





RESEARCH ARTICLE

Global solutions for 1D cubic defocusing dispersive equations: Part I

Mihaela Ifrim ¹ and Daniel Tataru ²

¹Department of Mathematics, University of Wisconsin - Madison, Madison, WI 53706, USA; E-mail: ifrim@wisc.edu.

²Department of Mathematics, University of California - Berkeley, Berkeley, CA, 94720, USA; E-mail: tataru@math.berkeley.edu.

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Abstract

This article is devoted to a general class of one-dimensional NLS problems with a cubic nonlinearity. The question of obtaining scattering, global in time solutions for such problems has attracted a lot of attention in recent years, and many global well-posedness results have been proved for a number of models under the assumption that the initial data are both *small* and *localized*. However, except for the completely integrable case, no such results have been known for small but not necessarily localized initial data.

In this article, we introduce a new, nonperturbative method to prove global well-posedness and scattering for L^2 initial data which are *small* and *nonlocalized*. Our main structural assumption is that our nonlinearity is *defocusing*. However, we do not assume that our problem has any exact conservation laws. Our method is based on a robust reinterpretation of the idea of Interaction Morawetz estimates, developed almost 20 years ago by the I-team.

In terms of scattering, we prove that our global solutions satisfy both global L^6 Strichartz estimates and bilinear L^2 bounds. This is a Galilean invariant result, which is new even for the classical defocusing cubic NLS.¹ There, by scaling, our result also admits a large data counterpart.

Contents

1	Introduction	1
2	Notations and preliminaries	5
3	Local well-posedness	9
4	Energy estimates and conservation laws	13
5	Interaction Morawetz identities	21
6	Frequency envelopes and the bootstrap argument	26
7	The frequency envelope bounds	28
8	Global bilinear and Strichartz estimates	42

1. Introduction

The question of obtaining scattering, global in time solutions for one-dimensional dispersive flows with quadratic/cubic nonlinearities has attracted a lot of attention in recent years, and many global well-posedness results have been proved for a number of models under the assumption that the initial data is both *small* and *localized*; without being exhaustive, see, for instance, [12, 13, 21, 18, 14]. The

¹There the global well-posedness was of course known, but not the Strichartz and bilinear L^2 bounds.

nonlinearities in these models are primarily cubic, though the analysis has also been extended via normal form methods to problems which also have nonresonant quadratic interactions; several such examples are [1, 15, 9, 16, 20]; see also further references therein.

In this article, we consider instead the much more difficult case where the initial data is just *small* but without any localization assumption. Here, it is natural to restrict the analysis to defocusing problems, as focusing one-dimensional cubic nonlinear Schrödinger (NLS) type problems typically admit small solitons, and thus, generically, the solutions do not scatter at infinity. Then one may formulate the following broad conjecture:

Conjecture. *One-dimensional dispersive flows with cubic defocusing nonlinearities, and small initial data have global in time, scattering solutions.*

The goal of this article is to prove the first global in time well-posedness result of this type. As part of our results, we also prove that our global solutions are scattering at infinity in a very precise, quantitative way, in the sense that they satisfy both L^6 Strichartz estimates and bilinear L^2 bounds. This is despite the fact that the nonlinearity is nonperturbative on large time scales.

1.1. Cubic NLS problems in one space dimension

One of the fundamental one-dimensional dispersive flows in one space dimension is the cubic NLS flow,

$$iu_t + u_{xx} = \pm u|u|^2, \quad u(0) = u_0. \quad (1.1)$$

Depending on the choice of signs, this comes in a defocusing (+) and a focusing (-) flavor. Both of these equations are important, not only by themselves but also as model problems for more complex one-dimensional dispersive flows, both semilinear and quasilinear.

The above cubic NLS flow is globally well-posed in L^2 both in the focusing and in the defocusing case, though the global behavior differs in the two cases.

Precisely, the focusing problem admits small solitons, so the solutions cannot in general scatter at infinity. If in addition the initial data is localized, then one expects the solution to resolve into a superposition of (finitely many) solitons and a dispersive part; this is called *the soliton resolution conjecture* and is known to hold in a restrictive setting, via the method of inverse scattering; see, for example, [2].

In the defocusing case, the inverse scattering approach also allows one to treat the case of localized data and show that global solutions scatter at infinity; see, for instance, [8]. This can also be proved in a more robust way, without using inverse scattering, under the assumption that the initial data is small and localized; see [14] and references therein. Much less is known in terms of scattering for nonlocalized L^2 data. However, if more regularity is assumed for the data, then we have the following estimate due to Planchon–Vega [22]; see also the work of Colliander–Grillakis–Tzirakis [3]:

$$\|u\|_{L^6}^6 + \|\partial_x |u|^2\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^3 \|u_0\|_{H^1}. \quad (1.2)$$

This allows one to estimate the L^6 Strichartz norm of the solution, that is, to prove some type of scattering or dispersive decay.

Because of the above considerations, our interest in this paper is in defocusing cubic problems. Precisely, we will consider a cubic nonlinear Schrödinger equation (NLS) type model in one space dimension

$$iu_t + u_{xx} = C(u, \bar{u}, u), \quad u(0) = u_0, \quad (1.3)$$

where u is a complex valued function, $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$. Here C is a trilinear translation invariant form, whose symbol $c(\xi_1, \xi_2, \xi_3)$ can always be assumed to be symmetric in ξ_1, ξ_3 ; see Section 2.3 for an expanded discussion of multilinear forms. The arguments u, \bar{u} and u of C are chosen so that our equation

(1.3) has the phase rotation symmetry, $u \rightarrow ue^{i\theta}$, as it is the case in most examples of interest. The symbol $c(\xi_1, \xi_2, \xi_3)$ will be required to satisfy a minimal set of assumptions:

(H1) Bounded and regular:

$$|\partial_\xi^\alpha c(\xi_1, \xi_2, \xi_3)| \leq c_\alpha, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{R}, \text{ for every multi-index } \alpha. \tag{1.4}$$

(H2) Conservative:

$$\Im c(\xi, \xi, \eta) = 0, \quad \xi, \eta \in \mathbb{R}, \text{ where } \Im z = \text{imaginary part of } z \in \mathbb{C}. \tag{1.5}$$

(H3) Defocusing:

$$c(\xi, \xi, \xi) \geq c > 0, \quad \xi \in \mathbb{R} \text{ and } c \in \mathbb{R}^+. \tag{1.6}$$

In selecting these assumptions, we have tried to strike a balance between the generality of the result on one hand and a streamlined exposition on the other hand.

The simplest example of such a trilinear form C is of course $C = 1$, which corresponds to the classical one-dimensional cubic NLS problem. But this problem is of course completely integrable and thus has infinitely many conservation laws. In particular, global well-posedness is straightforward, though our L^6 Strichartz and bilinear L^2 bounds are new even for this problem in the L^2 data setting. By contrast, the assumptions we impose on our model do not guarantee any exact conservation law at the L^2 level or at any other regularity level.

At the other end, both our use of the linear Schrödinger operator and the boundedness condition (H1) are nonoptimal, and we hope to relax both of these restrictions in subsequent work. However, using these restrictions brings the major expository advantage that our model has a Galilean invariance, in the sense that a Galilean transformation yields a problem that is in the same class, even though it is not exactly the same. This allows us to provide cleaner, shorter proofs for our results and to keep the focus on the main ideas.

1.2. The main result

Our main result asserts that global well-posedness holds for our problem for small L^2 data. In addition, our solutions not only satisfy uniform L^2 but also global space-time L^6 estimates, as well as bilinear L^2 bounds, as follows:

Theorem 1. *Under the above assumptions (H1), (H2) and (H3) on the symbol of the cubic form C , small initial data*

$$\|u_0\|_{L^2} \leq \epsilon \ll 1,$$

yields a unique global solution u for equation (1.3), which satisfies the following bounds:

(i) *Uniform L^2 bound:*

$$\|u\|_{L_t^\infty L_x^2} \lesssim \epsilon. \tag{1.7}$$

(ii) *Strichartz bound:*

$$\|u\|_{L_{t,x}^6} \lesssim \epsilon^{\frac{2}{3}}. \tag{1.8}$$

(iii) *Bilinear Strichartz bound:*

$$\|\partial_x(u\bar{u}(\cdot + x_0))\|_{L_t^2 H_x^{-\frac{1}{2}}} \lesssim \epsilon^2, \quad x_0 \in \mathbb{R}. \tag{1.9}$$

Here, we note that in the case $x_0 = 0$ the last bound gives

$$\|\partial_x |u|^2\|_{L_t^2 \dot{H}_x^{-\frac{1}{2}}} \lesssim \epsilon^2, \tag{1.10}$$

which is the more classical formulation of the bilinear L^2 bound. However, making this bound uniform with respect to the x_0 translation captures the natural separate translation invariance of this bound and is also quite useful in our proofs.

We also remark that all the bounds above are indeed Galilean invariant. As noted earlier, our main equation is not Galilean invariant, but the class of equations we are considering is. The estimates in the theorem do not represent the full strength of what we actually prove but are merely a simple but relevant sample. Our actual proof yields stronger frequency envelope bounds associated to a decomposition of the solution u on a unit frequency scale (rather than the more traditional dyadic decomposition); see Theorem 5 in Section 6.

Applied to the model cubic NLS problem (1.1), by scaling we have the following result which applies to the large data problem:

Theorem 2. *Consider the defocusing 1-d cubic NLS problem (1.1)(+) with L^2 initial data u_0 . Then the global solution u satisfies the following bounds:*

(i) *Uniform L^2 bound:*

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{L_x^2}. \tag{1.11}$$

(ii) *Strichartz bound:*

$$\|u\|_{L_{t,x}^6} \lesssim \|u_0\|_{L_x^2}. \tag{1.12}$$

(iii) *Bilinear Strichartz bound:*

$$\|\partial_x |u|^2\|_{L_t^2(\dot{H}_x^{-\frac{1}{2}} + cL_x^2)} \lesssim \|u_0\|_{L^2}^2, \quad c = \|u_0\|_{L^2}. \tag{1.13}$$

One may compare the above L^6 bound with the Planchon–Vega estimate (1.2), see [22], which applies only to H^1 solutions.

There are several ideas which play key roles in our analysis, all of which are used in a nonstandard fashion in the present work:

1. *Energy estimates via density-flux identities.* This is a classical idea in partial differential equations (PDEs), and particularly in the study of conservation laws, namely that the density-flux identities play a more fundamental role than just energy identities. The new twist in our context is that this analysis is carried out in a nonlocal setting, where both the densities and the fluxes involve translation invariant multilinear forms.

2. *The use of energy corrections.* This is an idea originally developed in the context of the so-called I-method [4] or more precisely the second generation I-method [7], whose aim was to construct more accurate almost conserved quantities. Here, we implement this idea at the level of density-flux identities, in a form closer to [19].

3. *Interaction Morawetz bounds.* These were originally developed in the context of the three-dimensional NLS problems by Colliander–Keel–Staffilani–Takaoka–Tao in [5] and have played a fundamental role in the study of many nonlinear Schrödinger flows (see, for example, [6, 23]) and also for one-dimensional quintic flows in the work of Dodson [10, 11]. Our take on this is somewhat closer to the one-dimensional approach of Planchon–Vega [22], though recast in the setting and language of nonlocal multilinear forms.

4. *Tao’s frequency envelope method.* This is used as a way to accurately track the evolution of the energy distribution across frequencies. Unlike the classical implementation relative to dyadic Littlewood–Paley decompositions, here we adapt and refine this notion for lattice decompositions instead. This is

also very convenient as a bootstrap tool (see, e.g., Tao [24], [25]) but with the added twist of also bootstrapping bilinear Strichartz bounds, as in the authors' paper [17].

1.3. An outline of the paper

In the next section, we begin by setting up the notations for function spaces and multilinear forms. More importantly, we also introduce our class of admissible frequency envelopes associated to lattice decompositions; this is based on the maximal function.

In Section 3, we carry out a preliminary step in the proof of our main result, namely we prove the small data local well-posedness result. This is independent of the global result and uses a contraction argument in a well-chosen function space defined via a wave packet type decomposition.

The goal of Section 4 is to recast energy identities for the mass and the momentum in density-flux form. We supplement this with two additional steps, where we first consider frequency localized mass and momentum densities, and then we improve their accuracy by adding a well chosen quartic correction.

In Section 5, we begin with the classical idea of Interaction Morawetz identities for the linear Schrödinger flow, and then we use our density-flux identities for the sharp frequency localized mass and momentum in order to obtain a set of refined Interaction Morawetz identities for our problem. For clarity of exposition, we consider separately the diagonal case, where the Interaction of equal frequency components is considered and the transversal case, which corresponds to separated frequency ranges.

The proof of our global result uses a complex bootstrap argument, involving both energy, Strichartz and bilinear L^2 bounds in a frequency localized setting and based on frequency envelopes. The bootstrap set-up is laid out in Section 6, which also contains a sharper, frequency envelope version of our result in Theorem 5. Our main estimates closing the bootstrap argument are carried out in Section 7, using the density-flux and Interaction Morawetz identities previously obtained.

Finally, in the last section of the paper we return from frequency localized bounds to global bounds in order to complete the proof of our main global result.

2. Notations and preliminaries

2.1. Lattice frequency decompositions

For our analysis, it will be convenient to localize functions in (spatial) frequency on the unit scale. For this, we consider a partition of unity

$$1 = \sum_{k \in \mathbb{Z}} p_k(\xi),$$

where p_k are smooth bump functions localized in $[k - 1, k + 1]$. Correspondingly, our solution u will be decomposed as

$$u = \sum_{k \in \mathbb{Z}} u_k, \quad u_k = P_k u.$$

The main estimates we will establish for our solution u will be linear and bilinear estimates for the functions u_k .

For a larger interval $A \subset \mathbb{Z}$, we denote

$$u_A = \sum_{k \in A} u_k.$$

2.2. Frequency envelopes

This is a tool which allows us to more accurately track the distribution of energy at various frequencies for the solutions to nonlinear evolution equations. In the present paper, they play a key bookkeeping

role in the proof of the linear and bilinear bounds for our solutions in the context of a complex bootstrap argument. In brief, given some standard decomposition of, say, an L^2 function

$$u = \sum u_k,$$

a frequency envelope for u is a sequence $\{c_k\}$ with the property that

$$\|u_k\|_{L^2} \lesssim c_k, \quad \|c_k\|_{\ell^2} \approx \|u\|_{L^2}.$$

In addition, one also limits how rapidly the sequence $\{c_k\}$ is allowed to vary. As originally introduced in work of Tao (see, e.g., [24]), in the context of dyadic Littlewood–Paley decompositions, one assumes that the sequence $\{c_k\}$ is slowly varying, in the sense that

$$\frac{c_j}{c_k} \leq 2^{\delta|k-j|}.$$

Here, we will instead work with a uniform lattice decomposition on the unit frequency scale. This requires a major revision of the above notion of ‘slowly varying’, which turns out to be far too weak for our purposes.

Instead, we want to strengthen this property in order to say that $c \approx Mc$ (the maximal function):

Definition 2.1. A lattice frequency envelope $\{c_k\}$ is said to have the maximal property if

$$Mc \leq Cc, \tag{2.1}$$

where Mc represents the maximal function of c ,

$$(Mc)_k = \sup_{j \geq 0} \frac{1}{2^j + 1} \sum_{l=k-j}^{k+j} c_l.$$

Here, C is a universal constant.

Frequency envelopes that have this property will be called *admissible*. An important observation is that admissible envelopes can always be found:

Lemma 2.2. Any ℓ^2 frequency envelope c^0 can be placed under a comparable maximal frequency envelope c , that is,

$$c^0 \leq c, \quad \|c\|_{\ell^2} \approx \|c^0\|_{\ell^2}. \tag{2.2}$$

Proof. We will use two properties of the maximal function:

- (i) $\|Mf\|_{L^2} \leq C\|f\|_{L^2}$.
 - (ii) $M(f + g) \leq Mf + Mg$.
- Given c^0 , we define c as

$$c = \sum_{k=0}^{\infty} (2C)^{-k} M^k c^0.$$

By property (i), this series converges in ℓ^2 , with

$$\|c\|_{\ell^2} \leq 2\|c^0\|_{\ell^2}.$$

Then by property (ii), we have

$$Mc \leq 2Cc.$$

The proof is concluded. □

For an interval $A \subset \mathbb{Z}$, we denote

$$c_A^2 = \sum_{k \in A} c_k^2.$$

Also for a dyadic integer n , we set

$$c_n^2 = \sum_{|k| \approx n} c_k^2.$$

Also given a translation invariant function space X , we denote by X_c the associated frequency envelope controlled norm

$$\|u\|_{X_c} = \sup_k c_k^{-1} \|u_k\|_X. \tag{2.3}$$

2.3. Multilinear forms and symbols

A key notion which is used throughout the paper is that of multilinear form. All our multilinear forms are invariant with respect to translations and have as arguments either complex valued functions or their complex conjugates.

For an integer $k \geq 2$, we will use translation invariant k -linear forms

$$(\mathcal{D}(\mathbb{R}))^k \ni (u_1, \dots, u_k) \rightarrow L(u_1, \bar{u}_2, \dots) \in \mathcal{D}'(\mathbb{R}),$$

where the nonconjugated and conjugated entries are alternating.

Such a form is uniquely described by its symbol $\ell(\xi_1, \xi_2, \dots, \xi_k)$ via

$$L(u_1, \bar{u}_2, \dots)(x) = (2\pi)^{-k} \int e^{i(x-x_1)\xi_1} e^{-i(x-x_2)\xi_2} \dots \ell(\xi_1, \dots, \xi_k) u_1(x_1) \bar{u}_2(x_2) \dots dx_1 \dots dx_k d\xi_1 \dots d\xi_k$$

or equivalently on the Fourier side

$$\mathcal{FL}(u_1, \bar{u}_2, \dots)(\xi) = (2\pi)^{-\frac{k-1}{2}} \int_D \ell(\xi_1, \dots, \xi_k) \hat{u}_1(\xi_1) \bar{\hat{u}}_2(\xi_2) \dots d\xi_1 \dots d\xi_{k-1},$$

where, with alternating signs,

$$D = \{\xi = \xi_1 - \xi_2 + \dots\}.$$

They can also be described via their kernel

$$L(u_1, \bar{u}_2, \dots)(x) = \int K(x - x_1, \dots, x - x_k) u_1(x_1) \bar{u}_2(x_2) \dots dx_1 \dots dx_k,$$

where K is defined in terms of the Fourier transform of ℓ

$$K(x_1, x_2, \dots, x_k) = (2\pi)^{-\frac{k}{2}} \hat{\ell}(-x_1, x_2, \dots, (-1)^k x_k).$$

All the symbols in this article will be assumed to be smooth, bounded and with bounded derivatives.

