AN ESTIMATE ON THE EIGENVALUES IN BIFURCATION FOR GRADIENT MAPPINGS

by RAFFAELE CHIAPPINELLI

(Received 21 December, 1995)

1. Introduction. Let H be a real Hilbert space and let $A : H \to H$ be a nonlinear operator such that $A(0) = 0$. We consider the eigenvalue problem

$$A(x) = \lambda x \qquad (\lambda \in \mathbb{R}, x \neq 0). \tag{1.1}$$

Recall that $\lambda_0 \in \mathbb{R}$ is said to be a *bifurcation point* for (1.1) if every neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times H$ contains solutions of (1.1).

In the variational case—namely, when A is the gradient of a functional ϕ on H—a classical theorem of Krasnosel'skii guarantees, under suitable assumptions on A and ϕ, the existence of *infinitely many* bifurcation points for (1). To that purpose, let us recall that a functional $\phi : H \to \mathbb{R}$ is said to be *weakly continuous* if it maps weakly convergent sequences of H into convergent sequences; while an operator $A : H \to H$ is said to be *completely continuous* if it is continuous and compact (i.e., it maps bounded sets into sets having compact closure). Krasnosels'kii result ([**6**], Theorem 6.2.2) can then be stated as follows.

THEOREM 0. *Assume that $\phi : H \to \mathbb{R}$ is weakly continuous and uniformly differentiable in a neighbourhood of* 0, *and assume that $A = \operatorname{grad} \phi : H \to H$ is completely continuous. Then, if A is differentiable at* 0, *every eigenvalue $\lambda_0 \neq 0$ of the derivative $A'(0)$ is a bifurcation point for* (1).

More precisely, for any sufficiently small $r > 0$ there exists $\lambda_r \in \mathbb{R}$, $x_r \in H$ with $\|x_r\| = r$ such that $A(x_r) = \lambda_r x_r$, and furthermore

$$\lambda_r \to \lambda_0 \qquad (r \to 0). \tag{1.2}$$

Recall that under the above assumptions $A'(0)$ is a (linear) compact self-adjoint operator, so that "generically" it has infinitely many nonzero eigenvalues. We refer to [**6**] for the concept of uniform differentiability, which is however easy to imagine.

Though the proof of the above Theorem is far from immediate, its basic lines run as follows. Finding an $x \in H$ with $\|x\| = r > 0$ which solves (1.1), i.e.

$$\operatorname{grad} \phi(x) = \lambda x \tag{1.3}$$

for some $\lambda \in \mathbb{R}$, is equivalent to finding a constrained critical point of ϕ on the sphere $S_r = \{x \in H : \|x\| = r\}$. These critical points are obtained in [**6**] by a "minimax" process—over a certain class of compact subsets of S_r—that we shall shortly recall later. Formula (1.2) for the eigenvalues is then a consequence of the differentiability of A in 0. Our point in this note is that, if the differentiability assumption is strengthened, then more precise information on the behaviour of λ_r as $r \to 0$ can be gained. To be clearer, consider Taylor's formula for A at 0:

$$A(x) = Tx + R(x), \tag{1.4}$$

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press

where $T = A'(0)$ and the remainder term $R(x)$ is $o(\|x\|)$ as $\|x\| \to 0$. Assuming $R(x) = O(\|x\|^p)$ as $\|x\| \to 0$ with $p > 1$, we prove that rather than (1.2) one gets

$$\lambda_r = \lambda_0 + O(r^{p-1}) \qquad (r \to 0).$$

We also show that such condition appears naturally in the applications of bifurcation theory to nonlinear elliptic eigenvalue problems like

$$\begin{cases} -\Delta u = \mu(u + f(x,u)), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and $f = f(x,s)$ is a function on $\Omega \times \mathbb{R}$ satisfying the usual growth assumptions in s: see e.g. [**4**], [**8**].

Remark. In the last decades, Theorem 0 has been established in greater generality. Essentially, it is sufficient to consider any isolated eigenvalue of finite multiplicity of the derivative $A'(0)$ without requiring any compactness: see e.g. [**10**]. We believe that our sharpening of the conclusion also holds in this generality.

