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Heat kernel asymptotics for real powers of Laplacians

Cipriana Anghel

Abstract. We describe the small-time heat kernel asymptotics of real powers Δ^r, $r \in (0,1)$ of a non-negative self-adjoint generalized Laplacian Δ acting on the sections of a Hermitian vector bundle $\mathcal{E}$ over a closed oriented manifold M. First, we treat separately the asymptotic on the diagonal of $M \times M$ and in a compact set away from it. Logarithmic terms appear only if n is odd and r is rational with even denominator. We prove the non-triviality of the coefficients appearing in the diagonal asymptotics, and also the non-locality of some of the coefficients. In the special case $r = 1/2$, we give a simultaneous formula by proving that the heat kernel of $\Delta^{1/2}$ is a polyhomogeneous conormal section in $\mathcal{E} \boxtimes \mathcal{E}^*$ on the standard blow-up space $\mathrm{M_{heat}}$ of the diagonal at time $t = 0$ inside $[0, \infty) \times M \times M$.

1 Introduction

Let Δ be a self-adjoint generalized Laplacian acting on the sections of a Hermitian vector bundle $\mathcal{E}$ over an oriented, compact Riemannian manifold M of dimension n. Denote by p_t the heat kernel of Δ, i.e., the Schwartz kernel of the operator $e^{-t\Delta}$. It is known since Minakshisundaram–Pleijel [21] that $p_t(x, y)$ has an asymptotic expansion as $t \searrow 0$ near the diagonal

$$p_t(x,y) \overset{t\searrow 0}{\sim} t^{-n/2} e^{-\frac{d(x,y)^2}{4t}} \sum_{j=0}^{\infty} t^j \Psi_j(x,y), \tag{1.1}$$

where $d(x, y)$ is the geodesic distance between x and y, and the Ψ_j's are recursively defined as solutions of certain ODE's along geodesics (see, e.g., [4, 5]). This asymptotic expansion applied to $\mathrm{D}^* \mathrm{D}$, where D is a twisted Dirac operator, plays a leading role in the heat kernel proofs of the Atiyah–Singer index theorem (see [6, 7, 12]).

Bär and Moroianu [2] studied the short-time asymptotic behavior of the heat kernel of $\Delta^{1/m}$, $m \in \mathbb{N}^*$, for a strictly positive self-adjoint generalized Laplacian Δ. They give explicit asymptotic formulæ separately in the case when $t \searrow 0$ along the diagonal $\mathrm{Diag} \subset M \times M$, and when t goes to 0 in a compact set away from the diagonal. The asymptotic behavior depends on the parity of the dimension n and of the root m.

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More precisely, logarithmic terms appear when n is odd and m is even. They use the Legendre duplication formula, and the more general Gauss multiplication formula for the Γ function (see, e.g., [22]). Another crucial argument in [2] is to use integration by parts in order to show that the Schwartz kernel q_{-s} of the pseudodifferential operator Δ^{-s}, $s \in \mathbb{C}$, defines a meromorphic function when restricted to the diagonal in $M \times M$.

1.1 Small-time heat asymptotic for real powers of Δ

The purpose of this paper is to study the short-time asymptotic of the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$, where $r \in (0,1)$ and Δ is a non-negative self-adjoint generalized Laplacian, like, for instance, $\Delta = \mathrm{D}^* \mathrm{D}$ for a Dirac operator D. We give separate formulæ as t goes to 0 in $[0,\infty) \times \mathrm{Diag}$, and when $t \searrow 0$ in $[0,\infty) \times K$, where $K \subset M \times M$ is a compact set disjoint from the diagonal. In Theorem 6.1, we obtain that $h_{t|[0,\infty)\times K} \in t \cdot \mathcal{C}^\infty([0,\infty) \times K)$ is a smooth function vanishing at least to order 1 at $\{t = 0\}$. The asymptotic along the diagonal depends on the parity of n (like in [2]) and on the rationality of r. In Theorem 7.1, the most interesting case occurs when logarithmic terms appear. This happens only if n is odd, $r = \frac{\alpha}{\beta}$ is rational, and the denominator β is even. In that case,

$$
\begin{aligned}
h_{t|_{\mathrm{Diag}}} \overset{t \searrow 0}{\sim} & \sum_{j=0}^{(n-1)/2} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + \sum_{\substack{j=1 \\ \alpha \nmid 2j+1}}^{\infty} t^{\frac{2j+1}{2r}} \cdot A_{\frac{2j+1}{2r}} \\
& + \sum_{\substack{j=1 \\ \beta \nmid j}}^{\infty} t^j \cdot A_j + \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} t^{l\frac{\beta}{2}} \log t \cdot B_l.
\end{aligned} \tag{1.2}
$$

Similar expansions are proved in Theorem 7.1 in all the other cases. Furthermore, we prove the non-triviality of the coefficients appearing in the diagonal asymptotics (Theorem 1.1), and also the non-locality of some of them (Theorem 1.3).

In the special case $r = 1/2$, Bär and Moroianu [2] described the small-time asymptotic behavior of h_t on the diagonal and away from it separately. In Theorem 1.4, we give an uniform description of the transition between the on- and off-diagonal behavior by proving that the heat kernel of $\Delta^{1/2}$ is a polyhomogeneous conormal section in $\mathcal{E} \boxtimes \mathcal{E}^*$ on the standard blow-up space $[[0,\infty) \times M \times M, \{t=0\} \times \mathrm{Diag}]$.

1.2 Comparison to previous results

Fahrenwaldt [11] studied the off-diagonal short-time asymptotics of the heat kernel of $e^{-tf(P)}$, where $f : [0,\infty) \longrightarrow [0,\infty)$ is a smooth function with certain properties, and P is a positive self-adjoint generalized Laplacian. The function $f(x) = x^r$, $r \in (0,1)$ does not satisfy the third condition in [11, Hypothesis 3.3], which seems to be crucial for the arguments and statements in that paper, so the results of [11] do not seem to apply here.

Duistermaat and Guillemin [10] give the asymptotic expansion of the heat kernel of e^{-tP}, where P is a scalar positive elliptic self-adjoint pseudodifferential operator. The order of P in [10] seems to be a positive integer. It is claimed in [1] that this asymptotic holds true in the context of fiber bundles. Furthermore, Grubb [16, Theorem 4.2.2]

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studied the heat asymptotics for e^{-tP} in the context of fiber bundles when the order of P is positive, not necessary an integer. In Theorem 7.1, we obtain the vanishing of some terms appearing in [16, Corollary 4.2.7] in our particular case when $P = \Delta^r$ is a real power of a self-adjoint non-negative generalized Laplacian Δ, $r \in (0,1)$. We also show that the remaining terms do not vanish in general.

Theorem 1.1 *For each $r \in (0,1)$, none of the coefficients in the small-time asymptotic expansion of h_t appearing in Theorem 7.1 vanishes identically for every generalized Laplacian Δ.*

The logarithmic coefficients B_l and the coefficients A_j for $j \notin \mathbb{Z}$ can be computed in terms of the heat coefficients for $e^{-t\Delta}$ appearing in (1.1). It is well known that the heat coefficients of a generalized Laplacian are locally computable in terms of the curvature of the connection on $\mathcal{E}$, the Riemannian metric of M and their derivatives (see, e.g., [5]). This is no longer the case for the coefficients of positive integer powers of t from Theorem 7.1 as we shall see now.