We remark that our notation is slightly nonstandard because of the alternation of complex conjugates, which is consistent with the setup of this paper. Another important remark is that, for k -linear forms, the cases of odd k , respectively, even k play different roles here, as follows:

i) The $2k + 1$ multilinear forms will be thought of as functions, for example, those which appear in some of our evolution equations.

ii) The $2k$ multilinear forms will be thought of as densities, for example, which appear in some of our density-flux pairs.

Correspondingly, to each $2k$ -linear form L we will associate a $2k$ -linear functional \mathbf{L} defined by

$$\mathbf{L}(u_1, \dots, u_{2k}) = \int_{\mathbb{R}} L(u_1, \dots, \bar{u}_{2k})(x) dx,$$

which takes real or complex values. This may be alternatively expressed on the Fourier side as

$$\mathbf{L}(u_1, \dots, u_{2k}) = (2\pi)^{1-k} \int_D \ell(\xi_1, \dots, \xi_{2k}) \hat{u}_1(\xi_1) \bar{\hat{u}}_2(\xi_2) \cdots \bar{\hat{u}}_{2k}(\xi_{2k}) d\xi_1 \cdots d\xi_{2k-1},$$

where, with alternating signs, the diagonal D_0 is given by

$$D_0 = \{0 = \xi_1 - \xi_2 + \cdots\}.$$

Note that in order to define the multilinear functional \mathbf{L} we only need to know the symbol ℓ on D_0 . There will be, however, more than one possible smooth extension of ℓ outside D_0 . This will play a role in our story later on.

2.4. Cubic Interactions in Schrödinger flows

Given three input frequencies ξ_1, ξ_2, ξ_3 for our cubic nonlinearity, the output will be at frequency

$$\xi_4 = \xi_1 - \xi_2 + \xi_3.$$

This relation can be described in a more symmetric fashion as

$$\Delta^4 \xi = 0, \quad \Delta^4 \xi := \xi_1 - \xi_2 + \xi_3 - \xi_4.$$

This is a resonant Interaction if and only if we have a similar relation for the associated time frequencies, namely

$$\Delta^4 \xi^2 = 0, \quad \Delta^4 \xi^2 := \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2.$$

Hence, we define the resonant set in a symmetric fashion as

$$\mathcal{R} := \{\Delta^4 \xi = 0, \Delta^4 \xi^2 = 0\}.$$

It is easily seen that this set may be characterized as

$$\mathcal{R} = \{\{\xi_1, \xi_3\} = \{\xi_2, \xi_4\}\}.$$

2.5. The Galilean symmetry

Here, we investigate how the equation (1.3) changes if we apply a Galilean transformation. In particular, we will justify our claim in the introduction that the transformed equation is of the same type.

We first recall the linear case. Suppose u solves the linear Schrödinger equation

$$(i\partial_t + \partial_x^2)u = f, \quad u(0) = u_0.$$

Given a frequency k , its Galilean transform v is defined by

$$v(t, x) := e^{-i(kx+k^2t)} u(t, x + 2kt)$$

and solves the linear Schrödinger equation

$$(i\partial_t + \partial_x^2)v = g, \quad v(0) = v_0,$$

where

$$v_0(x) = e^{-ikx}u_0, \quad g(t, x) = e^{-i(kx+k^2t)}f(t, x + 2kt).$$

Now, suppose that u solves equation(1.3). Then the above computation shows that v will solve a similar equation,

$$(i\partial_t + \partial_x^2)v = \tilde{C}(v, \bar{v}, v),$$

where

$$\tilde{C}(v, \bar{v}, v) = e^{-ikx}C(ve^{ikx}, \overline{ve^{ikx}}, ve^{ikx}).$$

This allows us to compute the symbol of \tilde{C} as

$$\tilde{c}(\xi_1, \xi_2, \xi_3) = c(\xi_1 - k, \xi_2 - k, \xi_3 - k).$$

This translated symbol is easily seen to have exactly the same properties as c .

3. Local well-posedness

Before approaching the global problem, an initial step is to establish local in time well-posedness. Since we only assume boundedness and smoothness on the symbol C , this is not an entirely straightforward matter. Our main result can be summarily stated as follows:

Theorem 3. *The evolution (1.3) is locally well-posed for small data in L^2 .*

Here, we need to clarify the meaning of well-posedness. For this problem, we will establish a semilinear type of well-posedness result. Precisely, for each initial data u_0 which is small in L^2 a unique solution exists in $C([0, 1]; L^2)$, with Lipschitz dependence on the initial data.

However, as it is often the case in the dispersive realm, we will not try to prove unconditional uniqueness and contend ourselves with having both existence and uniqueness of solutions in a ball in a restricted space $X \subset C([0, 1]; L^2)$.

A natural follow-up question here would be whether the same result holds for large data in our context. The answer is indeed affirmative; however, in this article we have chosen to only consider small data because this is all we need on one hand, and a large data result would require a more complex choice of the space X mentioned above, as well as a correspondingly more complex proof, on the other hand.

Another related question is whether a standard scaling argument could be used here. The scaling transformation would be the standard one for the cubic NLS problem,

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x).$$

For the initial data, this corresponds to

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x).$$

It is then easy to see that u_λ solves an equation of the same type as equation (1.3) but with the rescaled symbol

$$c_\lambda(\xi_1, \xi_2, \xi_3) = c(\xi_1/\lambda, \xi_2/\lambda, \xi_3/\lambda).$$

This satisfies the bound (1.4) uniformly only for $\lambda \geq 1$, so it cannot be used to reduce the large data problem to the small data problem. However, it can be used to obtain better life-span bounds for small data:

Corollary 3.1. *Assume that the initial data u_0 for equation (1.3) satisfies $\|u_0\|_{L^2} \leq \epsilon$. Then the solution u exists on $[0, T_\epsilon]$ with $T_\epsilon := c\epsilon^{-2}$, with similar bounds.*

The rest of this section is devoted to the proof of Theorem 3. The first step in our proof is to construct a suitable function space X where we seek the solutions.

Given a function u in $[0, 1] \times \mathbb{R}$, we start with a decomposition $u = \sum_{k \in \mathbb{Z}} u_k$ on the unit frequency scale and then a partition of unity in the physical space, also on the unit scale,

$$1 = \sum_{j \in \mathbb{Z}} \chi_j(x).$$

Finally, we define the norm of the space X for solutions

$$\|u\|_X^2 = \sum_{k \in \mathbb{Z}} \|u_k\|_{X_k}^2, \quad \|u_k\|_{X_k}^2 = \sum_{j \in \mathbb{Z}} \|\chi_j(t, x - 2tk)u_k\|_{L_t^\infty L_x^2}^2. \tag{3.1}$$

Here, the second argument of χ_j is consistent with the group velocity of frequency k waves. Indeed, if u were an L^2 solution to the homogeneous Schrödinger equation, then this would be nothing but a wave packet decomposition of u on the unit time scale. It is easily seen that we have the embedding

$$X \subset L_t^\infty L_x^2.$$

Remark 3.2. Due to the unit frequency localization of u_k and Bernstein’s inequality, we may freely replace the $L_t^\infty L_x^2$ norm in equation (3.1) by $L_{t,x}^\infty$.

Correspondingly, we define a similar space Y for the source term in a linear Schrödinger equation, namely

$$\|f\|_Y^2 = \sum_{k \in \mathbb{Z}} \|f_k\|_{Y_k}^2, \quad \|f_k\|_{Y_k}^2 = \sum_{j \in \mathbb{Z}} \|\chi_j(t, x - 2tk)f_k\|_{L_t^1 L_x^2}^2 \tag{3.2}$$

so that we have the duality relation

$$X = Y^*,$$

with equivalent norms.

Then for the small data local well-posedness result in X it suffices to establish the following two properties. The first is a linear mapping property:

Lemma 3.3. *The solution to the linear Schrödinger equation*

$$(i\partial_t + \partial_x^2)u = f, \quad u(0) = u_0 \tag{3.3}$$

in the time interval $[0, 1]$ satisfies

$$\|u\|_X \lesssim \|u_0\|_{L^2} + \|f\|_Y. \tag{3.4}$$

The second is an estimate for the nonlinearity:

Lemma 3.4. *For the cubic nonlinearity C , we have the bound*

$$\|C(u, \bar{u}, u)\|_Y \lesssim \|u\|_X^3. \tag{3.5}$$

Once we have these two lemmas, the proof of the local well-posedness result follows in a standard manner using the contraction principle in a small ball in X . However, for later use we also need to have a more precise, frequency envelope version of Theorem 3. This is as follows:

Theorem 4. *For each small initial data*

$$\|\mathbf{u}_0\|_{L^2} \leq \epsilon \ll 1,$$

there exists a unique solution u to equation (1.3) which is small in X . In addition, suppose c_k is an ℓ^2 normalized admissible frequency envelope so that

$$\|\mathbf{u}_0\|_{L_c^2} \lesssim \epsilon.$$

Then the solution u satisfies

$$\|u\|_{X_c} \lesssim \epsilon. \tag{3.6}$$

This requires stronger, frequency envelope versions of Lemmas 3.3, 3.4:

Lemma 3.5. *The solution to the linear Schrödinger equation*

$$(i\partial_t + \partial_x^2)u = f, \quad u(0) = \mathbf{u}_0 \tag{3.7}$$

in the time interval $[0, 1]$ satisfies

$$\|u\|_{X_c} \lesssim \|\mathbf{u}_0\|_{L_c^2} + \|f\|_{Y_c}. \tag{3.8}$$

The second is an estimate for the nonlinearity:

Lemma 3.6. *Let c_k be an ℓ^2 normalized admissible frequency envelope. Then for the cubic nonlinearity C we have the bound*

$$\|C(u, \bar{u}, u)\|_{Y_c} \lesssim \|u\|_{X_c}^3. \tag{3.9}$$

Proof of Lemmas 3.3, 3.5. We can freely localize on the unit scale in frequency and reduce the problem to the frequency localized estimate

$$\|u_k\|_{X_k} \lesssim \|\mathbf{u}_{0k}\|_{L^2} + \|f_k\|_{Y_k}. \tag{3.10}$$

We can further reduce the problem by applying a Galilean transformation by setting

$$v(t, x) = e^{-i(kx+k^2t)} u_k(t, x - 2kt), \quad \mathbf{v}_0(x) = e^{-ikx} \mathbf{u}_{0k}, \quad g(t, x) = e^{-i(kx+k^2t)} f_k(t, x - 2kt).$$

Here, the functions v, \mathbf{v}_0, g are now localized at frequency 0 and solve

$$(i\partial_t + \partial_x^2)v = g, \quad v(0) = \mathbf{v}_0,$$

whereas the bound (3.10) reduces to

$$\|v\|_{X_0} \lesssim \|\mathbf{v}_0\|_{L^2} + \|g\|_{Y_0}. \tag{3.11}$$

Inserting a harmless frequency localization P_0 , we represent v as

$$v(t) = e^{it\partial_x^2} P_0 \mathbf{v}_0 - i \int_0^t e^{i(t-s)\partial_x^2} P_0 g \, ds.$$

Here, by a slight abuse a notation, we allow P_0 to have slightly larger support. Finally, we localize spatially at both ends,

$$\chi_j v(t) = \sum_{l \in \mathbb{Z}} \left(\chi_j e^{it\partial_x^2} P_0 \chi_l v_0 + \int_0^t \chi_j e^{i(t-s)\partial_x^2} P_0 \chi_l g \, ds \right).$$

Here, the kernels for $e^{it\partial_x^2} P_0$ are uniformly Schwartz for $t \in [0, 1]$, so we get an L^2 bound with off-diagonal decay,

$$\|\chi_j e^{it\partial_x^2} P_0 \chi_l\|_{L^2 \rightarrow L^2} \lesssim \langle j - l \rangle^{-N}.$$

This implies that

$$\|\chi_j v\|_{L_t^\infty L_x^2} \lesssim \sum_{l \in \mathbb{Z}} \langle j - l \rangle^{-N} \left(\|\chi_l v_0\|_{L_x^2} + \|\chi_l g\|_{L_t^1 L_x^2} \right),$$

which in view of the off-diagonal decay implies the bound (3.11). □

Proof of Lemmas 3.4, 3.6. Here, the second lemma implies the first. We need to prove the estimate

$$\|P_k C(u, \bar{u}, u)\|_{Y_k} \lesssim c_k \|u\|_{X_c}^3.$$

By duality, this reduces to the integral bound

$$|I| \lesssim c_k \|u\|_{X_c}^3 \|v_k\|_{X_k}, \quad I = \int C(u, \bar{u}, u) \bar{v}_k \, dx dt. \tag{3.12}$$

Without any restriction in generality, we may assume that

$$\|u\|_{X_c} = 1, \quad \|v_k\|_{X_k} = 1.$$

We use the unit scale frequency decomposition to separate the above integral as

$$I = \sum_{k_1 - k_2 + k_3 = k} \int C_{k_1 k_2 k_3}(u_{k_1}, \bar{u}_{k_2}, u_{k_3}) \bar{v}_k \, dx dt,$$

where we have also localized the kernel of C near frequencies k_1, k_2, k_3 on the unit scale. The symbol of $C_{k_1 k_2 k_3}$ is smooth and bounded on the unit scale, so the above summands are essentially like products and may be indeed thought of as products via separation of variables. For bilinear products, we have the estimate

$$\|u_{k_1} v_{k_2}\|_{L_{x,t}^2} \lesssim \frac{1}{\langle k_1 - k_2 \rangle^{\frac{1}{2}}} \|u_{k_1}\|_{X_{k_1}} \|v_{k_2}\|_{X_{k_2}}. \tag{3.13}$$

This is obtained simply by examining the intersection of the supports of the bump functions traveling with speeds $2k_1$, respectively $2k_2$.

Denoting $\delta k^{hi} = \max |k_i - k_j|$, the relation $k_1 + k_3 = k_2 + k$ insures that we can group the four frequencies into two pairs at distance δk^{hi} . Then, using twice the above bilinear estimate, we have

$$|I| \lesssim \sum_{k_1 - k_2 + k_3 - k = 0} \frac{1}{\langle \delta k^{hi} \rangle} c_{k_1} c_{k_2} c_{k_3}.$$

Let n represent the dyadic size of δk^{hi} . Without loss of generality, by relabeling, suppose that

$$|k - k_3| \approx |k - k_2| \approx n, \quad |k - k_1| \lesssim n.$$

Then using the Cauchy–Schwartz inequality for the pair (k_2, k_3) for fixed k_1 we estimate

$$|I| \lesssim \sum_n \frac{1}{n} \sum_{|k_1-k| \lesssim n} c_{k_1} c_n^2, \quad c_n^2 := \sum_{|j-k| \approx n} c_j^2.$$

Now, we use the maximal function inequality for c , which gives

$$\frac{1}{n} \sum_{|k_1-k| \lesssim n} c_{k_1} \lesssim c_k.$$

We obtain

$$|I| \lesssim c_k \sum_n c_n^2 \approx c_k.$$

Thus, equation (3.12) is proved. □

For later use, we note that the frequency envelope bounds for u together with the bilinear L^2 bound (3.13) imply the following.

Corollary 3.7. *Let u be a solution for equation (1.3) in $[0, 1]$ as in Theorem 4. Then the following bounds hold:*

$$\|u_k\|_{L_{t,x}^6} \lesssim \epsilon c_k, \tag{3.14}$$

$$\|\partial_x(u_{k_1} \bar{u}_{k_2}(\cdot + x_0))\|_{L_{t,x}^2} \lesssim \epsilon^2 \langle k_1 - k_2 \rangle^{\frac{1}{2}} c_{k_1} c_{k_2}. \tag{3.15}$$

4. Energy estimates and conservation laws

4.1. Conservation laws for the linear problem

We begin our discussion with the linear Schrödinger equation

$$iu_t + u_{xx} = 0, \quad u(0) = u_0. \tag{4.1}$$

For this, we consider the following three conserved quantities, the mass

$$\mathbf{M}(u) = \int |u|^2 dx,$$

the momentum

$$\mathbf{P}(u) = 2 \int \Im(\bar{u} \partial_x u) dx,$$

as well as the energy

$$\mathbf{E}(u) = 4 \int |\partial_x u|^2 dx.$$

To these quantities, we associate corresponding densities

$$M(u) = |u|^2, \quad P(u) = i(\bar{u}\partial_x u - u\partial_x \bar{u}), \quad E(u) = -\bar{u}\partial_x^2 u + 2|\partial_x u|^2 - u\partial_x^2 \bar{u}.$$

The choice of densities here is not entirely straightforward. Symmetry is clearly a criterion, but further motivation is provided by the conservation law computation,

$$\partial_t M(u) = \partial_x P(u), \quad \partial_t P(u) = \partial_x E(u). \tag{4.2}$$

The symbols of these densities viewed as bilinear forms are

$$m(\xi, \eta) = 1, \quad p(\xi, \eta) = -(\xi + \eta), \quad e(\xi, \eta) = (\xi + \eta)^2.$$

More generally, we can start with a symbol $a(\xi, \eta)$ which is symmetric, in the sense that

$$a(\eta, \xi) = \overline{a(\xi, \eta)},$$

and then define an associated weighted mass density by

$$M_a(u) = A(u, \bar{u}).$$

We also define corresponding momentum and energy symbols p_a and e_a by

$$p_a(\xi, \eta) = -(\xi + \eta)a(\xi, \eta), \quad e_a(\xi, \eta) = (\xi + \eta)^2 a(\xi, \eta).$$

Then a direct computation yields the density-flux relations

$$\frac{d}{dt} M_a(u, \bar{u}) = \partial_x P_a(u, \bar{u}), \quad \frac{d}{dt} P_a(u, \bar{u}) = \partial_x E_a(u, \bar{u}).$$

4.2. Nonlinear density-flux identities for the mass and momentum

Here, we develop the counterpart of the linear analysis above for the nonlinear problem (1.3).