2. Asymptotic behaviour of the eigenvalues. In order to prove our result, we need recall the main points of the method followed in [**6**]. Let therefore (λ_i) be the nonzero eigenvalues of the compact self-adjoint operator $T = A'(0)$, repeated according to multiplicities, and let (e_i) be the corresponding orthonormal eigenvectors; we then have

$$(Tx, x) = \sum_{i=1}^{\infty} \lambda_i |(x, e_i)|^2 \qquad \forall x \in H. \tag{2.1}$$

Let λ_0 be one of the eigenvalues λ_i; suppose, to fix the ideas, $\lambda_0 > 0$. Denote by H_0 the eigenspace associated with λ_0 and by H_1 the sum of the eigenspaces associated to the eigenvalues $\lambda_i \geq \lambda_0$, i.e.

$$H_1 = \text{span}\{e_i : Te_i = \lambda_i e_i, \lambda_i \geq \lambda_0\}.$$

We thus have $(Tx, x) = \lambda_0 \|x\|^2$ if $x \in H_0$ and $(Tx, x) \geq \lambda_0 \|x\|^2$ if $x \in H_1$. Next set $R = \{x \in H : P_1 x \neq 0\}$, with P_1 the orthogonal projection on H onto H_1, and

$$M_r = \{F \subset S_r \cap R\,;\, F \text{ compact, noncontractible in } R\}. \tag{2.2}$$

(Recall that a subset F of a topological space R is said to be *contractible* (in R) to the point $x_0 \in R$ if there exists a continuous map $U : [0,1] \times F \to R$ such that $U(0,x) = x$ and $U(1,x) = x_0$ for any $x \in F$). We finally set

$$c_r = \sup_{F \in M_r} \min_{x \in F} \phi(x) \tag{2.3}$$

and note this is a good definition since M_r is nonempty, as we shall see later. Now, Krasnosels'skii proves that, for $r > 0$ less than some r_0, c_r is a critical value of ϕ on S_r: namely, there exist $x_r \in S_r$, $\lambda_r \in \mathbb{R}$ so that

$$\phi(x_r) = c_r \quad \text{and} \quad A(x_r) = \lambda_r x_r. \tag{2.4}$$

Moreover, the condition $R(x) = A(x) - Tx = o(\|x\|)$—expressing the differentiability of A at 0—is used to show that $\lambda_r \to \lambda_0$, which proves the bifurcation from λ_0. We have the following improvement to this result:

Theorem 1. *Under the same assumptions as in Theorem* 0, *suppose further that the*

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press

remainder term R satisfies $R(x)=O(\|x\|^p)$ for $x\to 0$ with $p>1$. Then the eigenvalues λ_r satisfy

$$\lambda_r=\lambda_0+O(r^{p-1}) \qquad (r\to 0). \tag{2.5}$$

Proof. Let $k>0$, $r_0>0$, $p>1$ be such that

$$\|R(x)\|\le k\,\|x\|^p \qquad (\|x\|<r_0). \tag{2.6}$$

Recall (e.g. [2], [9]) that $\phi(x)=\int_0^1(A(tx),x)\,dt$ and so

$$\phi(x)=\tfrac{1}{2}(Tx,x)+\int_0^1(R(tx),x)\,dt.$$

From (2.6) it then follows that

$$|\phi(x)-\tfrac{1}{2}(Tx,x)|\le\|x\|\int_0^1\|R(tx)\|\,dt\le\frac{k}{p+1}\|x\|^{p+1} \qquad (\|x\|<r_0).$$

Since $p>1$, one thus obtains

$$\tfrac{1}{2}(Tx,x)-\frac{k}{2}r^{p+1}\le\phi(x)\le\tfrac{1}{2}(Tx,x)+\frac{k}{2}r^{p+1} \tag{2.7}$$

for any $x:\|x\|\le r<r_0$. We can now estimate c_r and prove that

$$|c_r-\tfrac{1}{2}\lambda_0r^2|\le\frac{k}{2}r^{p+1} \qquad (r<r_0). \tag{2.8}$$

To this end, we need two basic properties of the family M_r [6]:

(i) $S_r^1\in M_r$, where $S_r^1=S_r\cap H_1$,

(ii) for any $F\in M_r$, there exists $z\in F$ such that $P_1z\in H_0$.

From (i) it follows that

$$c_r=\sup_{F\in M_r}\min_{x\in F}\phi(x)\ge\min_{x\in S_r^1}\phi(x).$$

On the other hand if $x\in S_r^1$, $r<r_0$, (2.7) yields (on recalling that $(Tx,x)\ge\lambda_0\|x\|^2$ on H_1)

$$\phi(x)\ge\tfrac{1}{2}\lambda_0r^2-\frac{k}{2}r^{p+1},$$

which gives one half of the estimate (8). To prove that

$$c_r\le\tfrac{1}{2}\lambda_0r^2+\frac{k}{2}r^{p+1}$$

we first let H_2 denote the orthogonal subspace to H_1; thus, writing $x=x_1+x_2$ with $x_1=P_1x\in H_1$, $x_2\in H_2$, we have (by virtue of (2.1)) $(Tx_2,x_2)\le\lambda_0\|x_2\|^2$.

