Canad. Math. Bull. Vol. **29** (4), 1986

UNIFORM PARACOMPACTNESS AND UNIFORM PARA-LINDELÖFNESS

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ABSTRACT. Relations between uniform paracompactness and uniform para-Lindelöfness of a uniform space and its uniform weight are established.

1. **Notations and definitions**. Let $(X, \mathcal{U})$ be a uniform space. All uniform spaces $(X, \mathcal{U})$ are assumed to be completely regular Hausdorff spaces, and their topologies are the associated ones with the uniformity $\mathcal{U}$.

DEFINITION 1. *The uniform weight of* $(X, \mathcal{U})$ *is the smallest cardinal number m such that* $\mathcal{U}$ *has a basis of cardinality m. It will be denoted by* $u(\mathcal{U})$.

DEFINITION 2. $(X, \mathcal{U})$ *is uniformly paracompact if for each open cover* $\mathcal{G}$ *of X there is an open cover* $\mathcal{G}'$, *which refines it, and* $U \in \mathcal{U}$ *such that each* $U[x]$ *intersects at most finitely many members of* $\mathcal{G}'$, *where x varies in X.*

DEFINITION 3. $(X, \mathcal{U})$ *is uniformly para-Lindelöf if for each open cover* $\mathcal{G}$ *of X there is an open cover* $\mathcal{G}'$ *of X, which refines it, and* $U \in \mathcal{U}$ *so that each* $U[x]$ *intersects at most countably many members of* $\mathcal{G}'$, *where x varies in X.*

DEFINITION 4. *Let* $(f_s)_{s \in S}$ *be a partition of unity of* $(X, \mathcal{U})$. *This partition is uniformly locally finite if there is* $U \in \mathcal{U}$ *so that each* $U[x]$ *intersects at most finitely many sets* $f_s^{-1}([0, 1])$. *where* $s \in S$ *and* $x \in X$.

DEFINITION 5. *A topological space Y is p-compact (where p is an infinite cardinal number) if every discrete closed subset of Y has cardinality less than p.*

A cardinal number is assumed to be the set of all ordinals less than it.

In a topological space Y, $\bar{A}$ denotes the closure of the subset A of Y.

2. **Main results**. The study of uniform paracompactness and uniform para-Lindelöfness were developed by Rice ([5]) and Hohti ([2]) for metric spaces. (In this case the uniformity is the associated with the metric.) Here we will consider general

Received by the editors January 5, 1984 and in revised from, May 2, 1985.

Key words: uniform, paracompact, uniformly para-Lindelöf, uniform weight. This paper was partially written with financial support from CNPq.

AMS Subject Classification: (1980). Primary 54D18. Secondary 54E15.

https://doi.org/10.4153/CMB-1986-062-1 Published online by Cambridge University Press

uniform spaces. Some of our results may be viewed as generalizations of theorems of Rice and Hohti.

LEMMA 1. *If* $(X, \mathcal{U})$ *is uniformly paracompact, then it is complete with respect to* $\mathcal{U}$.

LEMMA 2. *If* $(X, \mathcal{U})$ *is uniformly para-Lindelöf space, then* X *is paracompact.*

PROOF. Let $\mathcal{G}$ be an open covering of X, $\mathcal{G}'$ be an open refinement of $\mathcal{G}$ and $U \in \mathcal{U}$ so that each $U[x]$ intersects at most countably many members of $\mathcal{G}'$. On the other hand, let $\mathcal{C}$ be a locally finite open cover of X which refines $\{U[x] | x \in X\}$. (The existence of $\mathcal{C}$ is a consequence of the proof of A. H. Stone's theorem on full normality [4].)

For each $C \in \mathcal{C}$ let $\Omega_{1,G}, \ldots, \Omega_{n,G}, \ldots$ be members of $\mathcal{G}$ whose union contains C. Now, for each $n = 1, 2, 3, \ldots$, $\{C \cap \Omega_{n,G} | C \in \mathcal{C}\}$ is locally finite family of open subsets of X and, varying n on N, we get a σ-locally finite open refinement of $\mathcal{G}$, and the result follows from E. Michael's characterization of paracompactness ([3]).

Next we are going to construct a special kind of set that will be useful in the proofs of theorems 1 and 3 below.

CONSTRUCTION. Let $(X, \mathcal{U})$ be a non-discrete uniformly paracompact (respectively uniformly para-Lindelöf) $u(\mathcal{U}) = m$.