By applying Theorem 7.1 for $r \in (0,1)$ and a set of geometric data, namely a hermitic vector bundle $\mathcal{E}$ over an oriented, compact Riemannian manifold (M,g), a metric connection ∇ and an endomorphism $F \in \operatorname{End}\mathcal{E}$, $F^* = F$, we produce an endomorphism $A_l(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F) \in \mathcal{C}^\infty(M,\operatorname{End}\mathcal{E})$ for each index l appearing in (1.2).

Definition 1.1 (i) We say that a function A which associates to any set of geometric data $(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F)$ a section in $\mathcal{C}^\infty(M,\operatorname{End}\mathcal{E})$ is *locally computable* if for any two sets of geometric data $(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F)$, $(M',g',\mathcal{E}',h_{\mathcal{E}'},\nabla',F')$ which agree on an open set (i.e., there exist an isometry $\alpha : U \longrightarrow U'$ between two open sets $U \subset M$, $U' \subset M'$, and a metric isomorphism $\beta : \mathcal{E}_{|_U} \longrightarrow \mathcal{E}'_{|_{U'}}$ which preserves the connection and $\beta_x \circ F_x \circ \beta^{-1}_{\alpha(x)} = F'_{\alpha(x)}$), we have

$$\beta_x \circ A_x \circ \beta^{-1}_{\alpha(x)} = A_{\alpha(x)},$$

for any $x \in U$.

(ii) A scalar function a defined on the set of all geometric data $(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F)$ with values in $\mathbb{C}$ is called *locally computable* if there exists a locally computable function C as in (i) above such that $a = \int_M \operatorname{Tr} C \operatorname{dvol}_g$ for any $(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F)$.

(iii) A function A as in (i) is called *cohomologically locally computable* if there exists a locally computable function C as in (i) such that for any $(M,g,\mathcal{E},h_{\mathcal{E}},\nabla,F)$,

$$\left[\operatorname{Tr} A \operatorname{dvol}_g\right] = \left[\operatorname{Tr} C \operatorname{dvol}_g\right] \in H^n_{dR}(M).$$

Remark 1.2 (i) If a function A is locally computable, then the integral $a := \int_M \operatorname{Tr} A \operatorname{dvol}_g$ is locally computable.

(ii) A function A is cohomologically locally computable if and only if $a := \int_M \operatorname{Tr} A \operatorname{dvol}_g$ is locally computable.

Theorem 1.3 *If r is irrational, the heat coefficients A_j in Theorem 7.1 (and in particular in (1.2)) are not locally computable for integer $j \geq 1$. If $r = \frac{\alpha}{\beta}$ is rational, then A_j are not locally computable for $j \in \mathbb{N}\setminus\{l\beta : l \in \mathbb{N}\}$. All the other coefficients can be written in terms of the heat coefficients of $e^{-t\Delta}$, hence they are locally computable.*

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Consider the asymptotic expansion in [10, Corollary 2.2'] for a scalar *admissible* operator, i.e., an elliptic, self-adjoint, positive pseudodifferential operator P of positive *integer* order d:

$$e^{-tP} \overset{t\searrow 0}{\sim} \sum_{l=0}^{\infty} A_l(P)t^{(l-n)/d} + \sum_{k=1}^{\infty} B_k(P)t^k \log t.$$

Gilkey and Grubb [14, Theorem 1.4] proved that the coefficients $a_l(P)$ for $l \geq 0$ and $b_k(P)$ for $k \geq 1$ from the corresponding small-time heat trace expansion

$$\operatorname{Tr} e^{-tP} \overset{t\searrow 0}{\sim} \sum_{l=0}^{\infty} a_l(P)t^{(l-n)/d} + \sum_{k=1}^{\infty} b_k(P)t^k \log t \tag{1.3}$$

are generically non-zero in the above class of admissible operators. In Theorem 1.1, we prove the same type of statement. However, in our case, the order of the operator Δ^r is $2r$; thus, it is integer only for $r = 1/2$. Even in this case, the non-vanishing result in Theorem 1.1 is not a consequence of [14, Theorem 1.4] since, in our case, we do not consider the whole class of admissible operators of fixed integer order d in the sense of Gilkey and Grubb [14], but the smaller class of square roots of generalized Laplacians.

Furthermore, in [14, Theorem 1.7], it is proved that the coefficients $a_l(P)$ in (1.3) corresponding to $t^{(l-n)/d}$, for $(l-n)/d \in \mathbb{N}$, are not locally computable. Remark that the meaning of "locally computable" in [14] is different from our Definition 1.1. More precisely, in the definition of Gilkey and Grubb, a locally computable function A has to be a smooth function in the jets of the homogeneous components of the total symbol of the operator. A locally computable coefficient in the sense of Gilkey and Grubb [14] is clearly locally computable in the sense of Definition 1.1(ii).

For $r = 1/2$, Bär and Moroianu [2] remark that for odd $k = 1, 3, \ldots$, the coefficients A_k in (1.2) corresponding to t^k appear to be non-local. In Section 9, we clarify this remark by proving that they are indeed non-local in the sense of Definition 1.1 (i) (Theorem 1.3). In fact, we prove that the A_k's are not *cohomologically* local. By Remark 1.2 (ii), it also follows that the integrals $a_k := \int_M \operatorname{Tr} A_k \operatorname{dvol}_g$ are not locally computable in the sense of Definition 1.1 (ii). Therefore, the a_k's for odd k are also not locally computable in the sense of Gilkey and Grubb [14].

For $d = 1$, the non-local coefficients in the heat expansion (1.3) in [14] are $a_{n+1}, a_{n+2}, \ldots$, whereas in our case corresponding to $r = d/2 = 1/2$, the non-local coefficients are $a_1, a_3, \ldots$. Despite some formal resemblances, it appears therefore that the results of the present paper are quite different from those of [14].

1.3 The heat kernel as a conormal section

Recall that a smooth function f on the interior of a manifold with corners is said to be *polyhomogeneous conormal* if for any boundary hypersurface given by a boundary defining function θ, f has an expansion with terms of the form $\theta^k \log^l \theta$ toward $\{\theta = 0\}$ (only natural powers l are allowed). In [19], Melrose introduced the heat space M_H^2 by performing a parabolic blow-up of the diagonal in $M \times M$ at time $t = 0$. The new space is a manifold with corners with boundary hypersurfaces given by the boundary defining functions ρ and ω_0. Then the heat kernel p_t has the form $\rho^{-n}\mathcal{C}^\infty(M_H^2)$, and it vanishes rapidly at $\{\omega_0 = 0\}$ (see [19, Theorem 7.12]).

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In the special case $r = 1/2$, we are able to give a simultaneous formula for the asymptotic behavior of h_t as t goes to zero *both* on the diagonal and away from it. We can understand better the heat operator $e^{-t\Delta^{1/2}}$ on a *homogeneous* (rather than parabolic) *blow-up* heat space $\mathrm{M}_{\mathrm{heat}}$, the usual blow-up of $\{0\} \times \mathrm{Diag}$ in $[0,\infty) \times M \times M$. The new added face is called the *front face* and we denote it ff, whereas the lift of the old boundary is the *lateral boundary*, denoted lb.