4.2.1. The modified mass

To motivate what follows, we begin with a simpler computation for the L^2 norm of a solution u of equation (1.3):

$$\frac{d}{dt} \|u\|_{L^2}^2 = \int -iC(u, \bar{u}, u) \cdot \bar{u} + iu \cdot \overline{C(u, \bar{u}, u)} \, dx := \int C_m^4(u, \bar{u}, u, \bar{u}) \, dx.$$

A priori the symbol of the quartic form C_m^4 , defined on the diagonal $\Delta^4 \xi = 0$, is given by

$$c_m^4(\xi_1, \xi_2, \xi_3, \xi_4) = -ic(\xi_1, \xi_2, \xi_3) + i\bar{c}(\xi_2, \xi_3, \xi_4).$$

However, we can further symmetrize and replace it by

$$c_m^4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{i}{2} (-c(\xi_1, \xi_2, \xi_3) - c(\xi_1, \xi_4, \xi_3) + \bar{c}(\xi_2, \xi_3, \xi_4) + \bar{c}(\xi_2, \xi_1, \xi_4)).$$

In particular, we are interested in the behavior of $c_m^4(\xi_1, \xi_2, \xi_3, \xi_4)$ on the resonant set

$$\mathcal{R} = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 / \Delta^4 \xi = 0, \Delta^4 \xi^2 = 0\} = \{\{\xi_1, \xi_3\} = \{\xi_2, \xi_4\}\}.$$

On this set, we compute

$$\begin{aligned} c_m^4(\xi_1, \xi_1, \xi_3, \xi_3) &= \frac{i}{2} (-c(\xi_1, \xi_1, \xi_3) - c(\xi_1, \xi_3, \xi_3) + \bar{c}(\xi_1, \xi_3, \xi_3) + \bar{c}(\xi_1, \xi_1, \xi_3)) \\ &= \Im(c(\xi_1, \xi_1, \xi_3) + c(\xi_1, \xi_3, \xi_3)). \end{aligned}$$

Then we observe that our (H2) assumption on C shows that this expression vanishes. One might wonder here if we could not weaken this assumption by requiring that the sum of the two terms is zero, rather than each of them separately. This would indeed be the case if all we were interested in is the almost conservation of mass. However, we will later add localization weights which will act differently on the two terms.

The fact that c_m^4 vanishes on the resonant set \mathcal{R} implies (see Lemma 4.1 below) that we can smoothly divide

$$b_m^4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{ic_m^4(\xi_1, \xi_2, \xi_3, \xi_4)}{\Delta^4 \xi^2}$$

on $\Delta^4 \xi = 0$. We now use \mathbf{B}_m^4 as an energy correction. Then we obtain the modified energy relation

$$\frac{d}{dt} (\|u\|_{L^2}^2 + \mathbf{B}_m^4(u, \bar{u}, u, \bar{u})) = \mathbf{R}_m^6(u, \bar{u}, u, \bar{u}, u, \bar{u}), \tag{4.3}$$

where \mathbf{R}_m^6 is a symmetric 6-linear form. Here, the left-hand side may be viewed as a modified energy, while the right-hand side can potentially be estimated using the $L_{t,x}^6$ norm of u .

4.2.2. The modified mass and momentum density-flux pairs

The key idea here is that, corresponding to the above modified mass, we also want to write a conservation law for an associated mass density

$$M^\sharp(u) = M(u) + B_m^4(u, \bar{u}, u, \bar{u}). \tag{4.4}$$

However, when doing this, we remark that the symbol of B_m^4 was previously defined only on the diagonal $\Delta^4 \xi = 0$, whereas in order for the above expression to be well defined we need to extend it everywhere. For the purpose of this computation, we simply assume that we have chosen some smooth extension. A more careful choice will be considered later in Lemma 4.1.

Now, we compute

$$\partial_t M^\sharp(u) = \partial_x P(u) + C_m^4(u, \bar{u}, u, \bar{u}) + i(\Delta^4 \xi^2 B_m^4)(u, \bar{u}, u, \bar{u}) + R_m^6(u, \bar{u}, u, \bar{u}, u, \bar{u}).$$

By the choice of B_m^4 , the symbol of the quartic term above $c_m^4 + i\Delta^4 \xi^2 b_m^4$ vanishes on the diagonal $\{\Delta^4 \xi = 0\}$; therefore, we can express it smoothly in the form

$$c_m^4 + i\Delta^4 \xi^2 b_m^4 = i\Delta^4 \xi r_m^4. \tag{4.5}$$

Hence, the above relation can be written in the better form

$$\partial_t M^\sharp(u) = \partial_x (P(u) + R_m^4(u, \bar{u}, u, \bar{u})) + R_m^6(u, \bar{u}, u, \bar{u}, u, \bar{u}). \tag{4.6}$$

One may view here the relation (4.5) as a division problem, where c_m^4 vanishes on the resonant set \mathcal{R} . The symbols b_m^4 and r_m^4 are not uniquely determined by the relation (4.5), as we can change them by

$$b_m^4 \rightarrow b_m^4 + q\Delta^4 \xi, \quad r_m^4 \rightarrow r_m^4 + q\Delta^4 \xi^2,$$

for any smooth q . However, this is the only ambiguity. In particular r_m^4 is uniquely determined on the set $\Delta^4 \xi^2 = 0$, while b_m^4 is uniquely determined on the set $\Delta^4 \xi = 0$.

One could carry out a similar computation for the momentum, where the starting point is the relation

$$\partial_t P(u) = \partial_x E(u) + C_p^4(u, \bar{u}, u, \bar{u}).$$

Precisely, the symbol of C_p^4 is initially given by

$$c_p^4(\xi_1, \xi_2, \xi_3, \xi_4) = i(\xi_1 - \xi_2 + \xi_3 + \xi_4)c(\xi_1, \xi_2, \xi_3) - i(\xi_1 + \xi_2 - \xi_3 + \xi_4)\bar{c}(\xi_2, \xi_3, \xi_4).$$

However, we can further symmetrize it exactly as in the case of C_m^4 . Then it also vanishes on the resonant set \mathcal{R} , so it admits a (nonunique) representation of the form

$$c_p^4 + i\Delta^4 \xi^2 b_p^4 = i\Delta^4 \xi r_p^4. \tag{4.7}$$

Hence, as in the case of the mass, we define a quartic correction for the momentum density

$$P^\sharp(u) = P(u) + B_p^4(u, \bar{u}, u, \bar{u}).$$

This satisfies a conservation law of the form

$$\partial_t P^\sharp(u) = \partial_x (E(u) + R_p^4(u, \bar{u}, u, \bar{u})) + R_p^6(u, \bar{u}, u, \bar{u}, u, \bar{u}). \tag{4.8}$$

4.3. The choice for the density-flux corrections

Here, we consider the division problem in equation (4.5) and ask what should be a good balance between the symbols B_m^4 and R_m^4 . We recall that b_m^4 is uniquely determined on the diagonal $\Delta^4 \xi = 0$, but we can choose it freely away from the diagonal.

To move away from the diagonal, it is useful to do it in a Galilean invariant fashion. The expression $\Delta^4 \xi^2$ is not Galilean invariant, but we do have a suitable replacement, namely the expression

$$\tilde{\Delta}^4 \xi^2 := \Delta^4 \xi^2 - 2\xi_{avg} \Delta^4 \xi = \frac{1}{2}((\xi_1 - \xi_3)^2 - (\xi_2 - \xi_4)^2).$$

This is easily seen to be invariant with respect to translations. To measure the size of both $\Delta^4 \xi$ and $\tilde{\Delta}^4 \xi^2$, we introduce two parameters,

$$\begin{aligned} \delta \xi^{hi} &:= \max\{|\xi_1 - \xi_2| + |\xi_3 - \xi_4|, |\xi_1 - \xi_4| + |\xi_3 - \xi_2|\}, \\ \delta \xi^{med} &:= \min\{|\xi_1 - \xi_2| + |\xi_3 - \xi_4|, |\xi_1 - \xi_4| + |\xi_3 - \xi_2|\}, \end{aligned} \tag{4.9}$$

where $\delta \xi^{hi}$ measures the diameter of the full set of ξ 's whereas $\delta \xi^{med}$ measures the distance of the sets $\{\xi_1, \xi_3\}$ and $\{\xi_2, \xi_4\}$. With these notations, we have bounds from above as follows:

$$|\Delta^4 \xi| \lesssim \delta \xi^{med}, \quad |\tilde{\Delta}^4 \xi^2| \lesssim \delta \xi^{hi} \delta \xi^{med}. \tag{4.10}$$

We will think of the symbol $\Delta^4 \xi$ as being elliptic where approximate equality holds in the first relation and of $\tilde{\Delta}^4 \xi^2$ as being elliptic where approximate equality holds in the second relation. Based on this, we will decompose the phase space into three overlapping regions which can be separated using cutoff functions which are smooth on the unit scale:

- i) The full division region,

$$\Omega_1 = \{\delta \xi^{med} \lesssim 1\},$$

which represents a full unit size neighbourhood of the resonant set \mathcal{R} .

ii) The region

$$\Omega_2 = \{1 + |\Delta^4 \xi| \ll \delta \xi^{\text{med}}\},$$

where $\tilde{\Delta}^4 \xi^2$ must be elliptic, $|\tilde{\Delta}^4 \xi^2| \approx \delta \xi^{\text{hi}} \delta \xi^{\text{med}}$, and thus we will favor division by the symbol $\tilde{\Delta}^4 \xi^2$.

iii) The region

$$\Omega_3 = \{1 \ll \delta \xi^{\text{med}} \lesssim |\Delta^4 \xi|\},$$

we will instead divide by $\Delta^4 \xi$; this is compensated by the relatively small size of this region.

This decomposition leads us to the following division lemma:

Lemma 4.1. *Let c^4 be a bounded symbol which is smooth on the unit scale and which vanishes on \mathcal{R} . Then it admits a representation*

$$c^4 = \Delta^4 \xi \tilde{r}^4 - \tilde{\Delta}^4 \xi^2 b^4, \tag{4.11}$$

where \tilde{r}^4 and b^4 are also smooth on the unit scale, with the following properties:

i) Size

$$\begin{aligned} |\partial^\alpha \tilde{r}^4| &\lesssim \frac{1}{\langle \delta \xi^{\text{med}} \rangle}, \\ |\partial^\alpha b^4| &\lesssim \frac{1}{\langle \delta \xi^{\text{hi}} \rangle \langle \delta \xi^{\text{med}} \rangle}. \end{aligned} \tag{4.12}$$

ii) Support: b^4 is supported in $\Omega_1 \cup \Omega_2$ and \tilde{r}^4 is supported in $\Omega_1 \cup \Omega_3$.

Here and later in the paper, by ‘smooth on the unit scale’ we mean that the above functions and all their derivatives are bounded, with bounds as in equation (4.12), and where the implicit constant is allowed to depend on α , but not on anything else. As usual, only finitely many derivatives are needed on our analysis, but we do not take the extra step of determining how many.

To return to $\Delta^4 \xi$, we have the following straightforward observation:

Remark 4.2. Later, we will need similar decompositions but with $\tilde{\Delta}^4 \xi^2$ replaced by $\Delta^4 \xi^2$,

$$c^4 = \Delta^4 \xi r^4 - \Delta^4 \xi^2 b^4.$$

This is easily done via the substitution

$$r^4 = \tilde{r}^4 + 2\xi_{\text{avg}} b^4.$$

But in doing this, we loose the above bound for \tilde{r}^4 unless $|\xi_{\text{avg}}| \lesssim \delta \xi^{\text{hi}}$. Precisely, we obtain instead

$$|\partial^\alpha r^4| \lesssim \frac{1}{\langle \delta \xi^{\text{med}} \rangle} \left(1 + \frac{|\xi_{\text{avg}}|}{\langle \delta \xi^{\text{hi}} \rangle} \right). \tag{4.13}$$

Proof. Using a partition of unity which is smooth on the unit scale, we can reduce the problem to the case when c^4 is supported in exactly one of the regions Ω_1 , Ω_2 and Ω_3 . We consider each of these cases separately.

i) c^4 is supported in Ω_1 . To simplify notations here, we introduce new linear coordinates $(\eta_1, \eta_2, \eta_3, \eta_4)$, where

$$\eta_1 = \Delta^4 \xi, \quad \eta_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4, \quad \eta_3 = \xi_1 - \xi_2 - \xi_3 + \xi_4, \quad 2\eta_2 \eta_3 = \tilde{\Delta}^4 \xi^2.$$

For η_4 , we can choose in a symmetric fashion

$$\eta_4 = \xi_1 + \xi_2 + \xi_3 + \xi_4,$$

though this does not play any role in the sequel.

In these coordinates, we have

$$\Omega_1 = \{|\eta_1| + \min\{|\eta_2|, |\eta_3|\} \lesssim 1\} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{13},$$

where

$$\begin{aligned} \Omega_{11} &:= \{|\eta_1| + |\eta_2| + |\eta_3| \lesssim 1\}, & \Omega_{12} &:= \{|\eta_1| + |\eta_2| \lesssim 1 \lesssim |\eta_3|\}, \\ \Omega_{13} &:= \{|\eta_1| + |\eta_3| \lesssim 1 \lesssim |\eta_2|\}. \end{aligned}$$

Using another partition of unity which is smooth on the unit scale, the problem reduces to separately considering the case when c^4 is supported in each of these three sets.

Within the set Ω_{12} , we have $c^4(0, 0, \eta_3, \eta_4) = 0$; therefore, we can easily represent

$$c^4(\eta_1, \eta_2, \eta_3, \eta_4) = (c^4(\eta_1, \eta_2, \eta_3, \eta_4) - c^4(0, \eta_2, \eta_3, \eta_4)) + (c^4(0, \eta_2, \eta_3, \eta_4) - c^4(0, 0, \eta_3, \eta_4)),$$

where the first difference may be smoothly divided by η_1 and the second by η_2 , with the quotients contributing to \tilde{r}^4 , respectively b^4 . The set Ω_{13} can be dealt with in a similar fashion.

It remains to consider Ω_{11} , where we know that $c^4 = 0$ in $\eta_1 = \eta_2\eta_3 = 0$. Here, we write

$$c^4(\eta_1, \eta_2, \eta_3, \eta_4) = (c^4(\eta_1, \eta_2, \eta_3, \eta_4) - c^4(0, \eta_2, \eta_3, \eta_4)) + c^4(0, \eta_2, \eta_3, \eta_4).$$

Now, the first difference can be smoothly divided by η_1 , while the last term can be successively and smoothly divided by η_2 and η_3 .

ii) c^4 is supported in Ω_2 . Here we set

$$b^4 = \frac{c^4}{\bar{\Delta}^4 \xi^2}, \quad \tilde{r}^4 = 0,$$

and we observe that

$$\left| \partial^\alpha \frac{1}{\bar{\Delta}^4 \xi^2} \right| \lesssim \frac{1}{\delta \xi^{\text{med}} \delta \xi^{\text{hi}}}.$$

iii) c^4 is supported in Ω_3 . Here, we set

$$b^4 = 0, \quad \tilde{r}^4 = -\frac{c^4}{\Delta^4 \xi},$$

and we observe that

$$\left| \partial^\alpha \frac{1}{\Delta^4 \xi} \right| \lesssim \frac{1}{\delta \xi^{\text{med}}}. \quad \square$$

4.4. The Galilean invariance

While the assumptions (H1–3) on the cubic nonlinearity C are Galilean invariant, our density-flux identities are not. In order to rectify that, suppose heuristically that we are looking at linear waves concentrated around a frequency ξ_0 . This corresponds to a linear group velocity of $2\xi_0$, so in the

density-flux identities it would be natural to replace the operator ∂_t by $\partial_t + 2\xi_0\partial_x$. At the linear level, this is done by recentering the energy and momentum densities at ξ_0 ,

$$\begin{aligned} p_{\xi_0}(\xi_1, \xi_2) &= -\xi_1 - \xi_2 + 2\xi_0 = p + 2\xi_0m, \\ e_{\xi_0}(\xi_1, \xi_2) &= (\xi_1 + \xi_2 - 2\xi_0)^2 = e + 4\xi_0p + 4\xi_0^2m. \end{aligned} \tag{4.14}$$

Then the density-flux identities (4.2) become

$$(\partial_t + 2\xi_0\partial_x)M(u) = \partial_x P_{\xi_0}(u), \quad (\partial_t + 2\xi_0\partial_x)P_{\xi_0}(u) = \partial_x E_{\xi_0}(u). \tag{4.15}$$

Next, we consider the nonlinear setting. There M^\sharp is the same as before, but $P_{\xi_0}^\sharp$ is

$$P_{\xi_0}^\sharp = P^\sharp - 2\xi_0M^\sharp = P_{\xi_0} + B_{p, \xi_0}^4,$$

where the symbol for B_{p, ξ_0}^4 is given by

$$b_{p, \xi_0}^4 = b_p^4 + 2\xi_0b_m^4. \tag{4.16}$$

Then our density-flux identities have the form

$$(\partial_t + 2\xi_0\partial_x)M^\sharp(u) = \partial_x(p_{\xi_0}(u) + R_{m, \xi_0}^4(u, \bar{u}, u, \bar{u})) + R_{m, \xi_0}^6(u, \bar{u}, u, \bar{u}, u, \bar{u}), \tag{4.17}$$

$$(\partial_t + 2\xi_0\partial_x)P_{\xi_0}^\sharp(u) = \partial_x(e_{\xi_0}(u) + R_{p, \xi_0}^4(u, \bar{u}, u, \bar{u})) + R_{p, \xi_0}^6(u, \bar{u}, u, \bar{u}, u, \bar{u}), \tag{4.18}$$

where the symbols for R_{m, ξ_0}^4 and R_{p, ξ_0}^4 are defined by

$$r_{m, \xi_0}^4 = r_m^4 + 2\xi_0b_m^4, \quad r_{p, \xi_0}^4 = r_p^4 + 2\xi_0b_p^4 + 2\xi_0r_m^4 + 4\xi_0^2b_m^4.$$

4.5. Localized density-flux identities for mass and momentum

In our analysis later on, we will not use density-flux pairs for global estimates, but instead we will use them only in a frequency localized setting.