First let us assume that $m > \aleph_0$ and let B be a discrete closed subset of X of cardinality m, $\mathcal{G} = \{\Omega_b | b \in B\}$ be a discrete family of open sets such that $b \in \Omega_b$, $\forall b \in B$ and let $U \in \mathcal{U}$ so that each $U[x]$ intersects at most finitely many members of $\mathcal{G}$ (or countably many, respectively; X is paracompact by virtue of lemma 2.)

Let $U_* \in \mathcal{U}$ so that $U_* \circ U_* \subset U$ and let $\mathcal{A}$ denote the class of all subsets A of B verifying

$$U_*[a] \cap U_*[a'] = \emptyset \text{ if } a, a' \in A, a \neq a' .$$

Consider $\mathcal{A}$ ordered by inclusion. By Zorn's lemma $\mathcal{A}$ has a maximal element A_* and $|A_*| = m$.

Notice that if $m = \aleph_0$ there is a metric d on X and $\mathcal{U}$ is the uniformity subordinated to d. If there is no natural number $n = 1, 2, \ldots$ so that there is an infinite collection of pairwise disjoint open balls of radius $1/n$, then (X, d) is separable and thus Lindelöf. Furthermore, $(X, \mathcal{U})$ is uniformly para-Lindelöf.

THEOREM 1. *The collection of points of a uniformly paracompact space* $(X, \mathcal{U})$ *that admit no compact neighborhood is* m*-compact, where* m *is the uniform weight of* $\mathcal{U}$.

PROOF. The case $m = \alpha_0$ was proved by Rice in [5].

Let F be the collection of all points of X which admit no compact neighborhood. F is closed and if F is not m-compact, there is a closed discrete subset A, with cardinality m, and $U \in \mathcal{U}$ so that the $U[a]$, with $a \in A$, are pairwise disjoint. Let $\{U_i | i < m\}$ be a uniform basis of $\mathcal{U}$. Put $A = \{a_i | i < m\}$ and for each $i < m$ let V_i be a closed neighborhood of a_i contained in $U[a_i] \cap (\cap U_i[a_i])$ and let $\mathcal{V}_i$ be an open cover of V_i

https://doi.org/10.4153/CMB-1986-062-1 Published online by Cambridge University Press

which has no finite subcover of V_i. Put

$$\mathcal{V} = \bigcup_{i<m} \mathcal{V}_i \cup \left\{X \setminus \bigcup_{i<m} V_i\right\};$$

there is no $U' \in \mathcal{U}$ such that each $U'[y]$ is contained in the union of finitely many members of $\mathcal{V}$, which contradicts the uniform paracompactness.

COROLLARY. *If G is a topological group, $\mathcal{U}$ is the right uniformity of G and $(G, \mathcal{U})$ is uniformly paracompact, then G is locally compact or G is m-compact, where m is the uniform weight of $\mathcal{U}$.*

REMARK. Any locally compact topological group is uniformly paracompact with respect to its right uniformity.

THEOREM 2. *$(X, \mathcal{U})$ is uniformly paracompact if and only if every open cover of X has a uniformly locally finite partition of unity subordinate to it.*

THEOREM 3. *The collection of points of an uniformly para-Lindelöf space $(X, \mathcal{U})$ which admit no Lindelöf neighborhood is m-compact, where m is the uniform weight of $\mathcal{U}$.*

PROOF. The proof is analogous to that of theorem 1, with minor modifications; remember that X is paracompact.

REMARK. Let m be an infinite cardinal number and let X be a paracompact topological space which is m-compact. Then one of the two properties below is verifiable:

1) every closed discrete subset of X has cardinality less than $cf(m)$ (= cofinality of m); or

2) for each closed discrete subset F of cardinality $cf(m)$ there is $A \subset F$, with cardinality of A less than $cf(m)$ and a cardinal $p < m$ such that each point of $F - A$ has a p-compact neighborhood.

Indeed, let F be a closed discrete subset of X with cardinality $cf(m)$. Put $\gamma = cf(m)$ and let $(m_i)_{i<\gamma}$ be an increasing family of cardinals less than m and so that $\Sigma_{i<\gamma} m_i = m$. Chose $x_0 \in F$ with no m_0-compact neighborhood (if there is no such x_0 then put $A = \emptyset$ and $p = m_0$); now choose $x_1 \in F - \{x_0\}$ with no m_1-compact neighborhood (if there is no such x_1 put $A = \{x_0\}$ and $p = m_1$). Assume that for some $\theta < \gamma$ we have constructed $(x_i)_{i<\theta}$ so that they are pairwise distinct and x_i has no m_i-compact neighborhood. Let us construct x_θ; choose $x_\theta \in F - \{x_i | i < \theta\}$ with no m_θ-compact neighborhood (if this is not possible put and $A = \{x_i | i < \theta\}$ and $p = m_\theta$). This process ends before $cf(m)$, otherwise we have a family of pairwise distinct elements, $(x_i)_{i<\gamma}$, and each x_i has no m_i-compact neighborhood. Let $(V_i)_{i<\gamma}$ be a discrete family of closed sets, where each V_i is a neighborhood of x_i. So choose a discrete closed subset F_i of V_i with cardinality m_i. Then $\bigcup_{i<\gamma} F_i$ will be closed discrete subset of X of cardinality m, which is impossible.