Theorem 1.4 *If n is even, then the Schwartz kernel h_t of the operator $e^{-t\Delta^{1/2}}$ belongs to $\rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$, while if n is odd, $h_t \in \rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}}) + \rho \log\rho \cdot \omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$.*

Theorem 1.4 improves the results of [2] twofold. First, it holds true for non-negative generalized Laplacians. Second, while Bär–Moroianu describe the asymptotic behavior of h_t on the diagonal and away from it separately, this theorem also gives a precise, uniform description of the transition between these two regions by showing that h_t is a polyhomogeneous conormal section on $\mathrm{M}_{\mathrm{heat}}$ with values in $\mathcal{E} \boxtimes \mathcal{E}^*$.

Note that throughout the paper, integral kernels act on sections by integration with respect to the fixed Riemannian density from M in the second variable, so h_t does not contain a density factor. We feel that in the present context this exhibits more clearly the asymptotic behavior.

Based on the study of the case $r = 1/2$ and on the separate asymptotic expansions of the heat kernel h_t of Δ^r, $r \in (0,1)$ as t goes to 0 given in Theorems 6.1 and 7.1, we can conjecture that the heat kernel h_t is a polyhomogeneous conormal function for *all* $r \in (0,1)$ on a "*transcendental*" heat blow-up space M^r_{heat} depending on r. We leave this as a future project.

2 The heat kernel of a generalized Laplacian

Let $\mathcal{E}$ be a Hermitian vector bundle over a compact Riemannian manifold M of dimension n. Consider Δ to be a generalized Laplacian, i.e., a second-order differential operator which satisfies

$$\sigma_2(\Delta)(x,\xi) = |\xi|^2 \cdot \mathrm{id}_{\mathcal{E}}\,.$$

For example, if ∇ is a connection on $\mathcal{E}$ and $F \in \Gamma(\mathrm{End}\,\mathcal{E})$, $F^* = F$, then $\nabla^*\nabla + F$ is a symmetric generalized Laplacian on $\mathcal{E}$.

Suppose that Δ is self-adjoint. Since M is compact, the spectrum of Δ is discrete and $L^2(M,\mathcal{E})$ splits as an orthogonal Hilbert direct sum

$$L^2(M,\mathcal{E}) = \bigoplus_{\lambda\in\mathrm{Spec}\,\Delta}^{\perp} E_\lambda,$$

where E_λ is the eigenspace corresponding to the eigenvalue λ of Δ. Moreover, $\dim E_\lambda < \infty$ and by elliptic regularity, the eigensections are smooth (see, e.g., [8]). Let $e^{-t\Delta}$ be the *heat operator* defined as

$$e^{-t\Delta}\Phi = e^{-t\lambda}\Phi,$$

for any $\Phi \in E_\lambda$, $\lambda \in \mathrm{Spec}\,\Delta$.

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Definition 2.1 The *heat kernel* of a self-adjoint elliptic pseudodifferential operator P acting on the sections of $\mathcal{E}$ is the Schwartz kernel of the operator e^{-tP}.

If we denote by $\{\Phi_j\}$ an orthonormal Hilbert basis of Δ-eigensections, then the heat kernel $p_t(x,y)$ satisfies

$$p_t(x,y) = \sum_j e^{-t\lambda_j}\Phi_j(x) \otimes \Phi_j^*(y)$$

in $\mathcal{C}^\infty((0,\infty) \times M \times M)$.

Recall that the L^2-product of two sections $s_1, s_2 \in \Gamma(\mathcal{E})$ is given by

$$\langle s_1, s_2\rangle_{L^2(\mathcal{E})} = \int_M h_{\mathcal{E}}(s_1, s_2)\,\mathrm{dvol}_g,$$

where g is the metric on M and $h_{\mathcal{E}}$ is the Hermitian product on $\mathcal{E}$.

Let $y \in M$ be a fixed point. We work in geodesic normal coordinates defined by the exponential map

$$\exp_y : T_yM \longrightarrow M.$$

Since M is compact, there exists a global injectivity radius ε. For x close enough to y ($d(x,y) \le \varepsilon$), take $\mathrm{x} \in T_yM$ the unique tangent vector of length smaller than ε such that $x = \exp_y \mathrm{x}$. Let

$$\mathrm{j}(\mathrm{x}) = \frac{\exp_y^* dx}{d\mathrm{x}},$$

namely the pull-back of the volume form dx on M through the exponential map $\exp_y$ is equal with $\mathrm{j}(\mathrm{x})d\mathrm{x}$. More precisely,

$$\mathrm{j}(\mathrm{x}) = |\det\left(d_{\mathrm{x}}\exp_{x_0}\right)| = \det{}^{1/2}\left(g_{ij}(\mathrm{x})\right).$$

Denote by $\tau_x^y : \mathcal{E}_x \longrightarrow \mathcal{E}_y$ the parallel transport along the unique minimal geodesic $x_s = \exp_y(s\mathrm{x})$, where $s \in [0,1]$, which connects the points x and y. The heat kernel $p_t(x,y)$ belongs to the space $\mathcal{C}^\infty\left((0,\infty) \times M \times M, \mathcal{E}_x \otimes \mathcal{E}_y^*\right)$ and $p_t(x,y)$ satisfies the heat equation

$$(\partial_t + \Delta_x)\,p_t(x,y) = 0.$$

Furthermore, $\lim_{t\to 0} P_t s = s$, in $\|\cdot\|_0$, for any smooth section $s \in \Gamma(M,\mathcal{E})$, where

$$(P_t s)(x) = \int_M p_t(x,y)s(y)dg(y),$$

where $dg(y)$ is the Riemannian density of the metric g. The next theorem is due to Minakshisundaram and Pleijel (see, for instance, [4, 21]).

Theorem 2.1 *The heat kernel p_t has the following asymptotic expansion near the diagonal:*

$$p_t(x,y) \overset{t\searrow 0}{\sim} (4\pi t)^{-n/2} e^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{\infty} t^i \Psi_i(x,y),$$

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where $\Psi_i : \mathcal{E}_y \longrightarrow \mathcal{E}_x$ *are* $\mathcal{C}^\infty$ *sections defined near the diagonal. Moreover, the* Ψ_i*'s are given by the following explicit formulæ:*

$$\Psi_0(x,y) = \mathrm{j}^{-1/2}(\mathrm{x})\tau_y^x,$$
$$\tau_x^y \Psi_i(x,y) = -\mathrm{j}^{-1/2}(\mathrm{x}) \int_0^1 s^{i-1} \mathrm{j}^{-1/2}(x_s)\tau_{x_s}^y \, \Delta_x \, \Psi_{i-1}(x_s, y) ds.$$

The asymptotic sum in Theorem 2.1 can be understood using truncation and bounds of derivatives as in [5]. We prefer the interpretation given in [19], where the heat kernel p_t is shown to belong to $\rho^{-n}\mathcal{C}^\infty(M_H^2)$ on the parabolic blow-up space M_H^2 and to vanish rapidly at the temporal boundary face $\{\omega_0 = 0\}$ (see Section 10).