Here, we begin our discussion with a symmetric bilinear symbol $a(\xi, \eta)$. We are assuming it generates a real valued quadratic form $A(u, \bar{u})$, that is, that

$$a(\xi, \eta) = \bar{a}(\eta, \xi)$$

and that its symbol is bounded and uniformly smooth. Later, we will use such symbols a to localize our analysis to intervals I in frequency, either of unit size or larger.

Corresponding to such a , we define corresponding quadratic localized mass, momentum and energy by

$$m_a(\xi, \eta) := a(\xi, \eta), \quad p_a(\xi, \eta) := -(\xi + \eta)a(\xi, \eta), \quad e_a(\xi, \eta) := (\xi + \eta)^2a(\xi, \eta).$$

A direct computation yields the relation

$$\partial_t M_a(u) = P_a(u) + C_{m, a}^4(u), \tag{4.19}$$

where the symbol $C_{m,a}^4$ is given by

$$c_{m,a}^4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{i}{2} [c(\xi_1, \xi_2, \xi_3)m_a(\xi_1 - \xi_2 + \xi_3, \xi_4) + c(\xi_1, \xi_4, \xi_3)m_a(\xi_1 - \xi_4 + \xi_3, \xi_2) - \bar{c}(\xi_2, \xi_3, \xi_4)m_a(\xi_3, \xi_2 - \xi_3 + \xi_4) - \bar{c}(\xi_2, \xi_1, \xi_4)m_a(\xi_3, \xi_2 - \xi_1 + \xi_4)].$$

A similar identity applies in the case of the localized momentum, where we simply replace the symbol m_a by p_a .

As before, this symbol vanishes on the resonant set \mathcal{R} , so we can represent it as in the division relation (4.5),

$$c_{m,a}^4 + i\Delta^4 \xi^2 b_{m,a}^4 = i\Delta^4 \xi r_{m,a}^4, \tag{4.20}$$

as well as

$$c_{p,a}^4 + i\Delta^4 \xi^2 b_{p,a}^4 = i\Delta^4 \xi r_{p,a}^4. \tag{4.21}$$

Then, defining M_a^\sharp and P_a^\sharp as before,

$$M_a^\sharp(u) := M_a(u) + B_{m,a}^4(u, \bar{u}, u, \bar{u}), \tag{4.22}$$

$$P_a^\sharp(u) := P_a(u) + B_{p,a}^4(u, \bar{u}, u, \bar{u}), \tag{4.23}$$

we obtain density-flux identities akin to equation (4.6), namely

$$\partial_t M_a^\sharp(u) = \partial_x (P_a(u) + R_{m,a}^4(u)) + R_{m,a}^6(u), \tag{4.24}$$

and

$$\partial_t P_a^\sharp(u) = \partial_x (E_a(u) + R_{p,a}^4(u)) + R_{p,a}^6(u). \tag{4.25}$$

We will consider these relations together with their Galilean shifts obtaining relations of the form

$$(\partial_t + 2\xi_0 \partial_x) M_{a,\xi_0}^\sharp(u) = \partial_x (P_{a,\xi_0}(u) + R_{m,a,\xi_0}^4(u)) + R_{m,a,\xi_0}^6(u), \tag{4.26}$$

respectively

$$(\partial_t + 2\xi_0 \partial_x) P_{a,\xi_0}^\sharp(u) = \partial_x (E_{a,\xi_0}(u) + R_{p,a,\xi_0}^4(u)) + R_{p,a,\xi_0}^6(u). \tag{4.27}$$

These correspond to the algebraic division relations

$$c_{m,a}^4 + i\Delta^4 (\xi - \xi_0)^2 b_{m,a}^4 = i\Delta^4 \xi r_{m,a,\xi_0}^4, \tag{4.28}$$

respectively

$$c_{p,a,\xi_0}^4 + i\Delta^4 (\xi - \xi_0)^2 b_{p,a,\xi_0}^4 = i\Delta^4 \xi r_{p,a,\xi_0}^4, \tag{4.29}$$

where

$$c_{p,a,\xi_0}^4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{i}{2} [c(\xi_1, \xi_2, \xi_3)p_{a,\xi_0}(\xi_1 - \xi_2 + \xi_3, \xi_4) + c(\xi_1, \xi_4, \xi_3)p_{a,\xi_0}(\xi_1 - \xi_4 + \xi_3, \xi_2) - \bar{c}(\xi_2, \xi_3, \xi_4)p_{a,\xi_0}(\xi_1, \xi_2 - \xi_3 + \xi_4) - \bar{c}(\xi_2, \xi_1, \xi_4)p_{a,\xi_0}(\xi_3, \xi_2 - \xi_1 + \xi_4)]. \tag{4.30}$$

The symbols above are connected in the obvious way. Precisely, we have

$$r_{m,a,\xi_0}^4 = r_{m,a}^4 + 2\xi_0 b_{m,a}^4, \tag{4.31}$$

and

$$P_{a,\xi_0}^\# = P_a^\# + 2\xi_0 M_a^\#, \quad b_{p,a,\xi_0}^4 = b_{p,a}^4 + 2\xi_0 b_{m,a}^4, \tag{4.32}$$

and finally

$$r_{p,a,\xi_0}^4 = r_{p,a}^4 + 2\xi_0 b_{p,a}^4 + 2\xi_0^2 r_{m,a,\xi_0}^4. \tag{4.33}$$

To use these density-flux relations we need to have appropriate bounds for our symbols:

Proposition 4.3. *Let $J \subset \mathbb{R}$ be an interval of length r and $d(\xi_0, J) \lesssim r$. Assume that a is supported in $J \times J$, with bounded and uniformly smooth symbol. Then the relations (4.28) and (4.29) hold with symbols $b_{m,a}^4$, b_{p,a,ξ_0}^4 , r_{m,a,ξ_0}^4 and r_{p,a,ξ_0}^4 which can be chosen to have the following properties:*

- i) *Support: They are all supported in the region where at least one of the frequencies is in J .*
- ii) *Size:*

$$|b_{m,a}^4| \lesssim \frac{1}{\langle \delta \xi^{hi} \rangle \langle \delta \xi^{med} \rangle}, \quad |b_{p,a,\xi_0}^4| \lesssim \frac{r}{\langle \delta \xi^{hi} \rangle \langle \delta \xi^{med} \rangle}, \tag{4.34}$$

$$\begin{aligned} |r_{m,a,\xi_0}^4| &\lesssim \frac{1}{\langle \delta \xi^{med} \rangle} 1_{\Omega_1 \cup \Omega_3} + \frac{r}{\langle \delta \xi^{hi} \rangle \langle \delta \xi^{med} \rangle} 1_{\Omega_1 \cup \Omega_2}, \\ |R_{p,a,\xi_0}^4| &\lesssim \frac{r}{\langle \delta \xi^{med} \rangle} 1_{\Omega_1 \cup \Omega_3} + \frac{r^2}{\langle \delta \xi^{hi} \rangle \langle \delta \xi^{med} \rangle} 1_{\Omega_1 \cup \Omega_2}. \end{aligned} \tag{4.35}$$

- iii) *Regularity: Similar bounds hold for all derivatives.*

Proof. This is easily done by applying Lemma 4.1; see also Remark 4.2. □

5. Interaction Morawetz identities

5.1. The linear Schrodinger equation

The Interaction Morawetz inequality aims to capture the fact that the momentum moves to the right faster than the mass. Here, the left/right symmetry is broken due to the sign choice which is implicit in the choice of the momentum.

5.1.1. A global computation

To warm up, we start with two solutions u and v for the linear Schrödinger equation. To these we associate the Interaction functional

$$\mathbf{I}(u, v) = \int_{x>y} M(u)(x)P(v)(y) - P(u)(x)M(v)(y) \, dx dy$$

and compute $d\mathbf{I}/dt$ using the conservation laws (4.2). We have

$$\begin{aligned} \frac{d}{dt} \mathbf{I}(u, v) &= \int_{x>y} \partial_x P(u)(x)P(v)(y) + M(u)(x)\partial_y E(v)(y) \\ &\quad - \partial_x E(u)(x)M(v)(y) - P(u)(x)\partial_y P(v)(y) \, dx dy \\ &= \int M(u)E(v) + M(v)E(u) - 2P(u)P(v) \, dx := \int J^A(u, \bar{u}, v, \bar{v}) \, dx. \end{aligned}$$

Here, J^4 can be chosen² to have symbol

$$j^4(\xi_1, \xi_2, \xi_3, \xi_4) = 4(\xi_1 - \xi_4)(\xi_2 - \xi_3).$$

This is because of the following computation on the diagonal $\Delta^4\xi = 0$:

$$(\xi_1 + \xi_2)^2 + (\xi_3 + \xi_4)^2 - 2(\xi_1 + \xi_2)(\xi_3 + \xi_4) = (\xi_1 + \xi_2 - \xi_3 - \xi_4)^2 = 4(\xi_1 - \xi_4)(\xi_2 - \xi_3).$$

Thus, we have the positivity

$$J^4(u, \bar{u}, v, \bar{v}) = 4|\partial_x(u\bar{v})|^2.$$

The above computation is classically done using integration by parts; see [22]. However, it is more interesting to do it at the symbol level because we want to apply it in a more general context. Classically, this is done with $u = v$, but here we find it convenient to break the symmetry. Primarily, our v 's will be spatial translations of u .

5.1.2. A frequency localized bound

Here, we start with a symbol a which is localized on the unit scale near some frequency ξ_0 , and consider the Interaction Morawetz functional

$$\mathbf{I}_a(u, v) = \int_{x>y} M_a(u)(x)P_a(v)(y) - P_a(u)(x)M_a(v)(y) dx dy. \tag{5.1}$$

As above, its time derivative is

$$\frac{d}{dt}\mathbf{I}_a(u, v) = \mathbf{J}_a^4(u, \bar{u}, v, \bar{v}),$$

where J_a^4 has symbol

$$j_a(\xi_1, \xi_2, \xi_3, \xi_4) = 4a(\xi_1, \xi_2)\overline{a(\xi_3, \xi_4)}(\xi_1 - \xi_4)(\xi_2 - \xi_3).$$

This no longer has obvious positivity. However, if a has separated variables

$$a(\xi, \eta) = a_0(\xi)a_0(\eta), \tag{5.2}$$

then J_a is nonnegative,

$$\mathbf{J}_a^4(u, v) = 4 \int |K_a(u, \bar{v})|^2 dx,$$

where K_a has symbol $(\xi - \eta)a_0(\xi)a_0(\eta)$, that is,

$$K_a(u, \bar{v}) = \partial_x(A_0u\overline{A_0v}),$$

where A_0 is the multiplier associated to the symbol a_0 .

5.1.3. Interaction Morawetz for separated velocities

Here, we instead take two symbols a and b localized to two frequency intervals A and B so that $|A|, |B| \lesssim r$ and A and B have separation r (say A is to the left of B). Then we take the Interaction functional

$$\mathbf{I}_{AB} = \int_{x>y} M_A(u)(x)P_B(v)(y) - P_A(u)(x)M_B(v)(y) dx dy,$$

²Recall that a priori the symbol of j^4 is only determined uniquely on the diagonal $\Delta^4\xi = 0$.

or equivalently

$$\mathbf{I}_{AB} = \int_{x>y} M_A(x)P_{B,\xi_0}(y) - P_{A,\xi_0}(x)M_B(y) \, dx dy,$$

where ξ_0 is arbitrary but can be chosen more efficiently at distance $O(r)$ from both A and B .

Then we compute

$$\frac{d}{dt}\mathbf{I}_{AB} = \int M_A(u)E_B(v) + E_A(u)M_B(v)(x) - 2P_B(u)P_A(u) \, dx := \mathbf{J}_{AB}^4(u, \bar{u}, v, \bar{v}),$$

where \mathbf{J}_{AB}^4 has symbol

$$J_{AB}^4(\xi_1, \xi_2, \xi_3, \xi_4) = 2a(\xi_1, \xi_2)b(\xi_3, \xi_4)(\xi_1 - \xi_4)(\xi_2 - \xi_3).$$

Assuming that

$$a(\xi, \eta) = a_0(\xi)a_0(\eta), \quad b(\xi, \eta) = b_0(\xi)b_0(\eta),$$

we can write \mathbf{J}_{AB}^4 as

$$\mathbf{J}_{AB}^4 = \int |K_{AB}(u, \bar{v})|^2 \, dx,$$

where

$$K_{AB}(u, \bar{u}) = \partial_x(A_0u \overline{B_0v}).$$

Now, the differences $(\xi_1 - \xi_4)$ and $(\xi_2 - \xi_3)$ have size r so this leads to a bilinear L^2 bound for $A_0u \cdot \overline{B_0u}$,

$$\mathbf{J}_{AB} \approx r^2 \|A_0u \cdot \overline{B_0u}\|_{L^2}^2.$$

5.2. Nonlinear Interaction Morawetz estimates

Here, we consider the same Interaction Morawetz functional as above but now apply it to (two) solutions for the nonlinear equation (1.3).

5.2.1. A simple case

As a starting point, here we consider density-flux pairs as in equations (4.6), (4.8) to which we associate the nonlinear Interaction functional

$$\mathbf{I}(u, v) = \iint_{x>y} M^\sharp(u)(x)P^\sharp(v)(y) - P^\sharp(u)(x)M^\sharp(v)(y) \, dx dy. \tag{5.3}$$

Using the density-flux relations, we obtain

$$\frac{d\mathbf{I}}{dt} = \mathbf{J}^4 + \mathbf{J}^6 + \mathbf{J}^8 + \mathbf{K}^8, \tag{5.4}$$

where \mathbf{J}^4 is the same as above, while \mathbf{J}^6 and \mathbf{J}^8 are given by

$$\begin{aligned} \mathbf{J}^6(u, v) = \int & M(u)R_p^4(v) + B_m^4(u)E(v) - P(u)B_p^4(v) - R_m^4(u)P(v) + \\ & M(v)R_p^4(u) + B_m^4(v)E(u) - P(v)B_p^4(u) - R_m^4(v)P(u) \, dx, \end{aligned} \tag{5.5}$$

respectively

$$\mathbf{J}^8(u, v) = \int B_m^4(u)R_p^4(v) - R_m^4(u)B_p^4(v) + B_m^4(v)R_p^4(u) - R_m^4(v)B_p^4(u) dx. \tag{5.6}$$

Finally, we are also left with the double integral

$$\begin{aligned} \mathbf{K}^8 &= \iint_{x>y} M^\sharp(u)(x)R_p^6(v)(y) + P^\sharp(v)(y)R_m^6(u)(x) dx dy \\ &\quad - \iint_{x>y} M^\sharp(v)(y)R_p^6(u)(x) + P^\sharp(u)(x)R_m^6(v)(y) dx dy, \end{aligned} \tag{5.7}$$

whose leading part has order 8 but also contains terms of order 10, but we will treat it all perturbatively later.

It is instructive to consider the case of the cubic defocusing NLS. There $B_m^4 = 0, B_p^4 = 0$ and thus $R_m^6 = 0, R_p^6 = 0$. Further, $R_m^4 = 0$ but $R_p^4 = 1$. Thus, in particular we get

$$\mathbf{J}^6(u, u) = \int |u|^6 dx.$$

This is where the focusing/defocusing type of the equation comes in, as it determines the sign of \mathbf{J}^6 (relative to the sign of \mathbf{J}^4).

5.2.2. Nonlinear Interaction Morawetz: the localized diagonal case

Here, we use the frequency localized mass density-flux identity (4.17) and the corresponding momentum density-flux identity (4.18) in order to produce a localized Interaction Morawetz estimate. We consider a smooth symbol a as in equation (5.2), where a_0 is localized around a frequency ξ_0 on the unit scale.

Correspondingly, we have the localized mass and momentum densities

$$M_a^\sharp = M_a(u, \bar{u}) + B_{m,a}^4(u, \bar{u}, u, \bar{u}),$$

$$P_{a,\xi_0}^\sharp = P_{a,\xi_0}(u, \bar{u}) + B_{p,a,\xi_0}^4(u, \bar{u}, u, \bar{u}),$$

which satisfy the conservation laws

$$(\partial_t + 2\xi_0\partial_x)M_a^\sharp(u) = \partial_x(P_{a,\xi_0}(u) + R_{m,a,\xi_0}^4(u)) + R_{m,a,\xi_0}^6(u).$$

$$(\partial_t + 2\xi_0\partial_x)P_{a,\xi_0}^\sharp(u) = \partial_x(E_{a,\xi_0}(u) + R_{p,a,\xi_0}^4(u)) + R_{p,a,\xi_0}^6(u).$$

For these, we define the Interaction Morawetz functional

$$\mathbf{I}_a(u, v) = \iint_{x>y} M_a^\sharp(u)(x)P_{a,\xi_0}^\sharp(v)(y) - P_{a,\xi_0}^\sharp(u)(x)M_a^\sharp(v)(y) dx dy, \tag{5.8}$$

where, by writing it in a symmetric fashion, we have completely eliminated its dependence on ξ_0 .

The time derivative of \mathbf{I}_a is

$$\frac{d}{dt}\mathbf{I}_a = \mathbf{J}_a^4 + \mathbf{J}_a^6 + \mathbf{J}_a^8 + \mathbf{K}_a^8, \tag{5.9}$$

where all the terms are independent of ξ_0 .