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3. **Question**. Referee's question: Suppose that $(X, \mathcal{U})$ is uniformly paracompact and m is the least cardinality of a base for *any* uniformity compatible with X. Is there a uniformity $\mathcal{U}^*$ compatible with X such that $(X, \mathcal{U}^*)$ is uniformly paracompact and $u(\mathcal{U}^*) = m$?

The answer is *no*. But an inequality may be proved; instead of $u(\mathcal{U}^*) = m$ we have $u(\mathcal{U}^*) \leqq m^{\aleph_0}$. For locally compact spaces the answer is affirmative.

Let X be a paracompact space and U_* be its universal uniformity (the finest uniformity compatible with X). It is immediate that (X, U_*) is uniformly paracompact. We consider now two metrizable space Q (the rationals with the usual topology) and $\mathbb{R}^N$ (with the product topology). By virtue of Baire's theorem and lemma 1 there is no uniformity compatible with Q, $u(\mathcal{U}) = \aleph_0$, and such that $(Q, \mathcal{U})$ is uniformly paracompact. On the other hand, there is no uniformity $\mathcal{U}$ compatible with $\mathbb{R}^N$, with $u(\mathcal{U}) = \aleph_0$, such that $(\mathbb{R}^N, \mathcal{U})$ is uniformly paracompact by theorem 1. This second example shows that even for complete metrizable spaces the answer is no.

Before proving theorems 4 and 5 let us recall that if X is a paracompact space then the sets

$$\left\{ \bigcup_{Y \in C} Y \times Y \,\middle|\, C \in \mathcal{C} \right\},$$

where $\mathcal{C}$ is either the set of all open covers of X, or the set of all locally finite open covers of X, are basis for the universal uniformity of X.

Let $\mathcal{D}$ be a collection of open covers of X. For each $D \in \mathcal{D}$ let (D_n) be a fixed sequence of open covers of X, so that $D_0 = D$ and D_{n+1} Δ-refines D_n, $n = 0, 1, 2, \ldots$ Put

$$\mathcal{D}' = \{D_n \mid D \in \mathcal{D};\ n = 0, 1, 2, \ldots\}$$

and define $\mathcal{D}''$ as the set of all "finite intersections" of members of $\mathcal{D}'$. (If $A_1, \ldots, A_s$ belong to $\mathcal{D}'$ then

$$A_1 \cap \ldots \cap A_s = \{X_1 \cap \ldots \cap X_s \mid X_i \in A_i,\ i = 1, \ldots, s\}$$

is a finite intersection).

Then $\{\bigcup_{Y \in M} Y \times Y \mid M \in \mathcal{D}''\}$ is a base of a uniformity $\mathcal{U}_{\mathcal{D}}$ (maybe not compatible with X) on X.

THEOREM 4. *Let $(X, \mathcal{U})$ be a locally compact uniformly paracompact space and let m be the least cardinality of a base for any uniformity compatible with X. Then there is a uniformity $\mathcal{U}^*$ compatible with X such that $u(\mathcal{U}^*) = m$.*

PROOF. Let $\mathcal{U}_1$ be a compatible uniformity with $u(\mathcal{U}_1) = m$ (and $\mathcal{B}_1$ a base of this uniformity with $|\mathcal{B}_1| = m$) and let $(\mathcal{C}_n)$ be a sequence of locally finite open covers of X (whose members have compact closures) such that each $\mathcal{C}_{n+1}$ Δ-refines $\mathcal{C}_n$.

Denote by $\mathcal{U}^*$ the uniformity whose base is

$$\left\{ \mathcal{U} \cap \left(\bigcup_{Y \in \mathcal{C}_n} Y \times Y \right) \,\middle|\, \mathcal{U} \in \mathcal{B}_1,\ n = 1, 2, \ldots \right\}$$

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and the proof is completed.