Example 2.2 Let $\mathbb{T}^n = \left(S^1\right)^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be the n-dimensional torus with the standard product metric $g = d\theta_1^2 \otimes \cdots \otimes d\theta_n^2$. Consider the trivial bundle $\mathcal{E} = \underline{\mathbb{C}}$ over $\mathbb{T}^n$ with the standard metric $h_\mathcal{E}$, the trivial connection $\nabla = d$, and the zero endomorphism F. Let Δ_1 be the Laplacian on $\mathbb{T}^n$ given by the metric g. The eigenvalues of Δ_1 are $\{k_1^2 + \cdots + k_n^2 : k_1, \ldots, k_n \in \mathbb{Z}\}$. Let $\varphi_l(\xi) = \frac{1}{\sqrt{2\pi}} e^{il\xi}$ be the standard orthonormal basis of eigenfunctions of each Δ_{S^1}. Then, for $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$, the heat kernel p_t of Δ_1 is the following:

$$p_t(\theta,\theta) = \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} e^{-t(k_1^2+\cdots+k_n^2)} \varphi_{k_1}(\theta_1)\overline{\varphi_{k_1}(\theta_1)} \ldots \varphi_{k_n}(\theta_n)\overline{\varphi_{k_n}(\theta_n)}.$$

Since $\varphi_l(\xi)\overline{\varphi_l(\xi)} = \frac{1}{2\pi}$, for any $\xi \in S^1$, we get

$$p_t(\theta,\theta) = \frac{1}{(2\pi)^n} \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} e^{-t(k_1^2+\cdots+k_n^2)}.$$

Remark that the Fourier transform of the function $f_t : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f_t(x) = e^{-t|x|^2}$ is given by

$$\hat{f}_t(\xi) = \frac{\pi^{n/2}}{t^{n/2}} e^{-\frac{|\xi|^2}{4t}}.$$

Using the multidimensional Poisson formula (see, for instance, [3]), we obtain that

$$p_t(\theta,\theta) = \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} f_t(k) = \sum_{k\in\mathbb{Z}^n} \hat{f}_t(2\pi k) = \frac{\pi^{n/2}}{(2\pi)^n} t^{-n/2} + \frac{\pi^{n/2}}{(2\pi)^n} t^{-n/2} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{\pi^2|k|^2}{t}}.$$

Since the last sum is of order $\mathcal{O}\left(e^{-\frac{1}{t}}\right)$ as $t \to 0$, it follows that the first coefficient in the asymptotic expansion at small-time t of p_t is $\frac{\pi^{n/2}}{(2\pi)^n}$ and all the others vanish.

From now on, suppose that Δ is non-negative (i.e., $h_\mathcal{E}(\Delta f, f) \geq 0$, for any $f \in \mathcal{C}^\infty(M,\mathcal{E})$). For $s \in \mathbb{C}$, we define the complex powers $\Delta^{-s} \in \Psi^{-2s}(M,\mathcal{E})$ of Δ as

$$\Delta^{-s}\,\Phi = \begin{cases} \lambda^{-s}\Phi, & \text{if } \Phi \in E_\lambda,\ \lambda \neq 0, \\ 0, & \text{if } \Phi \in \operatorname{Ker}\Delta. \end{cases}$$

Remark that $(\Delta^s)_{s\in\mathbb{C}}$ is a holomorphic family of pseudodifferential operators. Let $r \in (0,1)$. We denote by h_t the heat kernel of Δ^r, namely the Schwartz kernel of the operator $e^{-t\Delta^r}$. We have seen that

$$p_t(x,x) \overset{t\searrow 0}{\sim} t^{-n/2}\sum_{j=0}^{\infty} t^j a_j(x,x), \tag{2.1}$$

with smooth sections $a_j(x,x) \in \mathcal{E}_x \otimes \mathcal{E}_x^*$.

3 The link between the heat kernel and complex powers of the Laplacian

Proposition 1 (Mellin Formula) *With the notations above, for $\Re s > 0$, we have*

$$\Delta^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(e^{-t\Delta} - \mathrm{P}_{\mathrm{Ker}\,\Delta}\right) dt,$$

where $\mathrm{P}_{\mathrm{Ker}\,\Delta}$ is the orthogonal projection onto the kernel of Δ.

Proof It is straightforward to check that both sides coincide on eigensections $\Phi \in E_\lambda$, $\lambda \in \operatorname{Spec}\Delta$. Since $\{\Phi_j\}_j$ is a Hilbert basis, the result follows. ■

We will write $\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ for the Schwartz kernel $\sum_k \varphi_k(x) \otimes \varphi_k^*(y)$, where $\{\varphi_k\}$ is an orthonormal basis in $\mathrm{Ker}\,\Delta$. Denote by q_{-s} the Schwartz kernel of the operator Δ^{-s}. Let us first study the poles and the zeros of q_{-s} away from the diagonal.

Proposition 2 *Let K be a compact in $M \times M \setminus \mathrm{Diag}$. Then, for $(x,y) \in K$, the function $s \longmapsto q_{-s}|_K \in \mathcal{C}^\infty(K, \mathcal{E} \boxtimes \mathcal{E}^*)$ is entire. Moreover, $q_{-s}|_K$ vanishes at each negative integer s.*

Proof For $\Re s > 0$, let $f_{x,y}(s) = \int_0^\infty t^{s-1}(p_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y))\,dt$. Remark that

$$\begin{aligned} f_{x,y}(s) &= \int_0^\infty t^{s-1}(p_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y))\,dt \\ &= \int_1^\infty t^{s-1}(p_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y))\,dt \\ &\quad + \int_0^1 t^{s-1} p_t(x,y)dt - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y) \cdot \int_0^1 t^{s-1}dt. \end{aligned}$$

Since $p_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ decays exponentially fast as t goes to ∞, the first integral is absolutely convergent in C^k norms. The heat kernel p_t vanishes with all of its derivatives as $t \searrow 0$ in the compact K, thus the second integral is also absolutely convergent. The last integral term is well-defined for $\Re s > 0$, and it extends to a meromorphic function on $\mathbb{C}$ with a simple pole in $s = 0$. Therefore, $s \mapsto f_{x,y}(s)$ extends to a meromorphic function on $\mathbb{C}$. By Proposition 1 and the identity theorem, the equality of meromorphic functions

$$\Gamma(s) q_{-s}(x,y) = f_{x,y}(s)$$

holds for any $s \in \mathbb{C}$. In particular, we obtain $q_0(x,y) = -\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$. Furthermore, $q_{-s}|_K$ is an entire function and vanishes in $s = -1, -2, \ldots$. ■

Remark 3.1 The fact that $q_{-s}|_K$ vanishes for negative integers s also follows from the fact that then Δ^{-s} is a differential operator.

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Now we check the behavior of q_{-s} along the diagonal. It is no longer holomorphic there, and the coefficients $a_j(x,x)$ from (2.1) appear as residues of $q_{-s}(x,x)$.

Proposition 3 *Let $x \in M$. Then the function $s \mapsto \Gamma(s)q_{-s}(x,x)$ has a meromorphic extension from the set $\{s \in \mathbb{C} : \Re s > \frac{n}{2}\}$ to $\mathbb{C}$ with simple poles in $s \in \{0\} \cup \{\frac{n}{2} - j : j \in \mathbb{N}\}$. The residue of $\Gamma(s)q_{-s}(x,x)$ in $s = \frac{n}{2} - j$, $j \neq \frac{n}{2}$, is $a_j(x,x)$. If n is even, then the residue of $\Gamma(s)q_{-s}(x,x)$ in $s = 0$ is $a_{\frac{n}{2}}(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)$. If n is odd, the residue in $s = 0$ is $-\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)$ and the meromorphic extension of $q_{-s}(x,x)$ vanishes at $s \in \{-1,-2,\dots\}$.*

Proof Consider the function $f_{x,x}(s) = \int_0^\infty t^{s-1}\left(p_t(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)\right)dt$ for $\Re s > \frac{n}{2}$. We have

$$\begin{aligned} f_{x,x}(s) &= \int_0^\infty t^{s-1}\left(p_t(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)\right)dt \\ &= \int_1^\infty t^{s-1}\left(p_t(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)\right)dt \\ &\quad + \int_0^1 t^{s-1}p_t(x,x)dt - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)\cdot\int_0^1 t^{s-1}dt. \end{aligned}$$

The first integral is absolutely convergent, as seen in the proof of Proposition 2. The last integral term is meromorphic with a simple pole at $s = 0$ with residue $-\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)$. Let us analyze the behavior of the second term $A_x(s) = \int_0^1 t^{s-1}p_t(x,x)dt$.