Here, the quartic contribution \mathbf{J}_a^4 is the same as in the linear case,

$$\mathbf{J}_a^4(u, v) = \int M_a(u)E_{a,\xi_0}(v) + M_a(v)E_{a,\xi_0}(u) - 2P_{a,\xi_0}(u)P_{a,\xi_0}(v) dx.$$

The sixth-order term \mathbf{J}_a^6 has the form

$$\begin{aligned} \mathbf{J}_a^6(u, v) = & \int M_a(u)R_{p,a,\xi_0}^4(v) + B_{m,a}^4(u)E_{a,\xi_0}(v) - P_{a,\xi_0}(u)B_{p,a,\xi_0}^4(v) - R_{m,a,\xi_0}^4(u)P_{a,\xi_0}(v) \\ & + M_a(v)R_{p,a,\xi_0}^4(u) + B_{m,a}^4(v)E_{a,\xi_0}(u) - P_{a,\xi_0}(v)B_{p,a,\xi_0}^4(u) - R_{m,a,\xi_0}^4(v)P_{a,\xi_0}(u) dx. \end{aligned} \tag{5.10}$$

Next, we have

$$\begin{aligned} \mathbf{J}_a^8(u, v) = & \tag{5.11} \\ & \int B_{m,a}^4(u)R_{p,a,\xi_0}^4(v) - R_{m,a,\xi_0}^4(u)B_{p,a,\xi_0}^4(v) + B_{m,a}^4(v)R_{p,a,\xi_0}^4(u) - R_{m,a,\xi_0}^4(v)B_{p,a,\xi_0}^4(u) dx. \end{aligned}$$

Finally, the 8-linear term \mathbf{K}_a^8 has the form

$$\begin{aligned} \mathbf{K}_a^8(u, v) = & \iint_{x>y} M_a^\#(u)(x)R_{p,a,\xi_0}^6(v)(y) + P_{a,\xi_0}^\#(v)(y)R_{m,a}^6(u)(x) \\ & - M_a^\#(v)(y)R_{p,a,\xi_0}^6(u)(x) - P_{a,\xi_0}^\#(u)(x)R_{m,a}^6(v)(y) dx dy. \end{aligned} \tag{5.12}$$

This also includes a 10-linear term.

Importantly, here we compute the symbol of \mathbf{J}_{a,ξ_0}^6 on the diagonal $\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 := \xi$. This will be essential later on in order to obtain bounds for the L^6 Strichartz norm.

Lemma 5.1. *The diagonal trace of the symbol j_a^6 is*

$$j_a^6(\xi) = a^2(\xi)c(\xi, \xi, \xi). \tag{5.13}$$

Proof. Since our symbol does not actually depend on ξ_0 , it suffices to compute it at $\xi = \xi_0$. The advantage is that $p_{a,\xi_0}(\xi_0) = e_{a,\xi_0}(\xi_0) = 0$, so we are left with the simpler expression

$$j_a^6(\xi_0) = m_a(\xi_0)r_{p,a,\xi_0}^4(\xi_0).$$

For r_{p,a,ξ_0}^4 , we have the relation

$$c_{p,a,\xi_0}^4 + i\Delta^4(\xi - \xi_0)^2 b_{p,a,\xi_0}^4 = i\Delta^4 \xi r_{p,a,\xi_0}^4.$$

We differentiate with respect to ξ_1 and then set all ξ 's equal to obtain

$$\partial_1 c_{p,a,\xi_0}^4(\xi_0) = i r_{p,a,\xi_0}^4(\xi_0).$$

It remains to compute the ξ_1 derivative of c_{p,a,ξ_0}^4 on the diagonal. This is a direct computation using the formula (4.30). Recalling that

$$p_a(\xi_1, \xi_2) = m_a(\xi_1, \xi_2)(-\xi_1 - \xi_2 + 2\xi_0),$$

it follows that

$$r_{p,a,\xi_0}^4(\xi_0) = m_a(\xi_0)c(\xi_0),$$

where we recall that c is real on the diagonal. Therefore,

$$j_{a,\xi_0}^6(\xi_0) = a^2(\xi_0)c(\xi_0),$$

as needed. □

5.2.3. Nonlinear Interaction Morawetz: the transversal case

Here, we return to the setting of Section 5.1.3 where we have two frequency intervals A, B with size at most M and separation also M and two smooth and bounded symbols a, b which are localized in the two intervals. Our Interaction Morawetz functional is given by

$$\mathbf{I}_{AB} = \int_{x>y} M_a^\#(u)(x)P_{b,\xi_0}^\#(v)(y) - P_{a,\xi_0}^\#(u)(x)M_b^\#(v)(y) \, dx dy, \tag{5.14}$$

and we observe as before that this does not depend on ξ_0 .

Using again the frequency localized mass density-flux identity (4.17) and the corresponding momentum density-flux identity (4.18) we produce a localized Interaction Morawetz estimate,

$$\frac{d}{dt} \mathbf{I}_{AB} = \mathbf{J}_{AB}^4 + \mathbf{J}_{AB}^6 + \mathbf{J}_{AB}^8 + \mathbf{K}_{AB}^8. \tag{5.15}$$

Here, the quartic contribution \mathbf{J}_{AB}^4 is the same as in the linear case

$$\mathbf{J}_{AB}^4 = \int M_a(u)(x)E_{b,\xi_0}(v)(x) + M_b(v)(x)E_{a,\xi_0}(u)(x) - 2P_{a,\xi_0}(u)(x)P_{b,\xi_0}(v)(x) \, dx$$

and captures the bilinear L^2 bound.

The sixth-order term \mathbf{J}_{AB}^6 has the form

$$\mathbf{J}_{AB}^6 = \int -(P_{a,\xi_0}B_{p,b,\xi_0}^4 + P_{b,\xi_0}R_{m,a,\xi_0}^4) + (M_aR_{p,b,\xi_0}^4 + E_{b,\xi_0}B_{m,a,\xi_0}^4) - \text{symmetric} \, dx,$$

where in the symmetric part we interchange both the indices a, b and the functions u, v .

Next, we have

$$\mathbf{J}_{AB}^8 = \int -R_{m,a,\xi_0}^4 B_{p,b,\xi_0}^4 + B_{m,a,\xi_0}^4 R_{p,b,\xi_0}^4 - \text{symmetric} \, dx.$$

Finally, the 8-linear term \mathbf{K}_{a,ξ_0}^8 has the form

$$\mathbf{K}_{AB}^8 = \iint_{x>y} M_a^\#(x)R_{p,b,\xi_0}^6 + P_{b,\xi_0}^\# R_{m,a,\xi_0}^6 - \text{symmetric} \, dx dy.$$

As before, this also includes a 10-linear term.

6. Frequency envelopes and the bootstrap argument

The primary goal of the proof of our main result in Theorem 1 is to establish a global $L_t^\infty L_x^2$ bound for small data solutions; by the local well-posedness result in Section 3, this implies the desired global well-posedness result. However, along the way, we will also establish bilinear L^2 and Strichartz bounds for the solutions. These will both play an essential role in the proof of Theorem 1 and will also establish the scattering properties of our global solutions.

Since the proof of our estimates loops back in a complex manner, it is most convenient to establish the bilinear L^2 and the L^6 Strichartz bounds in the setting of a bootstrap argument, where we already assume that the desired bilinear and Strichartz estimates hold but with weaker constants.

The setup for the bootstrap is most conveniently described using the language of frequency envelopes. This was originally introduced in work of Tao; see, for example, [24], but in the context of dyadic Littlewood–Paley decompositions. But here instead we work with a uniform decomposition on the unit scale, which requires a substantial revision of the notion of ‘slowly varying’, which we replace by the new notion of ‘maximal property’ introduced in Section 2.2.

To start with, we assume that the initial data is small size,

$$\|u_0\|_{L^2} \lesssim \epsilon.$$

We consider a frequency decomposition for the initial data on a unit spatial scale,

$$u_0 = \sum_{k \in \mathbb{Z}} u_{0,k}.$$

Then we place the initial data components under an admissible frequency envelope on the unit scale,

$$\|u_{0,k}\|_{L^2} \leq \epsilon c_k, \quad c \in \ell^2,$$

where the envelope $\{c_k\}$ is not too large,

$$\|c\|_{\ell^2} \approx 1.$$

Our goal will be to establish the following frequency envelope bounds for the solution:

Theorem 5. *Let $u \in C([0, T]; L^2)$ be a solution for the equation (1.3) with initial data u_0 which has L^2 size at most ϵ . Let $\{\epsilon c_k\}$ be an admissible frequency envelope for the initial data in L^2 , with c_k normalized in ℓ^2 . Then the solution u satisfies the following bounds:*

(i) *Uniform frequency envelope bound:*

$$\|u_k\|_{L_t^\infty L_x^2} \lesssim \epsilon c_k, \tag{6.1}$$

(ii) *Localized Strichartz bound:*

$$\|u_k\|_{L_{t,x}^6} \lesssim (\epsilon c_k)^{\frac{2}{3}}, \tag{6.2}$$

(iii) *Localized Interaction Morawetz:*

$$\|\partial_x |u_k|^2\|_{L_{t,x}^2} \lesssim \epsilon^2 c_k^2, \tag{6.3}$$

(iv) *Transversal bilinear L^2 bound:*

$$\|\partial_x (u_A \bar{u}_B(\cdot + x_0))\|_{L_{t,x}^2} \lesssim \epsilon^2 c_A c_B \langle \text{dist}(A, B) \rangle^{\frac{1}{2}}, \tag{6.4}$$

for all $x_0 \in \mathbb{R}$ whenever $|A| + |B| \lesssim \langle \text{dist}(A, B) \rangle$.

Here, equation (6.3) can be seen as a particular case of equation (6.4) when $A = B$ have unit length; we stated it separately in order to ease comparison with earlier work on Interaction Morawetz estimates.

To prove this theorem, we make a bootstrap assumption where we assume the same bounds but with a worse constant C , as follows:

(i) Uniform frequency envelope bound,

$$\|u_k\|_{L_t^\infty L_x^2} \lesssim C\epsilon c_k, \tag{6.5}$$

(ii) Localized Strichartz bound,

$$\|u_k\|_{L_{t,x}^6} \lesssim C(\epsilon c_k)^{\frac{2}{3}}, \tag{6.6}$$

(iii) Localized Interaction Morawetz,

$$\|\partial_x |u_k|^2\|_{L_{t,x}^2} \lesssim C\epsilon^2 c_k^2, \tag{6.7}$$

(iv) Transversal Interaction Morawetz,

$$\|\partial_x(u_{k_1} \overline{u_{k_2}(\cdot + x_0)})\|_{L_{t,x}^2} \lesssim C\epsilon^2 c_{k_1} c_{k_2} \langle k_1 - k_2 \rangle^{\frac{1}{2}} \tag{6.8}$$

uniformly for all $x_0 \in \mathbb{R}$.

Then we seek to improve the constant in these bounds. The gain will come from the fact that the C 's will always come paired with extra ϵ s.

We remark that the bootstrap hypothesis for the transversal bilinear L^2 bound (6.8) only requires unit size localization, unlike the corresponding conclusion (6.4). On one hand, this simplifies the continuity argument closing the bootstrap. On another hand, this is related to the fact that closing the bootstrap argument for global well-posedness only requires equation (6.4) for unit size sets. The full bound (6.4) is only used in the last section in order to obtain the global Strichartz and bilinear L^2 bounds, which are of course very interesting but secondary to the proof of the global result.

We also remark on the need to add translations to the bilinear L^2 estimates. This is because, unlike the linear bounds (6.5) and (6.6) which are inherently invariant with respect to translations, bilinear estimates are not invariant with respect to separate translations for the two factors. One immediate corollary of equation (6.8) is that for any multipliers L_1 and L_2 with smooth and bounded symbols we have

$$\|\partial_x(L_1(D)u_{k_1} \overline{L_2(D)u_{k_2}(\cdot + x_0)})\|_{L^2} \lesssim C\epsilon^2 c_{k_1} c_{k_2} \langle k_1 - k_2 \rangle^{\frac{1}{2}}. \tag{6.9}$$

This is essentially the only way we will use this translation invariance in our proofs.

For the rest of this section, we provide the continuity argument which shows that it suffices to prove Theorem 5 under the bootstrap assumptions (6.5)–(6.8).

For this, we denote by T the maximal time for which the bounds (6.5)–(6.8) hold in $[0, T]$. By the local well-posedness result, we have $T \geq 1$. Assume by contradiction that T is finite. Then the bootstrap version of the theorem implies that the bounds (6.1)–(6.4) hold in $[0, T]$. In particular, $u(T)$ will also be controlled by the same maximal envelope c_j coming from the initial data. By the local well-posedness result, this implies in turn that the bounds (6.5)–(6.8) hold in $[T, T + 1]$ with $C \approx 1$. Adding this to the bounds (6.1)–(6.4) in $[0, T]$, it follows that (6.5)–(6.8) hold in $[0, T + 1]$, thereby contradicting the maximality of T .

7. The frequency envelope bounds

The aim of this section is to prove the frequency envelope bounds in Theorem 5, given the bootstrap assumptions (6.5)–(6.8). In the proof, we will rely on our modified energy and momentum functionals, whose components we estimate first. The frequency localized energy estimate (6.1) will be an immediate

consequence of these bounds. For the Strichartz and L^2 bilinear bounds, we will then use the Interaction Morawetz identities, first in a localized diagonal setting and then in a transversal setting.

7.1. Spatial and space-time L^1 bounds

Here, we consider the corrections $B_{m,a}^4$ and errors $R_{m,a}^6$ and their momentum counterparts associated to a smooth bump function a selecting a frequency interval $A \subset \mathbb{Z}$. For $B_{m,a}^4$, we will prove a fixed time L^1 bound, while for $R_{m,a}^6$ we will prove a space-time L^1 bound. These bounds will be repeatedly used in each of the following subsections, first in the case when A has unit size and then in the case when A has a larger size. We begin with the $B_{m,a}^4$ bound.

Lemma 7.1. *Assume that the bootstrap bound (6.5) holds. Then we have the fixed time estimate*

$$\|B_{m,a}^4(u)\|_{L_x^1} \lesssim \epsilon^4 C^4 c_A^2. \tag{7.1}$$

The corresponding bound for the momentum follows as a corollary once we add an additional assumption in order to fix the momentum size:

Corollary 7.2. *Assume that the bootstrap bound (6.5) holds. Let $\xi_0 \in \mathbb{R}$, and*

$$n = \max_{k \in A} |k - \xi_0|.$$

Then we have the fixed time estimate

$$\|B_{p,a,\xi_0}^4(u)\|_{L_x^1} \lesssim n \epsilon^4 C^4 c_A^2, \tag{7.2}$$

Proof. The bounds (7.1) and (7.2) are similar, the only difference arises from the additional n factor in the size of the symbol p_A . So we will prove the first bound. Using our partition of unity in frequency on the unit scale, we expand

$$B_{m,a}^4(u) = \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}} B_{m,A}^4(u_{k_1}, \bar{u}_{k_2}, u_{k_3}, \bar{u}_{k_4}).$$

Here, we will separately estimate each term in L_x^1 based on the size of the symbol. By Proposition 4.3, for frequencies within a unit neighbourhood of $[k] = (k_1, k_2, k_3, k_4)$ the symbol $b_{m,a}^4$ and its derivatives can be estimated by

$$b_{m,a}^4[k] := \frac{1}{\langle \delta k^{hi} \rangle \langle \delta k^{med} \rangle}.$$

In addition, its support is contained in the region $\Omega_1 \cup \Omega_2$, where at least one frequency is in A . The region $\Omega_1 \cup \Omega_2$ can be described as the set of those quadruples $[k]$ so that

$$\text{either } |\Delta^4 k| \lesssim 1, \text{ or } |\Delta^4 k| \ll \delta k^{med}. \tag{7.3}$$

Without loss in generality, we assume that $k_1 \in A$. Then, using the above properties, we can estimate the L_x^1 bound in the lemma as

$$\|B_{m,a}^4(u)\|_{L_x^1} \lesssim \epsilon^4 \sum_{k_1 \in A} \sum_{[k] \in \Omega_1 \cup \Omega_2} \frac{1}{\langle \delta k^{hi} \rangle \langle \delta k^{med} \rangle} c_{k_1} c_{k_2} c_{k_3} c_{k_4}.$$

Fixing $k_1 \in A$, it suffices to show that

$$S_{k_1} := \sum_{k_2, k_3, k_4} \frac{1}{\langle \delta k^{hi} \rangle \langle \delta k^{med} \rangle} c_{k_2} c_{k_3} c_{k_4} \lesssim c_{k_1}. \tag{7.4}$$

This no longer has anything to do with the set A . For later use we have also removed the restriction $[k] \in \Omega_1 \cup \Omega_2$.

To discuss the possible configurations for $[k]$, we denote by $n_1 \leq n_2$ the dyadic size of δk^{med} , respectively δk^{hi} . By Galilean invariance, we set $k_1 = 0$, and then the rest of the indices may be reordered so that

$$|k_2| \lesssim n_1, \quad |k_3| \lesssim n_2, \quad |k_4| \approx n_2, \quad |k_3 - k_4| \lesssim n_1.$$

Then we have

$$S \lesssim \sum_{n_1 \leq n_2} \sum_{|k_2| \lesssim n_1} \sum_{|k_4| \approx n_2} \sum_{|k_3 - k_4| \lesssim n_1} \frac{1}{n_1 n_2} c_{k_2} c_{k_3} c_{k_4}.$$

Here, we use twice the envelope maximal function bound to write

$$\frac{1}{n_1} \sum_{|k_2| \lesssim n_1} c_{k_2} \lesssim c_0, \quad \frac{1}{n_1} \sum_{|k_3 - k_4| \lesssim n_1} c_{k_3} \lesssim c_{k_4}.$$

This gives

$$S \lesssim c_0 \sum_{n_1 \leq n_2} \frac{n_1}{n_2} \sum_{|k_4| \approx n_2} c_{k_4}^2 \approx c_0 \sum_{n_2} \sum_{|k_4| \approx n_2} c_{k_4}^2 \lesssim c_0.$$

This concludes the proof of equation (7.4) and therefore the proof of the lemma. □

Next, we turn our attention to $R_{m,a}^6$, which we estimate as follows:

Lemma 7.3. *Under our bootstrap assumptions (6.5)–(6.8), we have the space-time bound*

$$\|R_{m,a}^6\|_{L_{t,x}^1} \lesssim \epsilon^4 C^6 c_A^2. \tag{7.5}$$

As above, we also have a similar bound for the momentum:

Corollary 7.4. *Assume that the bootstrap bounds (6.5)–(6.8) hold. Let $\xi_0 \in \mathbb{R}$, and*

$$n = \max_{k \in A} |k - \xi_0|.$$

Then we have the space-time bound

$$\|R_{p,a,\xi_0}^6\|_{L_{t,x}^1} \lesssim n \epsilon^4 C^6 c_A^2. \tag{7.6}$$

Proof. As in the case of the earlier fixed time bound, we will focus on equation (7.5), as the proof of equation (7.6) is essentially the same. We recall $R_{m,a}^6$ is obtained from the cubic terms in the time derivative of $B_{m,a}^4$. We denote the four frequencies in $B_{m,a}^4$ by k_0, k_1, k_2, k_3 , where the k_0 factor gets differentiated in time. One of these four frequencies, call it k_A , must be in A .