Let now $F\in M_r$; by (ii), there exists $z\in F$ such that $z_1=P_1z\in H_0$ and so

$$(Tz,z)=(Tz_1,z_1)+(Tz_2,z_2)\le\lambda_0(\|z_1\|^2+\|z_2\|^2)=\lambda_0\|z\|^2.$$

On using this inequality in (2.7), we obtain

$$\min_{x\in F}\phi(x)\le\gamma(z)\le\tfrac{1}{2}\lambda_0r^2+\frac{k}{2}r^{p+1}$$

and since this is true for any $F\in M_r$, the result follows.

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press

We now go on to estimate λ_r, the eigenvalue corresponding to $x_r \in S_r$ with $\phi(x_r) = c_r$. One has $\lambda_r r^2 = (A(x_r), x_r)$ and so

$$\lambda_r - \lambda_0 = \frac{1}{r^2}[(A(x_r), x_r) - \lambda_0(x_r, x_r)]$$
$$= \frac{1}{r^2}(A(x_r) - Tx_r + Tx_r - \lambda_0 x_r, x_r)$$

whence, by virtue of (2.6),

$$|\lambda_r - \lambda_0| \le \frac{1}{r^2}\|A(x_r) - Tx_r\|\,\|x_r\| + \frac{2}{r^2}\left|\frac{(Tx_r, x_r)}{2} - \lambda_0\frac{r^2}{2}\right|$$
$$\le \frac{1}{r^2}kr^{p+1} + \frac{2}{r^2}\left|\frac{(Tx_r, x_r)}{2} - \phi(x_r) + c_r - \lambda_0\frac{r^2}{2}\right|$$

Finally, using (2.7) and the estimate (2.8) for c_r we have

$$|\lambda_r - \lambda_0| \le kr^{p-1} + 2kr^{p-1} = 3kr^{p-1} \qquad (r < r_0)$$

which concludes the proof of Theorem 1. □

3. An application. Let Ω be an open bounded subset of $\mathbb{R}^n$. Consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \mu(u + f(x, u)), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{3.1}$$

where $f = f(x, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function (i.e. continuous in t for a.a. $x \in \Omega$ and measurable in x for any $t \in \mathbb{R}$) satisfying the growth condition

$$|f(x, t)| \le a\,|t|^p \qquad (\text{a.a. } x \in \Omega, t \in \mathbb{R}) \tag{3.2}$$

for some $a \ge 0$ and some $p : 1 < p < \dfrac{n+2}{n-2}$ if $n > 2$, $1 < p < \infty$ if $n \le 2$.

We consider the real Sobolev space $H = W_0^{1,2}(\Omega)$, which is a Hilbert space under the scalar product $(u, v) = \int_\Omega \nabla u \,.\, \nabla v$, and recall that a weak solution of (P) is a function $u \in H$ such that

$$\int_\Omega \nabla u \,.\, \nabla v = \mu\left(\int_\Omega uv + \int_\Omega f(x, u)v\right) \qquad \forall v \in H. \tag{3.3}$$

If $u \ne 0$ we say that u is an *eigenfunction* of (3.1) associated to the eigenvalue μ; note that by (3.2) $f(x, 0) = 0$ for a.a. $x \in \Omega$, so that $u = 0$ is a solution for any $\mu \in \mathbb{R}$.

Let $0 < \mu_1 < \mu_2 \le \ldots$ denote the eigenvalues of the linear problem

$$\begin{cases} -\Delta u = \mu u, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}. \tag{3.4}$$

THEOREM 2. *Let μ_0 be an eigenvalue of* (3.4). *For any sufficiently small $r > 0$, there exists an eigenvalue μ_r of* (3.1) *with corresponding eigenfunction $u_r \in H$ such that $\|u\| = r$. We have $\mu_r \to \mu_0$ as $r \to 0$ (i.e., μ_0 is a bifurcation point for* (3.1)), *and more precisely*

$$\mu_r = \mu_0 + O(r^{p-1}) \quad \textit{as} \quad r \to 0. \tag{3.5}$$

Proof. We shall limit ourselves to sketch the relevant points; for a detailed discussion of semilinear elliptic problems like (3.1), standard references are e.g. **[4]**, **[8]**.