Using (2.1), we have that for $N \geq 0$,

$$t^{n/2}p_t(x,x) = \sum_{j=0}^{N} t^j a_j(x,x) + R_{N+1}(t,x),$$

where R_{N+1} is of order $\mathcal{O}(t^{N+1})$ as $t \to 0$. Furthermore, we obtain

$$\begin{aligned} A_x(s) = \int_0^1 t^{s-\frac{n}{2}-1}t^{\frac{n}{2}}p_t(x,x)dt &= \sum_{j=0}^{N}\int_0^1 t^{s-\frac{n}{2}-1}t^j a_j(x,x)dt + \int_0^1 t^{s-\frac{n}{2}-1}R_{N+1}(t,x)dt \\ &= \sum_{j=0}^{N} a_j(x,x)\frac{1}{s-\frac{n}{2}+j} + \int_0^1 t^{s-\frac{n}{2}-1}R_{N+1}(t,x)dt. \end{aligned}$$

Thus $s \mapsto A_x(s)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles in $\{\frac{n}{2} - j : \ j = \overline{0, N+1}\}$. Using again Proposition 1 and the identity theorem, we deduce the equality

$$\Gamma(s)q_{-s}(x,x) = f_{x,x}(s),$$

for any $s \in \mathbb{C}$. It follows that $\Gamma(s)q_{-s}(x,x)$ is meromorphic on $\mathbb{C}$ with simple poles in $s \in \{0\} \cup \{\frac{n}{2} - j : j \in \mathbb{N}\}$. Moreover, the residue of $\Gamma(s)q_{-s}(x,x)$ in a pole $\frac{n}{2} - j$ is $a_j(x,x)$, and the conclusion follows. ∎

For $p \in \mathbb{C}$ and $\varepsilon > 0$, let $B_\varepsilon(p)$ be the open disk centered in p of radius ε. We need the following technical result.

Proposition 4 *Consider $\alpha < \beta$, and let $\varepsilon > 0$, $l \in \mathbb{N}$.*

- *If K is a compact set disjoint from the diagonal, then the function $s \longmapsto \Gamma(s) q_{-s|_K}$ is uniformly bounded in $\{s \in \mathbb{C} : \alpha \leq \Re s \leq \beta\} \backslash B_\varepsilon(0)$ in the $\mathcal{C}^l$ norm on K.*
- *The function $s \longmapsto \Gamma(s) q_{-s|_{\mathrm{Diag}}}$ defined on $\{s \in \mathbb{C} : \alpha \leq \Re s \leq \beta\} \backslash \bigcup_{j \in \mathbb{N} \cup \{\frac{n}{2}\}} B_\varepsilon(\frac{n}{2} - j) \longrightarrow \mathcal{C}^l(\mathrm{Diag}, \mathcal{E} \otimes \mathcal{E}^*)$ is uniformly bounded.*

Proof With the same argument as in the proof of Proposition 2, the restriction of the $\mathcal{C}^l$ norm on K of the function $s \mapsto f_{x,y}(s)$ is absolutely convergent in $\{s \in \mathbb{C} : \alpha \leq \Re s \leq \beta\} \backslash B_\varepsilon(0)$, hence it is uniformly bounded.

As in the proof of Proposition 3, the $\mathcal{C}^l$ norm along Diag of $s \longmapsto f_{x,x}(s)$ converges absolutely in $\{s \in \mathbb{C} : \alpha \leq \Re s \leq \beta\} \backslash \bigcup_{j \in \mathbb{N} \cup \{\frac{n}{2}\}} B_\varepsilon(\frac{n}{2} - j)$, thus the conclusion follows. ∎

4 The behavior of quotients of Gamma functions along vertical lines

A fundamental result used in [2] is the Legendre duplication formula

$$\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{\sqrt{2\pi}} 2^{s-\frac{1}{2}} \Gamma\left(\frac{s+1}{2}\right),$$

together with the rapid decay of the Gamma function in vertical lines $\Re s = \tau$ (see, e.g., [22]). These results are replaced in our case by the following estimate.

Proposition 5 *The function $s \longmapsto \frac{\Gamma(s)}{\Gamma(rs)}$ decreases in vertical lines faster than $|s|^{-k}$, for any $k \geq 0$, uniformly in each strip $\{s \in \mathbb{C} : \alpha \leq \Re(s) \leq \beta\}$, for any $\alpha, \beta \in \mathbb{R}$.*

Proof For $z \in \mathbb{C} \backslash \mathbb{R}_-$, recall the Stirling formula (see, for instance, [23])

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \Omega(z),$$

where log is defined on its principal branch, and Ω is an analytic function of z. For $|\arg z| < \pi$ and $|z| \to \infty$, Ω can be written as

$$\Omega(z) = \sum_{j=1}^{N-1} \frac{B_{2j}}{2j(2j-1) z^{2j-1}} + R_N(z),$$

where B_{2j} are the Bernoulli numbers $\left(B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \text{etc.}\right)$. Moreover, the error term satisfies

$$|R_N(z)| \leq \frac{|B_{2N}|}{2N(2N-1)} \cdot \frac{\sec^{2N}\left(\frac{\arg z}{2}\right)}{|z|^{2N-1}};$$

thus, $R_N(z)$ is of order $\mathcal{O}\left(|z|^{-2N+1}\right)$ as $|z| \to \infty$ (see, for instance, [22, equation (2.1.6)]). For $s \notin (-\infty, 0)$, it follows that

$$\frac{\Gamma(s)}{\Gamma(rs)} = s^{-s(r-1)} e^{s(r-1)} r^{\frac{1}{2}-rs} e^{\Omega(s)-\Omega(rs)}.$$

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Let $s = a + ib$, $a \in \mathbb{R}$ fixed. As $|b| \to \infty$, the difference $|\Omega(s) - \Omega(rs)| \to 0$; thus, $|e^{\Omega(s)-\Omega(rs)}| \to 1$. Note that $|r^{\frac{1}{2}-rs}| = |r^{\frac{1}{2}-ra}|$ and $|e^{(r-1)s}| = e^{(r-1)a}$, so these terms are bounded. We show in Lemma 4.1 that for any $k \geq 0$, $|s|^k |s^s|$ goes to 0 as $\Re s = a$ is fixed and $|\operatorname{Im} s|$ tends to ∞. It follows that the quotient $\frac{\Gamma(s)}{\Gamma(rs)}$ indeed decreases in vertical lines faster than $|s|^{-k}$, for any $k \geq 0$, uniformly in vertical strips. ∎

Lemma 4.1 *Let $k \geq 0$. If $a \in \mathbb{R}$ is fixed and $|b| \to \infty$, then $|(a+ib)^{k+a+ib}|$ tends to zero.*

Proof Let $s = a + ib \notin (-\infty, 0)$ and set $\log(a+ib) = x + iy$. Then $x = \log\sqrt{a^2+b^2}$, $y = \arg s \in (-\pi, \pi)$; hence,

$$|s^{s+k}| = |e^{(k+a+ib)\log(a+ib)}| = e^{(k+a)x-by} = e^{(k+a)\log\sqrt{a^2+b^2}-b\arg s}.$$

Since $b = \tan\arg s \cdot a$, the exponent is equal to

$$\begin{aligned}(4.1)\qquad &(k+a)\log\sqrt{a^2+b^2} - b\arg s\\ &= (k+a)\log a + \frac{k+a}{2}\log\left(1+\tan^2\arg s\right) - a\tan\arg s\cdot\arg s.\end{aligned}$$

If $a > 0$, then $\arg s \nearrow \frac{\pi}{2}$ or $\arg s \searrow -\frac{\pi}{2}$, and in both cases $t := \tan\arg s$ tends to ∞. The exponent (4.1) behaves as the function $t \longmapsto \log(1+t^2) - t$; therefore, as $t \to \infty$, the exponent goes to $-\infty$ and the statement of the claim follows.