With k_A as above, we expand

$$R_{m,a}^6(u) = \sum_{k_A \in A} R_{m,a,k_A}^6(u).$$

Then it suffices to establish the bound

$$\|R_{m,a,k_A}^6(u)\|_{L_{t,x}^1} \lesssim \epsilon^4 C^6 c_{k_A}^2. \tag{7.7}$$

Here, by Galilean invariance we can set $k_A = 0$. We also drop the index A , as no localization associated to the set A will be used in the sequel. In particular, we replace $B_{m,a}^4$ by $B_{m,0}^4$ to emphasize that one of the frequencies in B^4 is assumed to be near zero.

To describe the size and localization of the symbol $b_{m,0}^4$, we introduce as before the notations δk^{med} and δk^{hi} for the distances between $k_0, k_1, k_2, k_3, n_1 < n_2$ for the dyadic size of δk^{med} and δk^{hi} , and $\Delta^4 k$ associated to the same indices. In the support of $b_{m,0}^4$, we must have

$$|\Delta^4 k| \lesssim 1 \quad \text{or} \quad |\Delta^4 k| \ll n_1, \quad 0 \in \{k_0, k_1, k_2, k_3\}. \tag{7.8}$$

In this region, the symbol of $b_{m,0}^4$ as well as its derivatives have size

$$|b_{m,0}^4| \lesssim \frac{1}{n_1 n_2}. \tag{7.9}$$

The time differentiation is producing three additional frequencies k_4, k_5, k_6 so that

$$k_0 = k_4 - k_5 + k_6. \tag{7.10}$$

Then equation (7.8) translates to

$$|\Delta^6 k| \lesssim 1 \quad \text{or} \quad |\Delta^6 k| \ll n_1 \tag{7.11}$$

relative to the indices k_1, \dots, k_6 .

Overall, for $R_{m,0}^6$ we have the decomposition

$$R_{m,0}^6(u) = \sum_{n_1 \leq n_2} \sum_{k_{0-7} \in \Gamma} R_{m,0}^6(u_{k_1}, \bar{u}_{k_2}, u_{k_3}, \bar{u}_{k_4}, u_{k_5}, \bar{u}_{k_6}),$$

where Γ describes the set of indices satisfying equations (7.8) and (7.10). To bound this sum in $L_{t,x}^1$, we consider several cases:

A. If all six frequencies are near 0, then we use the localized L^6 bound to obtain

$$\|R_{m,0}^6(u_{k_1}, \bar{u}_{k_2}, u_{k_3}, \bar{u}_{k_4}, u_{k_5}, \bar{u}_{k_6})\|_{L_{t,x}^1} \lesssim C^4 (\epsilon_0)^4,$$

which suffices.

B. Otherwise, we denote by $1 \ll n$ the minimal dyadic size of the interval containing all six k indices. Clearly, we have $n_1 \leq n_2 \leq n$. Also, due to equation (7.11) we must also have $|\Delta^6 k| \ll n$. This implies that within the set (k_1, \dots, k_6) there must be at least two disjoint pairs of frequencies at distance comparable to n . Applying two bilinear L^2 estimates and L^∞ bounds for the other two factors, we can bound

$$\|R_{m,0}^6(u)\|_{L_{t,x}^1} \lesssim \epsilon^6 C^4 S, \quad S := \sum_{n_1 \leq n_2} \sum_{k_{0-6} \in \Gamma} \frac{1}{n_1 n_2 n} c_{k_1} c_{k_2} c_{k_3} c_{k_4} c_{k_5} c_{k_6}. \tag{7.12}$$

It remains to bound the above sum S by

$$S \lesssim c_0^2. \tag{7.13}$$

There are several cases to consider:

B1. $k_0 = 0$. Relabeling, we may assume that

$$|k_1| \approx n_1, \quad |k_2 - k_3| \approx n_1, \quad |k_2|, |k_3| \approx n_2. \tag{7.14}$$

We distinguish further cases by comparing n_2 and n .

B1a. $n \gg n_2$. Then we may assume that

$$|k_4| \lesssim |k_5| \approx |k_6| \approx n. \tag{7.15}$$

For fixed k_4 , we can apply the Cauchy–Schwarz inequality for the pair of indices (k_5, k_6) , and also for (k_2, k_3) . We obtain

$$S \lesssim \sum_{n_1 \leq n_2 \ll n} \sum_{|k_1| \approx n_1, |k_4| \leq n} \frac{1}{n_2 n} c_{k_1} c_{k_4} c_{n_2}^2 c_n^2 = \sum_{n_2 \ll n} \sum_{|k_1| \leq n_2, |k_4| \leq n} \frac{1}{n_2 n} c_{k_1} c_{k_4} c_{n_2}^2 c_n^2.$$

Now, we use twice the envelope maximal bound for the k_1 , respectively k_4 summation to get

$$S \lesssim c_0^2 \sum_{n_2 \ll n} c_{n_2}^2 c_n^2 \approx c_0^2.$$

B1b. $k_0 = 0, n \approx n_2$. In this case, we can introduce another dyadic parameter $n_3 \leq n$ so that, after relabeling,

$$|k_4| \leq |k_5| \approx |k_6| \approx n_3. \tag{7.16}$$

Then applying Cauchy–Schwarz inequality exactly as above we arrive at

$$S \lesssim \sum_{n_3 \leq n_2} \sum_{|k_1| \leq n_2, |k_4| < n_3} \frac{1}{n_2^2} c_{k_1} c_{k_4} c_{n_2}^2 c_{n_3}^2,$$

where we can conclude again by applying twice the envelope maximal bound for the k_1 , respectively the k_4 summation relative to 0.

B2. $k_1 = 0, |k_0| \approx n_1$. In this case, we must also have

$$|k_2 - k_3| \approx n_1, \quad |k_2|, |k_3| \approx n_2. \tag{7.17}$$

Again, we compare n and n_2 :

B2a. $n \gg n_2$. Here, we can assume again that equation (7.15) holds. As in case B1a, we apply the Cauchy–Schwarz inequality for the pair of indices (k_5, k_6) and also for (k_3, k_2) , with the difference that now the difference $k_5 - k_6$ is no longer fixed, instead it varies in an n_1 range. Thus, we lose two n_1 factors, obtaining

$$S \lesssim c_0 \sum_{n_1 \leq n_2 \ll n} \sum_{|k_4| < n} \frac{n_1}{n_2 n} c_{k_4} c_{n_2}^2 c_n^2.$$

The n_1 summation is trivial now, and for the k_4 summation we use the envelope maximal bound to obtain

$$S \lesssim c_0^2 \sum_{n_2 \ll n} c_{n_2}^2 c_n^2 \approx c_0^2.$$

B2b. $n \approx n_2$. Here, we take two subcases.

B2b(i). If

$$|k_4| + |k_5| + |k_6| \lesssim n_1,$$

then we use the Cauchy–Schwarz inequality for the pair (k_2, k_3) to obtain

$$S \lesssim c_0 \sum_{n_1 \leq n_2} \frac{1}{n_2^2} c_{n_2}^2 \sum_{|k_4|+|k_5|+|k_6| \leq n_1} c_{k_4} c_{k_5} c_{k_6}.$$

Finally, we use the envelope maximal bound for the k_4 summation relative to 0 and for k_5 relative to k_6 to get

$$S \lesssim c_0^2 \sum_{n_1 \leq n_2} \frac{n_1^2}{n_2^2} c_{n_2}^2 c_{\leq n_1}^2 \lesssim c_0^2 \sum_{n_2} c_{n_2}^2 c_{\leq n_2}^2 \lesssim c_0^2,$$

which suffices.

B2b(ii). If instead

$$|k_4| + |k_5| + |k_6| \gg n_1,$$

then we can introduce n_3 as in equation (7.16), with $n_1 \ll n_3 \leq n_2$. Applying the Cauchy–Schwarz inequality for the pair of indices (k_5, k_6) , and (k_2, k_3) yields

$$S \lesssim c_0 \sum_{n_1 \ll n_3 \leq n_2} \sum_{|k_4| < n_3} \frac{n_1}{n_2^2} c_{k_4} c_{n_2}^2 c_{n_3}^2 \approx c_0 \sum_{n_3 \leq n_2} \sum_{|k_4| < n_3} \frac{n_3}{n_2^2} c_{k_4} c_{n_2}^2 c_{n_3}^2.$$

At this stage, we complete the argument by using the envelope maximal bound for the k_4 summation.

B3. $k_1 = 0, |k_0| \approx n_2 \gg n_1$. In this case, we may assume that

$$|k_2| \approx n_1, \quad |k_3| \approx n_2, \quad |k_0 - k_3| \approx n_1. \tag{7.18}$$

Next, we compare n_2 and n :

B3a. $n_2 \ll n$. Retaining k_0 as a summation index, we first apply the Cauchy–Schwarz inequality for the pair (k_5, k_6) to obtain

$$\begin{aligned} S &\lesssim c_0 \sum_{n_1 \ll n_2 \ll n} \sum_{|k_0| \approx n_2} \sum_{|k_2| \approx n_1} \sum_{|k_0 - k_3| \approx n_1} \sum_{|k_4| \leq n} \frac{1}{n_1 n_2 n} c_{k_2} c_{k_3} c_{k_4} c_n^2 \\ &\lesssim c_0 \sum_{n_1 \ll n_2 \ll n} \sum_{|k_3| \approx n_2} \sum_{|k_2| \approx n_1} \sum_{|k_4| \leq n} \frac{1}{n_2 n} c_{k_2} c_{k_3} c_{k_4} c_n^2 \\ &= c_0 \sum_{n_2 \ll n} \sum_{|k_3| \approx n_2} \sum_{|k_2| \ll n_2} \sum_{|k_4| \leq n} \frac{1}{n_2 n} c_{k_2} c_{k_3} c_{k_4} c_n^2. \end{aligned}$$

Now, we use the envelope maximal bound for k_4 relative to 0 and for k_2 relative to k_3 . This yields

$$S \lesssim c_0^2 \sum_{n_2 \ll n} \sum_{|k_3| \approx n_2} c_{k_3}^2 c_n^2 \approx c_0^2.$$

B3b. $n = n_2$. In this case, we dispense with k_0 as a summation index, retaining instead the relation

$$|k_3 - k_4 + k_5 - k_6| \lesssim n_1.$$

At least one of the frequencies k_4, k_5, k_6 must have size n_2 , say $|k_6| \approx n_2$. Then we use the Cauchy–Schwarz inequality for the (k_3, k_6) pair, losing an n_1 factor due to the relation above, and arriving at

$$S \lesssim c_0 \sum_{n_1 \leq n_2} \sum_{|k_2| \approx n_1} \sum_{|k_4|, |k_5| \leq n_2} \frac{1}{n_2^2} c_{k_2} c_{k_4} c_{k_5} c_{n_2}^2 \approx c_0 \sum_{|k_2|, |k_4|, |k_5| \leq n_2} \frac{1}{n_2^2} c_{k_2} c_{k_4} c_{k_5} c_{n_2}^2.$$

Finally, we use the envelope maximal bound for k_2 relative to 0 and for k_4 relative to k_5 to obtain

$$S \lesssim c_0^2 c_{\leq n_2}^2 c_{n_2}^2 \lesssim c_0^2.$$

This concludes the proof of the lemma.

B4. $k_2 = 0$. Here, we can assume that

$$|k_1| \approx n_1, \quad |k_3| \approx |n_2|,$$

but the size of k_0 is both not set and not needed. Instead, we will simply rely on equation (7.11) and consider two subcases.

B4a. $n_2 \ll n$, where we can assume that equation (7.15) holds. Here, we first use the maximal function for c_{k_1} to estimate

$$S \leq c_0^2 \sum_{n_1 \leq n_2 \ll n} \frac{1}{n_2 n} c_{k_3} c_{k_4} c_{k_5} c_{k_6},$$

where we retain the constraint relative to k_3, k_4, k_5, k_6 ,

$$|\Delta^4 k| \lesssim k_1.$$

Here, we can fix $\Delta^4 k$ at the expense of another n_1 factor. Then fixing k_3 and k_4 fixes the difference $k_5 - k_6$, so applying the Cauchy–Schwarz inequality with respect to k_5, k_6 we arrive at

$$S \leq c_0^2 \sum_{n_1 \leq n_2 \ll n} \frac{n_1}{n_2 n} c_{k_3} c_{k_4} c_n^2.$$

Finally, using Hölder’s inequality for k_3 and k_4 , which have size n_2 , respectively $\leq h$ yields

$$S \leq c_0^2 \sum_{n_1 \leq n_2 \ll n} \frac{n_1}{n_2 n} \sqrt{n_2 n} c_{n_2} c_{\leq n} c_n^2 \lesssim c_0^2 \sum_n c_{\leq n}^2 c_n^2 \lesssim c_0^2.$$

B4b. $n_2 \approx n$. Here, the case $n_1 \approx n_2$ is straightforward, as we can directly apply once the Cauchy–Schwarz inequality for two size n frequencies, twice Hölder’s inequality and once the maximal function bound for the three remaining frequencies of size $\leq n$. We are left with the more interesting case when $n_1 \ll n$. There, using again the maximal function for c_{k_1} we estimate

$$S \leq c_0^2 \sum_{n_1 \leq n_2 \ll n} \frac{1}{n^2} c_{k_3} c_{k_4} c_{k_5} c_{k_6},$$

where for the four remaining indices we have $|\Delta^4 k| \lesssim n_1 \ll n$. Here, $|k_3| \approx n$, so there must be at least one other frequency of size n . Then, as in the previous case, we apply once the Cauchy–Schwarz inequality for the two size n frequencies, and twice Hölder’s inequality for the two remaining frequencies of size $\lesssim n$. This concludes the proof of the lemma. □

7.2. The energy estimate

Our objective here is to prove the bound (6.1). We remark that once this is proved, we may drop the C^4 factor in Lemma 7.1. By the Galilean invariance, it suffices to prove the desired bound (6.1) at $k = 0$. For this, we consider a symbol $a(\xi_1, \xi_2)$ of the form

$$a(\xi_1, \xi_2) = a_0(\xi_1)a_0(\xi_2), \tag{7.19}$$

with a_0 localized near frequency 0 on the unit scale. Then

$$\mathbf{M}_a(u) = \|A_0(D)u\|_{L^2}^2,$$

and we need to bound this quantity uniformly in time,

$$\mathbf{M}_a(u) \lesssim c_0^2 \epsilon^2. \tag{7.20}$$

For this, we use the density-flux relation (4.26) with $\xi_0 = 0$, which yields

$$\frac{d}{dt} M_a^\sharp(u) = \partial_x(P_a(u) + R_{m,a}^4(u)) + R_{m,a}^6(u),$$

where

$$M_a^\sharp(u, \bar{u}) = M_a(u, \bar{u}) + B_{m,a}^4(u).$$

To prove equation (7.20), we integrate the above density-flux relation in t, x to obtain:

$$\int M_a(u) + B_{m,a}^4(u) dx \Big|_0^T = \int_0^T \int_{\mathbb{R}} R_{m,a}^6(u) dx dt. \tag{7.21}$$

Finally, we can estimate the contributions of $B_{m,a}^4$ and $R_{m,a}^6$ using Lemma 7.1, respectively Lemma 7.3.

Remark 7.5. For later use, we observe that once the energy bounds (6.1) have been established, then they can be used instead of the bootstrap assumption (6.5) in the proof of Lemma 7.1. This leads to a stronger form of equations (7.1), (7.2), with the constant C removed:

$$\|B_{m,A}^4(u)\|_{L_t^\infty L_x^1} + \|B_{p,A}^4(u)\|_{L_t^\infty L_x^1} \lesssim c_A^2 \epsilon^4. \tag{7.22}$$

7.3. The localized Interaction Morawetz

Our objective here is to prove the bounds (6.2) and (6.3) using our bootstrap assumptions. By the Galilean invariance, it suffices to do this at $k = 0$. This will be achieved using our Interaction Morawetz identity (5.9) with $v = u$ and with a localized at frequency 0, exactly as in equation (7.19). For such a , we can simply set $\xi_0 = 0$. It will suffice to estimate the quantities in equation (5.9) as follows:

$$|\mathbf{I}_a(u, u)| \lesssim \epsilon^4 c_0^4, \tag{7.23}$$

$$\mathbf{J}_a^4(u, u) \approx \|\partial_x |A_0(D)u|^2\|_{L_x^2}^2, \tag{7.24}$$

$$\int_0^T \mathbf{J}_a^6(u, u) dt \approx \|A_0(D)^{\frac{2}{3}}u\|_{L_{t,x}^6}^6 + O(\epsilon^5 C^6 c_0^4), \tag{7.25}$$

$$\int_0^T \mathbf{J}_a^8(u, u) dt = O(\epsilon^5 C^6 c_0^4), \tag{7.26}$$

$$\int_0^T \mathbf{K}_a^8(u, u) dt = O(\epsilon^5 C^8 c_0^4). \tag{7.27}$$

This allows us to estimate the localized Interaction Morawetz term, as well as the localized L^6 norm as in equations (6.2) and (6.3), provided that ϵ is small enough. There is nothing to do for \mathbf{J}_a^4 , so we consider the remaining contributions:

7.3.1. The \mathbf{I}_a bound

The Interaction Morawetz functional \mathbf{I}_a is as in equation (5.8), with $\xi_0 = 0$,

$$\mathbf{I}_a = \iint_{x>y} M_a^\sharp(u)(x)P_a^\sharp(v)(y) - P_a^\sharp(u)(x)M_a^\sharp(v)(y) dx dy \tag{7.28}$$

with

$$M_a^\sharp(u) = M_a(u) + B_{m,a}^4(u), \quad P_a^\sharp(u) = P_a(u) + B_{p,a}^4(u).$$

For $B_{m,a}^4$ and $B_{p,a}^4$, we have the $L_t^\infty L_x^1$ bound (7.22). For $M_a(u)$ and $P_a(u)$, we have the straightforward uniform in time bounds

$$\|M_a(u)\|_{L_t^\infty L_x^1} + \|P_a(u)\|_{L_t^\infty L_x^1} \lesssim \epsilon^2 c_a^2. \tag{7.29}$$

Combining this with equation (7.22), the estimate (7.23) immediately follows.