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press

Define operators T, R in H by the rules

$$(Tu, v) = \int_\Omega uv, \qquad (R(u), v) = \int_\Omega f(x, u)v$$

for $u, v \in H$; then (3.3) becomes

$$(u, v) = \mu[(Tu, v) + (R(u), v)] \qquad \forall v \in H$$

so that u is a weak solution of (3.1) if and only if

$$A(u) := Tu + R(u) = \lambda u$$

where $\lambda = \mu^{-1}$ ($\mu \neq 0$). A is the gradient of the functional

$$\phi(u) = \frac{1}{2}\int_\Omega u^2 + \int_\Omega F(x, u)$$

where $F(x, t) = \int_0^t f(x, s)\, ds$. We now check that

$$\|R(u)\| \leq c\, \|u\|^p \tag{3.6}$$

for some $c > 0$ and all $u \in H$ ($\|\,.\,\|$ is the norm in H). Indeed, using (3.2) and Holder's inequality with exponents $p + 1$, $(p + 1)/p$ one has

$$\left|\int_\Omega f(x, u)v\right| \leq a \int_\Omega |u|^p\, |v| \leq a\left(\int_\Omega |u|^{p+1}\right)^{p/(p+1)}\left(\int_\Omega |v|^{p+1}\right)^{1/(p+1)}$$
$$= a\, \|u\|^p_{L^{p+1}(\Omega)}\, \|v\|_{L^{p+1}(\Omega)} \leq c\, \|u\|^p\, \|v\|$$

where the last inequality is a consequence of the Sobolev embedding $H \subset L^{p+1}(\Omega)$; (3.6) now follows readily from the definition of R. The compactness of this embedding—together with well-known properties of the Nemitskii operator in L^p spaces ([**4**], [**8**])—also shows that R is not only completely continuous but in fact *strongly continuous*, namely it maps weakly convergent sequences into strongly convergent ones. This in turn implies that ϕ is weakly continuous and uniformly differentiable on each bounded subset of H (see e.g. [**1**], Lemma 3.2 and [**9**], Theorem 4.2).

We thus see that all conditions of Theorem 1 are satisfied and—since the eigenvalues of T are clearly the *reciprocals* of the eigenvalues μ_n of (3.4)—we deduce that if μ_0 is one such eigenvalue, then for small $r > 0$ there exist a solution pair $(\lambda_r, u_r) \in \mathbb{R} \times H$ of (3.1) with $\|u_r\| = r$ while $\lambda_r = \mu_0^{-1} + O(r^{p-1})$ as $r \to 0$. This yields immediately the corresponding formula (3.5) for $\mu_r = \lambda_r^{-1}$.

Remark. Recall (see e.g. [**7**], Remark 3.3) that the Nemitskii operator induced by a Caratheodory function maps $L^{p+1}(\Omega)$ into $L^q(\Omega)$ ($q = (p + 1)/p$) if and *only if* there exist $a > 0$ and $b \in L^q(\Omega)$ such that

$$|f(x, t)| \leq a\, |t|^p + b(x) \qquad (\text{a.a. } x \in \Omega, t \in \mathbb{R}). \tag{3.7}$$

In the applications to eigenvalue problems, one takes $b = 0$ in (3.6), so that $f(x, 0) = 0$ for a.a. $x \in \Omega$. In this sense we can say that (3.2) is the natural condition to consider in the present context.

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press

REFERENCES

1. H. Amann, Liusternik-Schnirelmann theory and nonlinear eigenvalue problems, *Math. Ann.* **199** (1972), 55–72.

2. M. S. Berger, *Nonlinearity and functional analysis* (Academic Press, New York, 1977).

3. H. Brezis, *Analyse fonctionnelle, theorie et applications* (Masson, Paris, 1983).

4. D. G. De Figueiredo, *Lectures on the Ekeland variational principle with applications and detours* (Tata Inst. of Fundamental Research, Bombay, 1989).

5. J. Dieudonne, *Foundations of modern analysis* (Academic Press, New York, 1969).

6. M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations* (Pergamon Press, New York, 1964).

7. G. Prodi and A. Ambrosetti, *Analisi non lineare* (Scuola Normale Superiore, Pisa, 1973).

8. P. H. Rabinowitz, Variational methods for nonlinear eigenvalue problems, *Eigenvalues of nonlinear problems* pp. 141–195. (Cremonese, Rome, 1974).

9. M. M. Vainberg, *Variational methods for the study of nonlinear operators* (Holden-Day, San Francisco, 1964).

10. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems* (Springer, Berlin, 1989).

DIPARTIMENTO DI MATEMATICA
UNIVERSITÁ DI SIENA
53100 SIENA (ITALY)
E-MAIL: chiappinelli@unisi.it

https://doi.org/10.1017/S0017089500032080 Published online by Cambridge University Press