If $a < 0$, then $\arg s \searrow \frac{\pi}{2}$ or $\arg s \nearrow -\frac{\pi}{2}$. In the first case when $\arg s \searrow \frac{\pi}{2}$, it follows that $t = \tan\arg s \to -\infty$. The exponent (4.1) behaves as $\pm\log(1+t^2) + t$; hence, the conclusion follows. While if $\arg s \nearrow -\frac{\pi}{2}$, then $t \to \infty$, and the exponent (4.1) behaves as $\pm\log(1+t^2) - t$; thus, the exponent tends again to $-\infty$. Therefore, $|s^{k+s}|$ goes to zero, which ends the proof. ∎

5 Link between the complex powers of Δ and the heat kernel of Δ^r

Proposition 6 (Inverse Mellin Formula) *For $\Re\tau > 0$, the operators $e^{-t\Delta^r}$ and Δ^{-s} are related by the following formula:*

$$e^{-t\Delta^r} - \mathrm{P}_{\operatorname{Ker}\Delta} = \frac{1}{2\pi i}\int_{\Re s=\tau} t^{-s}\Gamma(s)\,\Delta^{-rs}\,ds.$$

Proof The equality holds on each eigensection Φ_j corresponding to an eigenvalue $\lambda_j \in \operatorname{Spec}\Delta$. Since $\{\Phi_j\}_j$ is a Hilbert basis, the result follows. ∎

Set $\tau > \frac{n}{2r}$. Then the Schwartz kernel q_{-rs} of Δ^{-rs} is continuous and by the inverse Mellin formula, we get an identity which relates the Schwartz kernels h_t and q_{-rs}:

$$\begin{aligned} h_t(x,y) - \mathrm{P}_{\operatorname{Ker}\Delta}(x,y) &= \tfrac{1}{2\pi i}\int_{\Re s=\tau} t^{-s}\Gamma(s) q_{-rs}(x,y)ds\\ &= \tfrac{1}{2\pi i}\int_{\Re s=\tau} t^{-s}\frac{\Gamma(s)}{\Gamma(rs)}\cdot\Gamma(rs)q_{-rs}(x,y)ds.\end{aligned}$$

Now let $k > 0$. By changing τ to $\tau + \varepsilon$ (for a small $\varepsilon > 0$) if needed, we can assume that $\tau - k \notin \{\frac{n}{2} - j : j \in \mathbb{N}\} \cup \{0\}$. Using Propositions 4 and 5, we can apply the residue

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formula and move the line of integration to the left:

$$h_t(x,y) = \frac{1}{2\pi i}\int_{\Re s=\tau-k} t^{-s}\frac{\Gamma(s)}{\Gamma(rs)}\cdot\Gamma(rs)q_{-rs}(x,y)ds \tag{5.1}$$
$$+\sum_{s\in-\mathbb{N}\cup\{\frac{n-2j}{2r}:\ j\in\mathbb{N}\}} \operatorname{Res}_s\left(t^{-s}\Gamma(s)q_{-rs}(x,y)\right) + \mathrm{P}_{\operatorname{Ker}\Delta}(x,y).$$

Notice that $-\mathbb{N}\cup\{\frac{n-2j}{2r}:\ j\in\mathbb{N}\}$ is the set of all possible poles of $s\mapsto\Gamma(s)q_{-rs}(x,y)$, but some of them might actually be regular points. We will study the sum (5.1) in detail in Theorems 6.1 and 7.1.

Let K be a compact set in $M\times M\setminus\operatorname{Diag}$ and $l\in\mathbb{N}$. Remark that the integral term in (5.1) is of order $\mathcal{O}\left(t^{k-\tau}\right)$ in $\mathcal{C}^l(K,\mathcal{E}\boxtimes\mathcal{E}^*)$. Indeed,

$$\left\|\int_{\Re s=\tau-k} t^{-s}\Gamma(s)q_{-rs|_K}ds\right\|_l \le t^{-\tau+k}\cdot\int_{s=\tau-k+iu}\left\|\frac{\Gamma(s)}{\Gamma(rs)}\cdot\Gamma(rs)q_{-rs|_K}\right\|_l du,$$

and using again Propositions 4 and 5, the claim follows. Furthermore, when k goes to ∞, we get

$$h_{t|_K} \overset{t\searrow 0}{\sim} \sum_{\alpha=0}^{\infty} t^{\alpha}\cdot\operatorname{Res}_{s=-\alpha}\left(\Gamma(s)q_{-rs|_K}\right) + t^0\cdot\mathrm{P}_{\operatorname{Ker}\Delta|_K}, \tag{5.2}$$

The meaning of the asymptotic sign in (5.2) is that if we set h_t^N to be the right-hand side in (5.2) restricted to $\alpha\le N$, then the difference $|\partial_t^j\left(h_{t|_K}-h_t^N\right)|$ is of order $\mathcal{O}(t^{N+1-j})$ in $\mathcal{C}^l(K,\mathcal{E}\boxtimes\mathcal{E}^*)$, for any $N,j\in\mathbb{N}$.

Remark that using again Propositions 4 and 5, the integral term in (5.1) is of order $\mathcal{O}\left(t^{k-\tau}\right)$ in $\mathcal{C}^l(\operatorname{Diag},\mathcal{E}\otimes\mathcal{E}^*)$. Therefore when k tends to ∞, we obtain

$$h_{t|_{\operatorname{Diag}}} \overset{t\searrow 0}{\sim} \sum_{\alpha\in(-\mathbb{N})\cup\{\frac{n-2j}{2r}:j\in\mathbb{N}\}} t^{-\alpha}\cdot\operatorname{Res}_{s=\alpha}\left(\Gamma(s)q_{-rs|_{\operatorname{Diag}}}\right) + t^0\cdot\mathrm{P}_{\operatorname{Ker}\Delta|_{\operatorname{Diag}}}, \tag{5.3}$$

in the sense of the following:

Definition 5.1 Consider $l\in\mathbb{N}$ and let $A,B\subset\mathbb{R}$. We say that $h_{t|_{\operatorname{Diag}}} \overset{t\searrow 0}{\sim} \sum_{\alpha\in A} t^{\alpha}c_{\alpha} + \sum_{\beta\in B} t^{\beta}\log t\cdot c_{\beta}$ if for any $k,N\in\mathbb{N}$, the difference

$$\partial_t^j\left(h_{t|_{\operatorname{Diag}}} - \sum_{\alpha\le N} t^{\alpha}c_{\alpha} - \sum_{\beta\le N} t^{\beta}\log t\cdot c_{\beta}\right)$$

is of order $\mathcal{O}(t^{N+1-j}\log t)$ in $\mathcal{C}^l(\operatorname{Diag},\mathcal{E}\otimes\mathcal{E}^*)$.