7.3.2. The \mathbf{J}_a^6 bound

This is a 6-linear expression whose expression we recall from equation (5.10),

$$\mathbf{J}_a^6 = 2 \int -(P_a B_{p,a}^4 + P_a R_{m,a}^4) + (M_a R_{p,a}^4 + E_a B_{m,a}^4) dx, \tag{7.30}$$

where again we have set $\xi_0 = 0$.

We first discuss the symbol localization properties for \mathbf{J}_a^6 with respect to the six entries at frequencies k_1, k_2, k_3, k_4, k_5 and k_6 . Here, we a priori have two frequencies close to 0, say $k_5 = k_6 = 0$, namely those arising from M_a, P_a and E_a , all of which have smooth and bounded symbols. In the symbols for B_a^4 and R_a^4 , on the other hand, we have at least one frequency equal to zero, say $k_1 = 0$, and the near-diagonal property $\Delta^4 k = 0$.

Next, we consider the size of the symbols, where we use Proposition 4.3. This gives the following symbol bounds regardless of the p or m index:

$$|b_a^4| \lesssim \frac{1}{\langle \delta k^{hi} \rangle \langle \delta k^{med} \rangle}, \quad |r_a^4| \lesssim \frac{1}{\langle \delta k^{med} \rangle}$$

and similarly for their derivatives. We split the analysis in two cases, depending on whether all frequencies are equal (i.e., $\delta k^{hi} \lesssim 1$) or not.

A. *The case of separated frequencies, $\delta k^{hi} \gg 1$.* To fix the notations, suppose that $|k_2| \approx \delta k^{med} \approx n_1$ and $|k_3| \approx |k_4| \approx \delta k^{hi} \approx n_2$, where $n_1 \leq n_2$ represent dyadic scales. Then we can apply two bilinear L^2 bounds (6.7) for the frequency pairs $(k_1 = 0, k_4)$ and (k_2, k_3) and simply estimate the k_5 and k_6 factors in L^∞ by Bernstein’s inequality. This yields the bound for the corresponding portion of \mathbf{J}_a^6

$$\left| \int_0^T \mathbf{J}_a^{6,unbal}(u) dt \right| \lesssim \epsilon^6 C^6 c_0^3 \sum_{k_2, k_3, k_4} \frac{1}{n_1 n_2} c_{k_2} c_{k_3} c_{k_4}.$$

Since $k_4 - k_3 = k_2$, for fixed k_2 we can apply the Cauchy–Schwartz inequality with respect to the k_3 and k_4 indices to obtain

$$\left| \int_0^T \mathbf{J}_a^{6,unbal}(u) dt \right| \lesssim \epsilon^6 C^6 c_0^3 \sum_{|k_2| \approx n_1 \leq n_2} \frac{1}{n_1 n_2} c_{k_2} c_{n_2}^2.$$

Finally, using the maximal function property for c_{k_2} we arrive at

$$\left| \int_0^T \mathbf{J}_a^{6,unbal}(u) dt \right| \lesssim \epsilon^6 C^6 c_0^4 \sum_{n_2} \frac{\log n_2}{n_2} c_{n_2}^2,$$

which suffices.

B. The case of equal frequencies, $\delta k^{hi} \lesssim 1$. Here, we have $|k_j| \lesssim 1$ for all j , and the symbol of j_a^6 is smooth and bounded. The important feature here is the symbol of the 6-linear form J_0^6 on the diagonal

$$\{\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6\},$$

which we would like to be positive. But we know this by equation (5.13), which shows that this equals

$$j_a^6(\xi) = a_0^4(\xi) c(\xi, \xi, \xi).$$

It follows that we can write the symbol j_a^6 in the form

$$j_a^6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = b_0(\xi_1) b(\xi_2) b(\xi_3) b(\xi_4) b(\xi_5) b(\xi_6) + j_a^{6,rem}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6),$$

where $b_0(\xi) = a_0(\xi)^{\frac{2}{3}} c(\xi, \xi, \xi)^{\frac{1}{6}}$ and $j_0^{6,rem}$ vanishes when all ξ 's are equal. Then we can write $j_0^{6,rem}$ as a linear combination of terms $\xi_{even} - \xi_{odd}$ with smooth coefficients. The first term yields the desired L^6 norm,

$$\mathbf{J}_a^6(u) = \|B_0(D)u\|_{L_x^6}^6 + \mathbf{J}_a^{6,rem}.$$

On the other hand the contribution $\mathbf{J}_a^{6,rem}$ of the second term be estimated using a bilinear L^2 bound (6.7), three L^6 bounds (6.6) and one L^∞ via Bernstein's inequality,

$$\left| \int_0^T \mathbf{J}_a^{6,rem}(u) dt \right| \lesssim \|J_a^{6,rem}(u)\|_{L_{t,x}^1} \lesssim (C \epsilon^2 c_0^2) C^3 (\epsilon c_0)^2 C \epsilon c_0 = C^5 \epsilon^5 c_0^5,$$

which suffices.

7.3.3. The bound for \mathbf{J}_0^8

We recall that \mathbf{J}_0^8 has an expression of the form

$$\mathbf{J}_a^8 = \int B_{m,a}^4(u) R_{p,a}^4(u) - R_{m,a}^4(u) B_{p,a}^4(u) + B_{m,a}^4(u) R_{p,a}^4(u) - R_{m,a}^4(u) B_{p,a}^4(u) dx; \quad (7.31)$$

see equation (5.11) where we set $\xi_0 = 0$. For this, we need to show that

$$\left| \int_0^T \mathbf{J}_a^8 dt \right| \lesssim \epsilon^6 c_0^4.$$

This is an 8-linear term which has two factors, both of which are 4-linear terms with output at frequency 0 and one factor at frequency 0. But the symbols are not the same, that is, we have more decay in B_a^4 than in R_a^4 .

As usual, we localize the entries of \mathbf{J}_a^8 on the unit frequency scale and estimate each term separately. We denote the four frequencies in B_a^4 by k_1, k_2, k_3, k_4 with $k_1 = 0$ and the four frequencies in R_a^4 by

l_1, l_2, l_3, l_4 with $l_1 = 0$. These are constrained by the relations $\Delta^4 k = 0, \Delta^4 l = 0$. In addition, their symbols are bounded, along with their derivatives, as follows:

$$|b_a^4| \lesssim \frac{1}{\langle \delta k^{hi} \rangle \langle \delta k^{med} \rangle}, \quad |r_a^4| \lesssim \frac{1}{\langle \delta l^{med} \rangle}.$$

We consider several cases:

A) All eight frequencies are close to zero. Then we use six $L^6_{t,x}$ Strichartz bounds as in equation (6.6) and two L^∞ bounds obtained from the energy via Bernstein’s inequality.

B) Some frequencies are away from zero. Denote by $n_1 \leq n_2$ the dyadic separations for the k_j frequencies in B_4 , and by $o_1 \leq o_2$ the dyadic separations for the l_j frequencies in R_4 . We consider two cases depending on how n_2 and o_2 compare.

B1) $n_2 \lesssim o_2$. Then the R^4 frequencies are in two o_2 separated clusters with distance below o_1 within each cluster. We use two bilinear L^2 bounds there, and L^∞ bounds for all the B_a^4 factors to estimate

$$\left| \int_0^T \mathbf{J}_a^8(u) dt \right| \lesssim \epsilon^8 C^6 c_0^2 \sum \frac{1}{n_1 n_2} c_{k_2} c_{k_3} c_{k_4} \frac{1}{o_1 o_2} c_{l_2} c_{l_3} c_{l_4}.$$

Suppose k_2 and l_2 are the smaller frequencies in each group so that $|k_2| \approx n_1$ and $|l_2| \approx o_1$. For fixed k_2 , respectively l_2 , we apply the Cauchy–Schwarz inequality for the pairs (k_3, k_4) , respectively (l_3, l_4) . We obtain

$$\left| \int_0^T \mathbf{J}_a^8(u) dt \right| \lesssim \epsilon^8 C^6 c_0^2 \sum \frac{1}{n_1 n_2} c_{k_2} c_{n_2}^2 \frac{1}{o_1 o_2} c_{l_2} c_{o_2}^2.$$

Now, we use the maximal function to also fix k_2 and l_2 ,

$$\left| \int_0^T \mathbf{J}_a^8(u) dt \right| \lesssim \epsilon^8 C^6 c_0^4 \sum_{n_2 \leq o_2} \frac{\log n_2}{n_2} c_{n_2}^2 \frac{\log o_2}{o_2} c_{o_2}^2 \lesssim \epsilon^8 C^6 c_0^4.$$

B2) $o_2 \ll n_2$. Here, we proceed exactly as before but using instead two bilinear L^2 bounds in B_a^4 . Following the same steps, we arrive at

$$\left| \int_0^T \mathbf{J}_a^8(u) dt \right| \lesssim \epsilon^8 C^6 c_0^4 \sum_{o_2 \leq n_2} \frac{\log n_2}{n_2^2} c_{n_2}^2 \log o_2 c_{o_2}^2 \lesssim \epsilon^8 C^6 c_0^4.$$

Here, the denominators are unbalanced compared to the previous case but in a favourable way.

7.3.4. The bound for \mathbf{K}_a^8

We recall that \mathbf{K}_a^8 has the form

$$\begin{aligned} \mathbf{K}_a^8(u) = & \iint_{x>y} M_a^\sharp(u)(x) R_{p,a}^6(u)(y) + P_a^\sharp(u)(y) R_{m,a}^6(u)(x) \\ & - M_a^\sharp(u)(y) R_{p,a}^6(u)(x) - P_a^\sharp(u)(x) R_{m,a}^6(u)(y) dx dy. \end{aligned} \tag{7.32}$$

The time integral of $\mathbf{K}_a^8(u)$ is estimated directly using the $L^1_{t,x}$ bound for R^6 in Lemma 7.3 and the uniform L^1_x bound for M^\sharp and P^\sharp , provided by Lemma 7.1 together with the simpler bound (7.29).

7.4. Near parallel interactions

Here, we briefly discuss the bilinear L^2 bound (6.4) in the case when the sets A and B are of size ≤ 1 and at distance ≤ 1 . This can be viewed on one hand as a slight generalization of the argument in the

previous subsection, where instead of $v = u$ we take $v = u(\cdot + x_0)$. The only difference in the proof is that, because of the translations, we can no longer use the defocusing property to control the sign of the diagonal \mathbf{J}^6 contribution. However, this is not a problem because the localized L^6 norm of u_k has already been estimated in the previous subsection.

7.5. The transversal bilinear L^2 estimate

Here, we prove the bilinear L^2 bound (6.4). This repeats the same analysis as before but using the Interaction Morawetz functional associated to two separated frequency intervals A and B , of size at most n and with n separation. Here, we no longer take $v = u$, and instead we let $v = u(\cdot + x_0)$. The parameter $x_0 \in \mathbb{R}$ is arbitrary and the estimates are uniform in x_0 .

Since x_0 does not play any role in the analysis, we simply drop it from our notations. To further simplify the notations in what follows, we take advantage of the Galilean invariance to translate the problem in frequency so that 0 is roughly halfway between the intervals A and B . This will allow us to set $\xi_0 = 0$ in equation (5.14), and to assume that both A and B are within distance n from the origin. We consider mass m_a, m_b and momentum forms p_a, p_b , where a and b are bump functions, smooth on the unit scale, selecting the sets A and B .

The Interaction functional takes the form (see equation (5.14))

$$\mathbf{I}_{AB}(u, v) = \iint_{x>y} M_a^\#(u)(x)P_b^\#(v)(y) - P_a^\#(u)(x)M_b^\#(v)(y) \, dx dy. \tag{7.33}$$

Its time derivative is given (see equation (5.9)) by

$$\frac{d}{dt}\mathbf{I}_{AB} = \mathbf{J}_{AB}^4 + \mathbf{J}_{AB}^6 + \mathbf{J}_{AB}^8 + \mathbf{K}_{AB}^8. \tag{7.34}$$

Following the same pattern as in the earlier case of the localized Interaction Morawetz case, we will estimate each of these terms as follows:

$$|\mathbf{I}_{AB}(u, v)| \lesssim n\epsilon^4 c_A^2 c_B^2, \tag{7.35}$$

$$\mathbf{J}_{AB}^4(u, v) \approx \|\partial_x(u_A \bar{v}_B)\|_{L_x^2}^2, \tag{7.36}$$

$$\left| \int_0^T \mathbf{J}_{AB}^6 \, dt \right| \lesssim n(\epsilon^6 C^6 + \epsilon^4) c_A^2 c_B^2, \tag{7.37}$$

$$\left| \int_0^T \mathbf{J}_{AB}^8 \, dt \right| \lesssim n\epsilon^8 C^8 c_A^2 c_B^2, \tag{7.38}$$

$$\left| \int_0^T \mathbf{K}_{AB}^8 \, dt \right| \lesssim n\epsilon^6 C^8 c_A^2 c_B^2. \tag{7.39}$$

7.5.1. The fixed time estimate for \mathbf{I}_{AB}

Here, we prove the bound (7.35), which is a consequence of fixed time L^1 estimates for the energy densities, namely

$$\|M_a^\#(u)\|_{L_x^1} \lesssim \epsilon^2 c_A^2, \quad \|P_a^\#(u)\|_{L_x^1} \lesssim n\epsilon^2 c_A^2, \tag{7.40}$$

and the similar estimates with a replaced by b and u replaced by v . This is obvious for the quadratic part of the above densities, where we note that the n factor for the momentum bound arises due to the

distance $o(n)$ between the set A and the origin. It remains to consider the quartic terms, where we can use Lemma 7.1 together with Corollary 7.2.

7.5.2. The bound for J_{AB}^6

Here, we prove the bound for J_{AB}^6 in equation (7.37). We recall that J_{AB}^6 has the form

$$J_{AB}^6 = \int M_a(u)R_{p,b}^4(v) - P_b(v)R_{m,a}^4(u) + B_{m,a}^4(u)E_b(v) - B_{p,b}^4(v)P_a(u) - \text{symmetric } dx,$$

where the symmetric term is obtained by interchanging the indices a and b and also u and v . The symbols for the M, P and E factors have size 1, n and n^2 , respectively, with a similar balance between the B_m^4 and B_p^4 terms, respectively the R_m^4 and R_p^4 terms. So it suffices to consider one R^4 term and one B^4 term.

A) The B^4 term $B_{m,a}^4(u)E_b(v)$. Here, we denote by l_1, l_2 the E_b frequencies and by k_1, k_2, k_3, k_4 the $B_{m,a}^4$ frequencies, where

$$\Delta^2 l + \Delta^4 k = 0.$$

The symbol for E_b has size n^2 , with both frequencies in B . The symbol for $B_{m,a}^4(u)$ has size $(\langle \delta k^{med} \rangle \langle \delta k^{hi} \rangle)^{-1}$ and support in the region, where $|\Delta^4 k| \ll 1 + k^{med}$, and at least one of the frequencies is in A . We denote the dyadic sizes of k^{med} and k^{hi} by $n_1 \leq n_2$. Without any loss in generality, we may assume that k_1, k_2, k_3, k_4 are chosen so that

$$k_1 \in A, \quad |k_1 - k_2| \approx n_1, \quad |k_1 - k_3| \approx n_2, \quad |k_1 - k_4| \approx n_2, \quad |k_3 - k_4| \approx n_1. \quad (7.41)$$

Depending on the size of n relative to n_1, n_2 , we consider two cases:

A1) $n_2 \ll n$. Since A and B are n -separated, within the set of six frequencies we can find two pairs of n -separated frequencies. Then we can apply twice the bilinear L^2 bound and estimate the remaining factors in L^∞ . We arrive at the frequency envelope bound

$$\left| \int_0^T J_{AB}^6 dt \right| \lesssim \epsilon^6 C^6 n \sum \frac{1}{n_1 n_2} c_{l_1} c_{l_2} c_{k_1} c_{k_2} c_{k_3} c_{k_4},$$

where the summation indices are restricted as discussed above. Then, applying the Cauchy–Schwarz inequality for the pair (l_1, l_2) we obtain

$$\begin{aligned} \left| \int_0^T J_{AB}^6 dt \right| &\lesssim \epsilon^6 C^6 n \sum \frac{1}{n_1 n_2} \sum_{|\Delta^4 k| < n_1} c_{k_1} c_{k_2} c_{k_3} c_{k_4} \sum_{l_1, l_2 \in B}^{\Delta l = -\Delta^4 k} c_{l_1} c_{l_2} \\ &\lesssim \epsilon^6 n C^6 c_B^2 \sum \frac{1}{n_1 n_2} \sum_{|\Delta^4 k| < n_1} c_{k_1} c_{k_2} c_{k_3} c_{k_4}. \end{aligned}$$

Hence, it remains to estimate the last sum above as follows:

$$S_A := \sum_{n_1 < n_2} \frac{1}{n_1 n_2} \sum_D c_{k_1} c_{k_2} c_{k_3} c_{k_4} \lesssim c_A^2, \quad (7.42)$$

where the summation set D is described by equation (7.3). In this estimate, the parameter n no longer appears. Recalling that $k_1 \in A$, we fix k_1 and split

$$S_A = \sum_{k_1 \in A} c_{k_1} S_{k_1}, \quad S_{k_1} := \sum_D \sum_{n_1 < n_2} \frac{1}{n_1 n_2} \sum_D c_{k_2} c_{k_3} c_{k_4}.$$

Then it suffices to show that

$$S_{k_1} \lesssim c_{k_1},$$

which is exactly the bound (7.4) proved earlier.

