation fails only among points in $Q_\infty - Q$ and Z. Consider $eh + (1 - e)f$. It is easily verified that

$$||g - [eh + (1 - e)f]|| < \epsilon$$

for all t. This completes the proof.

THEOREM 3. *The uniformly closed ring generated by a J_2 operator is algebraically isomorphic and isometric to the algebra of all functions g on the spectrum K of A which are continuous complex valued on $K \cap Z$ and continuous second order matrix valued on $K \cap Q_\infty$, and which have the additional property that there*

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exists a continuous complex valued function f such that $g(t) = f(t)$ for all t in $K \cap Z$ and

$$g(\lambda, \mu, 0) = \begin{pmatrix} f(\lambda) & 0 \\ 0 & f(\mu) \end{pmatrix}$$

or all $(\lambda, \mu, 0)$ in $K \cap Q_\infty$.

This theorem which generalizes the previous theorems illustrates the fundamental distinction between separated and non-separated operators.

We conclude this section with certain remarks concerning the equality of $C(A, A^*)$ and $R(A, A^*)$. Further details are found in **(1)**. The equality $R(A, A^*) = C(A, A^*)$ is equivalent to the statement: Every open set in spectrum A differs from a closed open set by a null set if A is separated. Otherwise, $R(A, A^*)$ is never equal to $C(A, A^*)$.

5. A counter-example. This section is independent of the rest of the paper.

It is known that all operators A satisfying $A^2 = 0$ are J_2 operators. (A proof may be found in **(1)**.) However, the corresponding statement for operators A satisfying $A^n = 0$ is not valid. In fact, we give an example of an operator A satisfying $A^3 = 0$ on a space of $\aleph_0$ dimensions which is irreducible.

Let $a_1, b_1, b_2 \ldots b_n \ldots c_1, c_2 \ldots c_n \ldots$ be a basis of the Hilbert space H. Let

$$Aa_1 = b_1, \; Ab_1 = c_1, \; Ab_{i+1} = c_i + c_{i+1} \qquad i \geqslant 1, \; Ac_i = 0.$$

Clearly A is bounded and $A^3 = 0$. Suppose $H = E \oplus F$ where E and F reduce A. We show that either E or F must be H.

LEMMA 5.1. *If $\lambda_1 a_1 + \sum \mu_i b_i + \sum \nu_J c_J \in E$ and $\lambda'_1 a_1 + \sum \mu'_i b_i + \sum \nu'_J c_J \in F$ then either λ_1 or $\lambda'_1 = 0$.*

Proof. Let A^2 operate on each. Then $\lambda_1 c_1 \in E$ and $\lambda'_1 c_1 \in F$. Hence λ_1 or $\lambda'_1 = 0$, otherwise E and F would not be orthogonal.

It follows that E or F is orthogonal to a_1. Thus $a_1 \in E$ or F. Suppose without loss of generality that $a_1 \in E$. Then $b_1 = Aa_1 \in E$ and $c_1 = Ab_1 \in E$.

LEMMA 5.2. *$b_2 \in E$.*

Proof. Suppose

$$\lambda_2 b_2 + \sum_{i>2} \lambda_i b_i + \sum \mu_i c_i \in F.$$

Then

$$A\left[\lambda_2 b_2 + \sum_{i>2} \lambda_i b_i + \sum \mu_i c_i\right] \in F.$$

In the latter expression the coefficient of c_1 is λ_2. Since $c_1 \in E$, $\lambda_2 = 0$. Since this is true for all points in F, b_2 is orthogonal to F and is hence in E.

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Now $c_2 = Ab_2 - c_1 \in E$. By induction all the b's and c's are in E thus showing that $E = H$.

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