6 The asymptotic expansion of h_t away from the diagonal

Theorem 6.1 *The Schwartz kernel h_t of the operator $e^{-t\Delta^r}$ is $\mathcal{C}^\infty$ on $[0,\infty)\times(M\times M\setminus\operatorname{Diag})$. Furthermore, let $K\subset M\times M\setminus\operatorname{Diag}$ be a compact set. Then the Taylor series of $h_{t|_K}$ as $t\searrow 0$ is the following:*

$$h_{t|_K} \overset{t\searrow 0}{\sim} \sum_{j=1}^{\infty} t^j q_{rj|_K}\frac{(-1)^j}{j!}.$$

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Moreover, if $r = \frac{\alpha}{\beta}$ is rational with α, β coprime, then the coefficient of t^j vanishes for $j \in \beta\mathbb{N}^*$.

Proof Let $j \in \mathbb{N}$. Using Propositions 4 and 5, $(-s)(-s-1)\dots(-s-j+1)t^{-s-j}\frac{\Gamma(s)}{\Gamma(rs)}\Gamma(rs)q_{-rs}{}_{|K}$ is L^1 integrable on $\Re s = \tau - k$ in $\mathcal{C}^l(K, \mathcal{E} \boxtimes \mathcal{E}^*)$, for sufficiently large k and for any $l \in \mathbb{N}$. It follows that h_t is $\mathcal{C}^\infty$ on $(0,\infty) \times (M \times M \backslash \mathrm{Diag})$. By Proposition 2, the function $s \mapsto q_{-rs}(x,y)$ is entire for any $(x,y) \in K$. Since $\mathrm{Res}_{s=-j}\Gamma(s) = \frac{(-1)^j}{j!}$, using (5.2) we get

$$h_{t|_K} \overset{t\searrow 0}{\sim} \sum_{j=0}^{\infty} t^j q_{rj}{}_{|_K} \frac{(-1)^j}{j!} + \mathrm{P}_{\mathrm{Ker}\,\Delta|_K}.$$

We obtained in the proof of Proposition 2 that $q_{0|_K} = -\mathrm{P}_{\mathrm{Ker}\,\Delta|_K}$; thus,

$$h_{t|_K} \overset{t\searrow 0}{\sim} \sum_{j=1}^{\infty} t^j q_{rj}{}_{|_K} \frac{(-1)^j}{j!},$$

and therefore $h_{t_{|_K}}$ is $\mathcal{C}^\infty$ also at $t=0$, and vanishes at order 1. Moreover, using again Proposition 2, if $r = \frac{\alpha}{\beta}$ is rational and j is a non-zero multiple of β, then $q_{rj}{}_{|_K} \equiv 0$ and the conclusion follows. ■

7 The asymptotic expansion of h_t along the diagonal

To obtain the coefficients in the asymptotic of h_t along the diagonal as $t \searrow 0$, we need to compute the residues from (5.3). Some of them are related to the heat coefficients a_j's of p_t due to Proposition 3. We will distinguish three cases. If n is even, $\Gamma(s)q_{-rs}(x)$ has simple poles in $\{\frac{n}{2r}, \frac{n-2}{2r}, \dots, \frac{2}{2r}\} \cup \{0, -1, \dots\}$ and the residues will give rise to real powers of t. If n is odd and either r is irrational or r is rational with odd denominator, $\Gamma(s)q_{-rs}(x)$ has simple poles in $\{0,-1,\dots\} \cup \{\frac{n-2j}{2r} : j = 0,1,\dots\}$. Otherwise, if n is odd and r is rational with even denominator, then there exist some double poles which give rise to logarithmic terms in the asymptotic expansion of h_t.

Theorem 7.1 *Let $a_j(x,x)$ be the coefficients in (2.1) of the heat kernel p_t of the non-negative self-adjoint generalized Laplacian Δ. The asymptotic expansion of the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$, $r \in (0,1)$ along the diagonal when $t \searrow 0$ is the following:*

(1) *If n is even, then*

$$h_{t|_{\mathrm{Diag}}} \overset{t\searrow 0}{\sim} \sum_{j=0}^{n/2-1} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + a_{n/2} + \sum_{j=1}^{\infty} t^j \cdot A_j.$$

If $r = \frac{\alpha}{\beta}$ is rational, for $j = l\beta$, $l \in \mathbb{N}^*$, we obtain that $q_{rj}(x,x) = (-1)^j \cdot j! \cdot a_{\frac{n}{2}+l\alpha}(x,x)$, and the coefficient of $t^{l\beta}$ can be described more precisely as

$$A_{l\beta} = a_{\frac{n}{2}+l\alpha}.$$

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(2) *If n is odd and either $r \in \mathbb{R}\backslash\mathbb{Q}$ or the denominator of r is odd, then*

$$h_{t|_{\mathrm{Diag}}} \overset{t\searrow 0}{\sim} \sum_{j=0}^{(n-1)/2} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + \sum_{j=1}^{\infty} t^{j} \cdot A_j + \sum_{j=1}^{\infty} t^{\frac{2j+1}{2r}} \cdot A_{\frac{2j+1}{2r}}.$$

Moreover, if $r = \frac{\alpha}{\beta}$ is rational and β is odd, then $A_{l\beta} \equiv 0$ for any $l \in \mathbb{N}^*$.

(3) *If n is odd, $r = \frac{\alpha}{\beta}$ is rational and its denominator β is even, then*

$$h_{t|_{\mathrm{Diag}}} \overset{t\searrow 0}{\sim} \sum_{j=0}^{(n-1)/2} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + \sum_{\substack{j=1\\ \alpha \nmid 2j+1}}^{\infty} t^{\frac{2j+1}{2r}} \cdot A_{\frac{2j+1}{2r}} + \sum_{\substack{j=1\\ \frac{\beta}{2} \nmid j}}^{\infty} t^{j} \cdot A_j$$
$$+ \sum_{\substack{l=1\\ l \text{ odd}}}^{\infty} t^{l\frac{\beta}{2}} \cdot A_{l\frac{\beta}{2}} + \sum_{\substack{l=1\\ l \text{ odd}}}^{\infty} t^{l\frac{\beta}{2}} \log t \cdot B_{l\frac{\beta}{2}}.$$

In all these cases, the coefficients are

$$A_{-\frac{n-2j}{2r}}(x) = \frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)} \cdot \frac{1}{r} \cdot a_j(x,x), \qquad A_j(x) = \frac{(-1)^j}{j!} \cdot q_{rj}(x,x),$$

$$A_{\frac{2j+1}{2r}}(x) = \frac{\Gamma\left(-\frac{2j+1}{2r}\right)}{\Gamma\left(-\frac{2j+1}{2}\right)} \cdot \frac{1}{r} \cdot a_{\frac{n+2j+1}{2}}(x,x), \quad B_{l\frac{\beta}{2}}(x) = \frac{(-1)^{l\frac{\beta}{2}}}{r\left(l\frac{\beta}{2}\right)!\Gamma\left(-l\frac{\beta}{2}\cdot r\right)} \cdot a_{\frac{n+l\alpha}{2}}(x,x),$$

$$A_{l\frac{\beta}{2}}(x) = \frac{(-1)^{l\frac{\beta}{2}}}{(l\frac{\beta}{2})!\Gamma(-rl\frac{\beta}{2})} \cdot \mathrm{FP}_{s=-l\frac{\beta}{2}} \left(\Gamma(rs) q_{-rs}(x,x)\right) + \mathrm{FP}_{s=-l\frac{\beta}{2}} \left(\frac{\Gamma(s)}{\Gamma(rs)}\right) \cdot \frac{a_{\frac{n+l\alpha}{2}}(x,x)}{r}.$$