A2) $n_2 \gtrsim n$. This time, within the set of four k frequencies we can find two pairs of n_2 -separated frequencies. Applying twice the bilinear L^2 bound and estimating the remaining factors in L^∞ , we arrive at

$$\left| \int_0^T \mathbf{J}_{AB}^6 dt \right| \lesssim \epsilon^6 C^6 n^2 \sum \frac{1}{n_1 n_2^2} c_{l_1} c_{l_2} c_{k_1} c_{k_2} c_{k_3} c_{k_4},$$

Applying the Cauchy–Schwarz inequality for the pair (l_1, l_2) now yields

$$\left| \int_0^T \mathbf{J}_{AB}^6 dt \right| \lesssim \epsilon^6 C^6 n^2 c_B^2 \sum \frac{1}{n_1 n_2^2} \sum_{|\Delta^4 k| < n} c_{k_1} c_{k_2} c_{k_3} c_{k_4}.$$

Since $n \lesssim n_2$, we can conclude again using the bound (7.42) which was already proved in (A1).

B. The R^4 terms are also all similar, so to fix the notations we will discuss the expression $P_b(u)R_{m,a}^4(u)$. We denote again the six frequencies by l_1, l_2 for P_b , respectively by k_1, k_2, k_3, k_4 for $B_{m,a}^4$. The symbol of p_B is supported in $B \times B$ and has size n . The symbol $R_{m,a}^4$ has size

$$|r_{m,a}^4([k])| \lesssim \frac{n + \delta k^{hi}}{\langle \delta k^{med} \rangle \langle \delta k^{hi} \rangle}.$$

The bound for the portion containing the n term in the denominator is identical to the one in case A, so in the sequel we dismiss this term and simplify the above bound to

$$|r_{m,a}^4([k])| \lesssim \frac{1}{\delta k^{med}}.$$

Retaining the notations $n_1 \leq n_2$ for the dyadic sizes of δk^{med} and δk^{hi} , we may also restrict our analysis to the case when $n_2 \gg n$. This is similar to case A above. We get the better n_2^{-1} factor from the bilinear L^2 bounds, which allows us to reduce the problem to proving exactly the bound (7.42), but for a larger set of indices

$$k_1 \in A, \quad |k_1 - k_2| \lesssim n_1, \quad |k_1 - k_3| \approx n_2, \quad |k_1 - k_4| \approx n_2, \quad |k_3 - k_4| \lesssim n_1. \quad (7.43)$$

But this still follows from equation (7.4).

7.5.3. The bound for \mathbf{J}_{AB}^8

Here, we prove the bound (7.38). We recall that \mathbf{J}_{AB}^8 has the form

$$\mathbf{J}_{AB}^8 = \iint B_{m,a}^4(u)R_{p,b}^4(v) - B_{p,b}^4(v)R_{m,a}^4(u) + B_{m,b}^4(v)R_{p,a}^4(u) - B_{p,a}^4(u)R_{m,b}^4(v) dx dt.$$

All terms here are similar, so it suffices to consider the first one. To avoid a lengthy proof which would largely repeat the arguments in the proof of equation (7.37), we make a simple observation, namely that the proof of the bound for this term becomes a corollary of the previous bound if we can establish a representation

$$B_{m,a}^4(u) \approx \sum_{l_1, l_2 \in A} u_{l_1} w_{l_2}$$

so that, for each k which is M -separated from l_2 , the function w_{l_2} satisfies a bilinear L^2 bound of the form

$$\|w_{l_2} u_k\|_{L^2_{t,x}} \lesssim M^{-\frac{1}{2}} C^4 \epsilon^4 c_{l_2} c_k. \tag{7.44}$$

If that is true, then w_{l_2} would play exactly the role of u_{l_2} in the \mathbf{J}_{AB}^6 estimate.

Indeed, we may represent

$$B^4_{m,a}(u) = \sum_{l_1, l_2 \in A} \sum_{k_2 - k_3 + k_4 = l_2} B^4_{m,a}(u_{l_1}, u_{k_2}, u_{k_3}, u_{k_4}).$$

Here, the symbol for $B^4_{m,a}$ and its derivatives have size $\lesssim \frac{1}{n_1 n_2}$ in a unit region around frequency (l_1, k_2, k_3, k_4) . Hence, we may separate variables and represent $B^4_{m,a}(u_{l_1}, u_{k_2}, u_{k_3}, u_{k_4})$ as the sum of a rapidly convergent series

$$B^4_{m,a}(u_{l_1}, u_{k_2}, u_{k_3}, u_{k_4}) = \sum_j D^j u_{l_1} B^{4,j}_{m,a}(u_{k_2}, u_{k_3}, u_{k_4}) := \sum D^j u_{l_1} w^j_{l_2},$$

where the symbols for D^j , respectively $B^{4,j}_{m,a}$ have unit size, respectively $\lesssim \frac{1}{n_1 n_2}$ with rapid decay in j . Then it remains to prove the estimate (7.44) for the functions $w^j_{l_2}$.

Indeed, at least one of the k 's must be M -separated from k , so using a bilinear L^2 bound we have

$$\|w^j_{l_2} u_k\|_{L^2} \lesssim M^{-\frac{1}{2}} \epsilon^4 C^4 c_k j^{-10} \sum_{k_2 - k_3 + k_4 = l_2} \frac{1}{n_1 n_2} c_{k_2} c_{k_3} c_{k_4}.$$

It remains to estimate the last sum. Suppose k_2 is within distance n_1 from l_2 , then we use the maximal function to estimate

$$\sum_{k_2 - k_3 + k_4 = l_2} \frac{1}{n_1 n_2} c_{k_2} c_{k_3} c_{k_4} \lesssim c_{l_2} \sup_{k_2} \sum_{k_2 - k_3 + k_4 = l_2} \frac{1}{n_2} c_{k_3} c_{k_4} \lesssim c_{l_2},$$

as needed.

7.5.4. The bound for \mathbf{K}_{AB}^8

This is immediate by combining the bound (7.40) with the R^6 bounds in Lemma 7.3 and Corollary 7.4.

8. Global bilinear and Strichartz estimates

Our objective in this last section is to supplement the unit frequency scale bilinear L^2 and Strichartz estimates with their more global counterparts:

Theorem 6. *The global small data solutions u for equation (1.3) in Theorem 5 satisfy the following bounds:*

- *Strichartz estimate:*

$$\|u\|_{L^6_{t,x}} \lesssim \epsilon^4, \tag{8.1}$$

- *Bilinear L^2 bound:*

$$\|\partial_x |u|^2\|_{L^2_t H^{\frac{1}{2}}_x} \lesssim \epsilon^4. \tag{8.2}$$

Proof. We successively consider the two estimates:

A. *The global L^6 bound.* We prove the global L^6 bound using the previous localized estimates. We aim to estimate the integral

$$I = \iint_{\mathbb{R} \times \mathbb{R}} |u|^6 dx dt$$

by taking a suitable frequency decomposition. Given six unit frequency regions indexed by k_1, k_2, k_3, k_4, k_5 and k_6 , they can only contribute to the above integral iff $\Delta^6 k = 0$. We divide them as follows:

1. The diagonal case $|k_i - k_j| \lesssim 1$.
2. The nondiagonal case. we index these frequencies by the dyadic size $n \gg 1$ of the set of frequencies, that is, so that

$$\max |k_i - k_j| \approx n.$$

Within this range, we organize frequencies in intervals A_1, \dots, A_6 of size $n/100$. Of these intervals, at least two pairs must be n -separated in order to contribute to the above integral.

Based on this, we split I as

$$I = I_0 + \sum_n I_n,$$

where

$$I_0 = \sum_{|k_i - k_j| \lesssim 1} \iint u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} u_{k_5} \bar{u}_{k_6} dx dt,$$

and

$$I_n = \sum \iint u_{A_1} \bar{u}_{A_2} u_{A_3} \bar{u}_{A_4} u_{A_5} \bar{u}_{A_6} dx dt,$$

where the last sum is indexed over the sets A_j of size $n/100$, with largest distance $\approx n$ and at least two distances $\geq n/10$.

For the diagonal part, we use the L^6 bound (6.2) to estimate

$$|I_0| \lesssim \epsilon^4 \sum_k c_k^4 \lesssim \epsilon^4,$$

which suffices.

For the off-diagonal part we apply two bilinear L^2 bounds for the separated intervals (gaining $n^{-\frac{1}{2}}$ each time) and two L^∞ bounds via Bernstein's inequality (losing $n^{\frac{1}{2}}$ each time) to bound the corresponding term by

$$|I_n| \lesssim \epsilon^6 \sum c_{A_1} c_{A_2} c_{A_3} c_{A_4} c_{A_5} c_{A_6}.$$

We retain only the separated parts and apply the Cauchy-Schwarz inequality to estimate

$$|I_n| \lesssim \epsilon^6 \sum_{d(A_1, A_2) > n/10} c_{A_1}^2 c_{A_2}^2 \lesssim \epsilon^6 \sum_{|k_1 - k_2| \approx n} c_{k_1}^2 c_{k_2}^2.$$

Then summation over n yields

$$\sum_n |I_n| \lesssim \epsilon^6 \sum_{k_1, k_2} c_{k_1}^2 c_{k_2}^2 \lesssim \epsilon^6,$$

which again suffices.

B. The global bilinear L^2 bound. Here, we prove the estimate (8.2). Expanding relative to the dyadic difference n of the two input frequencies, we have

$$\partial_x(|u|^2) = \partial_x w_0 + \sum_n \partial_x w_n,$$

where

$$w_0 = \sum_{|k_1 - k_2| \leq 1} u_{k_1} \bar{u}_{k_2},$$

$$w_n = \sum_{\substack{d(A_1, A_2) \approx n \\ |A_1|, |A_2| \approx n}} u_{A_1} \bar{u}_{A_2}.$$

We use equation (6.3) to estimate w_0 as

$$\|\partial_x w_0\|_{L^2_{t,x}}^2 \lesssim \epsilon^4 \sum_k c_k^4 \lesssim \epsilon^4.$$

On the other hand, for w_n we get

$$\|\partial_x w_n\|_{L^2_t H^{-\frac{1}{2}}_x}^2 \lesssim n \sum_{\substack{d(A_1, A_2) \approx n \\ |A_1|, |A_2| \approx n}} \sum_{\substack{d(A_3, A_4) \approx n \\ |A_3|, |A_4| \approx n}} \int u_{A_1} \bar{u}_{A_2} u_{A_3} \bar{u}_{A_4} dx.$$

Denoting by $n_0 \geq n$ the largest distance between two A_j 's, we have two pairs of intervals with separation $O(n_0)$; therefore, applying twice the bilinear L^2 bound we obtain

$$\|\partial_x w_n\|_{H^{-\frac{1}{2}}}^2 \lesssim \epsilon^4 \sum_{n_0 \geq n} \frac{n}{n_0} \sum_{\substack{d(A_1, A_2) \approx n \\ |A_1|, |A_2| \approx n}} \sum_{\substack{d(A_3, A_4) \approx n \\ |A_3|, |A_4| \approx n}} c_{A_1} c_{A_2} c_{A_3} c_{A_4}.$$

We separate the cases when $n_0 \approx n$ and $n_0 \gg n$. In the first, diagonal case we simply bound the corresponding part of the sum by

$$\epsilon^4 \sum_{\substack{d(A_1, A_2) \approx n \\ |A_1|, |A_2| \approx n}} c_{A_1}^2 c_{A_2}^2.$$

In the off-diagonal case, we apply the Cauchy–Schwarz inequality separately for the pairs A_1, A_2 and A_3, A_4 to obtain a bound

$$\epsilon^4 \frac{n}{n_0} \sum_{\substack{d(B_1, B_2) \approx n_0 \\ |B_1|, |B_2| \approx n_0}} c_{B_1}^2 c_{B_2}^2.$$

Incorporating the first case into the second, we arrive at

$$\|\partial_x w_n\|_{L_t^2 H_x^{-\frac{1}{2}}}^2 \lesssim \epsilon^4 \sum_{n_0 \geq n} \frac{n}{n_0} \sum_{|B_1|, |B_2| \approx n_0}^{d(B_1, B_2) \approx n_0} c_{B_1}^2 c_{B_2}^2.$$

Finally, using orthogonality in frequency we have

$$\begin{aligned} \left\| \sum_n \partial_x w_n \right\|_{L_t^2 H_x^{-\frac{1}{2}}}^2 &\lesssim \epsilon^4 \sum_n \sum_{n_0 \geq n} \frac{n}{n_0} \sum_{|B_1|, |B_2| \approx n_0}^{d(B_1, B_2) \approx n_0} c_{B_1}^2 c_{B_2}^2 \\ &\lesssim \epsilon^4 \sum_{n_0} \sum_{|B_1|, |B_2| \approx n_0}^{d(B_1, B_2) \approx n_0} c_{B_1}^2 c_{B_2}^2 \\ &\lesssim \epsilon^4. \end{aligned}$$

The proof of the theorem is concluded. □

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References

- [1] T. Alazard and J.-M. Delort, ‘Global solutions and asymptotic behavior for two dimensional gravity water waves’, *Ann. Sci. Éc. Norm. Supér. (4)* **48**(5) (2015), 1149–1238.
- [2] M. Borghese, R. Jenkins and K. D. T.-R. McLaughlin, ‘Long time asymptotic behavior of the focusing nonlinear Schrödinger equation’, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **35**(4) (2018), 887–920.
- [3] J. Colliander, M. Grillakis and N. Tzirakis, ‘Tensor products and correlation estimates with applications to nonlinear Schrödinger equations’, *Comm. Pure Appl. Math.* **62**(7) (2009), 920–968.
- [4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, ‘Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation’, *Math. Res. Lett.* **9**(5–6) (2002), 659–682.
- [5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, ‘Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \mathbb{R}^3 ’, *Comm. Pure Appl. Math.* **57**(8) (2004), 987–1014.
- [6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, ‘Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \mathbb{R}^3 ’, *Ann. of Math. (2)* **167**(3) (2008), 767–865.
- [7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, ‘Resonant decompositions and the I -method for the cubic nonlinear Schrödinger equation on \mathbb{R}^2 ’, *Discrete Contin. Dyn. Syst.* **21**(3) (2008), 665–686.
- [8] P. Deift and X. Zhou, ‘Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space’, *Comm. Pure Appl. Math.* **56**(8) (2003), 1029–1077. Dedicated to the memory of Jürgen K. Moser.
- [9] J.-M. Delort, ‘Semiclassical microlocal normal forms and global solutions of modified one-dimensional KG equations’, *Ann. Inst. Fourier (Grenoble)* **66**(4) (2016), 1451–1528.
- [10] B. Dodson, ‘Global well-posedness and scattering for the defocusing, L^2 critical, nonlinear Schrödinger equation when $d = 1$ ’, *Amer. J. Math.* **138**(2) (2016), 531–569.
- [11] B. Dodson, ‘Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation’, *Ann. PDE* **3**(1) (2017), Paper No. 5, 35.
- [12] N. Hayashi and P. I. Naumkin, ‘Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations’, *Amer. J. Math.* **120**(2) (1998), 369–389.
- [13] N. Hayashi and P. I. Naumkin, ‘Large time asymptotics for the fractional nonlinear Schrödinger equation’, *Adv. Differential equations* **25**(1–2) (2020), 31–80.
- [14] M. Ifrim and D. Tataru, ‘Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension’, *Nonlinearity* **28**(8) (2015), 2661–2675.

- [15] M. Ifrim and D. Tataru, ‘Two dimensional water waves in holomorphic coordinates II: Global solutions’, *Bull. Soc. Math. France* **144**(2) (2016), 369–394.
- [16] M. Ifrim and D. Tataru, ‘The lifespan of small data solutions in two dimensional capillary water waves’, *Arch. Ration. Mech. Anal.* **225**(3) (2017), 1279–1346.
- [17] M. Ifrim and D. Tataru, ‘Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation’, *Ann. Sci. Éc. Norm. Supér. (4)* **52**(2) (2019), 297–335.
- [18] J. Kato and F. Pusateri, ‘A new proof of long-range scattering for critical nonlinear Schrödinger equations’, *Differential Integral equations* **24**(9–10) (2011), 923–940.
- [19] H. Koch and D. Tataru, ‘Energy and local energy bounds for the 1-d cubic NLS equation in $H^{-\frac{1}{4}}$ ’, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29**(6) (2012), 955–988.
- [20] H. Lindblad, J. Lührmann and A. Soffer, ‘Asymptotics for 1D Klein–Gordon equations with variable coefficient quadratic nonlinearities’, *Arch. Ration. Mech. Anal.* **241**(3) (2021), 1459–1527.
- [21] H. Lindblad and A. Soffer, ‘Scattering and small data completeness for the critical nonlinear Schrödinger equation’, *Nonlinearity* **19**(2) (2006), 345–353.
- [22] F. Planchon and L. Vega, ‘Bilinear virial identities and applications’, *Ann. Sci. Éc. Norm. Supér. (4)* **42**(2) (2009), 261–290.
- [23] E. Ryckman and M. Visan, ‘Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in \mathbb{R}^{1+4} ’, *Amer. J. Math.* **129**(1) (2007), 1–60.
- [24] T. Tao, ‘Global regularity of wave maps. II. Small energy in two dimensions’, *Comm. Math. Phys.* **224**(2) (2001), 443–544.
- [25] T. Tao, ‘Global well-posedness of the Benjamin–Ono equation in $H^1(\mathbb{R})$ ’, *J. Hyperbolic Differ. Equ.* **1**(1) (2004), 27–49.