THEOREM 5. *Let $(X, \mathcal{U})$ be a uniformly paracompact space and let m be the least cardinality of a base for any uniformity compatible with X. If the set of points of X with no compact neighborhood is m-compact, then there is a uniformity $\mathcal{U}^*$ compatible with X such that $u(\mathcal{U}^*) \leqq m^{\underline{m}}$.*

PROOF. Let $\mathcal{U}_1$ be an uniformity compatible with X such that $u(\mathcal{U}_1) = m$ and let F be the set of points of X with no compact neighborhood. Furthermore, fix a base $\mathcal{B}_1$ of $\mathcal{U}_1$ with $|\mathcal{B}_1| = m$. It follows that, for each $\mathcal{U} \in \mathcal{B}_1$, $\{\mathcal{U}[x] \mid x \in F\}$ is an open cover of F; fix a subcover of cardinality $< m$. The union of these subcovers when $\mathcal{U}$ varies in $\mathcal{B}_1$ is a "base" (in X) for the topology of the subspace F. Let $\mathcal{A}$ denote the set of all open covers of F by members of this "base" such that the cardinality of the cover is $< m$. If $F = X$, then the set $\{\bigcup_{Y \in A} Y \times Y \mid A \in \mathcal{A}\}$ is a base of the finest uniformity compatible with X and the cardinality of this set is $\leqq m^{\underline{m}}$.

Assume $X \backslash F \neq \emptyset$. For each $A \in \mathcal{A}$, put $\Omega_A = \bigcup_{Y \in A} Y$ and let $\mathcal{C}_A$ be an open cover of $X \backslash \Omega_A$ by sets with compact closures. It follows that $A \cup \mathcal{C}_A$ is an open cover of X.

Put $\mathcal{D} = \{A \cup \mathcal{C}_A \mid A \in \mathcal{A}\}$ and consider $\mathcal{U}_{\mathcal{D}}$ (constructed before) and let $\mathcal{U}^*$ be the uniformity generated by $\mathcal{U}_{\mathcal{D}}$ and $\mathcal{U}_1$; $\mathcal{U}^*$ is compatible with X.

$(X, \mathcal{U}^*)$ is uniformly paracompact. Indeed, let $\mathcal{C}$ be a locally finite open cover of X and $\mathcal{C}_1$ a locally finite open cover of X each member of which intersects only finitely many members of $\mathcal{C}$. There is $A \in \mathcal{A}$ that refines $\{\Omega \in \mathcal{C}_1 \mid \Omega \cap F \neq \emptyset\}$, let $D \in \mathcal{D}'$ be a Δ-refinement of $A \cup \mathcal{C}_A$, then $\mathcal{U} = \bigcup_{Y \in D} Y \times Y$ belongs to $\mathcal{U}_{\mathcal{D}}$ (and hence to $\mathcal{U}^*$). Fix $x \in X$; if $\mathcal{U}[x]$ is contained in some member of $\mathcal{C}_A$, it has compact closure and intersects only finitely many members of $\mathcal{C}$; on the other hand, if $\mathcal{U}[x]$ is contained in some member of A (hence is contained in some member of $\mathcal{C}_1$) and intersects only finitely many members of $\mathcal{C}$, by hypothesis.

Finally we will show two results on locally compact metrizable spaces.

1) A locally compact metric space (X, d) need not be uniformly paracompact with respect to the metric uniformity associated to d. As a modification of Rice's example ([5], p. 361) there is a locally compact metric space (X, d) which is not uniformly para-Lindelöf with respect to the metric uniformity associated to d. Let Y be an uncountable set and put $X =]0, 1[\times Y$ with the following metric d

$$d((r, y), (s, z)) = \begin{cases} 1 & \text{if } r \neq s \\ r & \text{if } r = s \text{ and } y \neq z \\ 0 & \text{otherwise} \end{cases}$$

X is a discrete topological space and no open ball is countable (hence it is not uniformly para-Lindelöf with respect to the metric uniformity associated to d).

2) A locally compact metrizable Lindelöf space X has a metric d compatible with the topology so that (X, d) is uniformly paracompact. Indeed, let (X_*, d_*) be a metric space so that X_* is the (one point) Alexandroff compactification of X. Consider the

product space $\mathbb{R} \times X_*$ (where $\mathbb{R}$ are the reals with the usual metric); then $\mathbb{R} \times X_*$ is uniformly paracompact and X is homeomorph to the subspace $\{(t, y) \in R \times X_* \mid td(y, \infty) = 1\}$, which is closed in $\mathbb{R} \times X_*$ (and hence uniformly paracompact).

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