Proof We compute the coefficients from (5.3) by using Proposition 3. ■

7.1 The case when *n* is even

For $j \in \{0, 1, \ldots, n/2 - 1\}$, we have

$$\mathrm{Res}_{s=\frac{n-2j}{2r}} \left(t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{-rs}(x,x)\right) = t^{-\frac{n-2j}{2r}} \cdot \frac{\Gamma(\frac{n-2j}{2r})}{\Gamma(\frac{n-2j}{2})} \cdot \frac{a_j(x,x)}{r}. \tag{7.1}$$

The residue in $s = 0$ is given by

$$\mathrm{Res}_{s=0} \left(t^{-s}\Gamma(s) q_{-rs}(x,x)\right) = \mathrm{Res}_{s=0} \left(t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{-rs}(x,x)\right)$$
$$= r \cdot \frac{1}{r}\left(a_{\frac{n}{2}}(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x)\right) = a_{\frac{n}{2}}(x,x) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,x),$$

thus the coefficient of t^0 in the asymptotic expansion (5.3) is $a_{\frac{n}{2}}(x,x)$.

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7.1.1 The case when *n* is even and *r* is irrational

Let $j \in \mathbb{N}^*$. Then

$$\operatorname{Res}_{s=-j}\left(t^{-s}\Gamma(s)q_{-rs}(x,x)\right) = t^j \frac{(-1)^j}{j!} \cdot q_{rj}(x,x). \tag{7.2}$$

Therefore, in this case, the asymptotic expansion of h_t is the following:

$$h_t(x,x) \overset{t\searrow 0}{\sim} \sum_{j=0}^{n/2-1} t^{-\frac{n-2j}{2r}} \frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)} \frac{a_j(x,x)}{r} + a_{\frac{n}{2}}(x,x) + \sum_{j=1}^{\infty} t^j \frac{(-1)^j}{j!} q_{rj}(x,x).$$

7.1.2 The case when *n* is even and $r = \frac{\alpha}{\beta}$ is rational with $(\alpha,\beta) = 1$

Some of the coefficients $q_{rj}(x,x)$ from (7.2) can be expressed in terms of the a_k's from (2.1). Remark that $\frac{\Gamma(s)}{\Gamma(rs)}$ has simple poles in $\{-1,-2,\dots\}\backslash\{\frac{-1}{r},\frac{-2}{r},\dots\}$. For $j \in \mathbb{N}^*$, $s := -\frac{j}{r} \in \{-1,-2,\dots\}$ if and only if j is a multiple of α, which is equivalent to $s = \frac{-l\alpha}{r} = -l\beta$ for some $l \in \mathbb{N}^*$. In this case, we obtain

$$\begin{aligned}\operatorname{Res}_{s=-l\beta}\left(t^{-s}\Gamma(s)q_{-rs}(x,x)\right) &= \operatorname{Res}_{s=-l\beta}\left(t^{-s}\frac{\Gamma(s)}{\Gamma(rs)}\Gamma(rs)q_{-rs}(x,x)\right)\\ &= t^{l\beta} r \cdot \frac{1}{r} a_{\frac{n}{2}+l\alpha}(x,x) = t^{l\beta} a_{\frac{n}{2}+l\alpha}(x,x).\end{aligned}$$

Hence, for rational $r = \frac{\alpha}{\beta}$, if $j = l\beta$, $l \in \mathbb{N}^*$, we conclude that

$$q_{rj}(x,x) = (-1)^j \cdot j! \cdot a_{\frac{n}{2}+l\alpha}(x,x), \tag{7.3}$$

and $h_t(x,x)$ has the following asymptotic expansion as $t \searrow 0$:

$$\sum_{j=0}^{n/2-1} t^{-\frac{n-2j}{2r}} \frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)} \frac{a_j(x,x)}{r} + a_{\frac{n}{2}}(x,x) + \sum_{\substack{j=1\\ \beta\nmid j}}^{\infty} t^j \frac{(-1)^j}{j!} q_{rj}(x,x) + \sum_{l=1}^{\infty} t^{l\beta} a_{\frac{n}{2}+l\alpha}(x,x).$$

7.2 The case when *n* is odd

For $j \in \{0,1,\dots,(n-1)/2\}$, the coefficient of $t^{-\frac{n-2j}{2r}}$ is computed as in (7.1). Furthermore, in $s = 0$,

$$\begin{aligned}\operatorname{Res}_{s=0}\left(t^{-s}\Gamma(s)q_{-rs}(x,x)\right) &= \operatorname{Res}_{s=0}\left(t^{-s}\frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs)q_{-rs}(x,x)\right)\\ &= r \cdot \frac{-1}{r} \cdot \mathrm{P}_{\operatorname{Ker}\Delta}(x,x) = -\mathrm{P}_{\operatorname{Ker}\Delta}(x,x);\end{aligned}$$

hence, there is no free term in the asymptotic expansion of h_t as t goes to zero.

Now we have to compute the residues of the function $t^{-s}\Gamma(s)q_{-rs}(x,x)$ in $s \in \{-1,-2,\dots\}$ and $s \in \{\frac{-1}{2r},\frac{-3}{2r},\dots\}$.

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7.2.1 The case when *n* is odd and *r* is irrational

Then these sets are disjoint; thus, all poles of the function $\Gamma(s)q_{-rs}(x)$ are simple. For $j \in \mathbb{N}^*$, the coefficient of t^j is obtained as in (7.2). Furthermore, for $j \in \mathbb{N}$, we get

$$\text{(7.4)}\qquad \operatorname{Res}_{s=-\frac{2j+1}{2r}}\left(t^{-s}\frac{\Gamma(s)}{\Gamma(rs)}\cdot\Gamma(rs)q_{-rs}(x,x)\right)=t^{\frac{2j+1}{2r}}\cdot\frac{\Gamma(-\frac{2j+1}{2r})}{\Gamma(-\frac{2j+1}{r})}\cdot\frac{a_{\frac{n+2j+1}{2}}(x,x)}{r}.$$

Therefore, the small-time asymptotic expansion of h_t is the following:

$$h_t(x,x)\overset{t\searrow 0}{\sim}\sum_{j=0}^{n/2-1}t^{-\frac{n-2j}{2r}}\cdot\frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)}\cdot\frac{a_j(x,x)}{r}+\sum_{j=1}^{\infty}t^j\cdot\frac{(-1)^j}{j!}q_{rj}(x,x)$$
$$+\sum_{j=0}^{\infty}t^{\frac{2j+1}{2r}}\cdot\frac{\Gamma\left(-\frac{2j+1}{2r}\right)}{\Gamma\left(-\frac{2j+1}{2}\right)}\cdot\frac{a_{\frac{n+2j+1}{2}}(x,x)}{r}.$$

7.2.2 The case when *n* is odd and $r=\frac{\alpha}{\beta}$ is rational

Consider the sets

$$A:=\{-1,-2,\dots\},\qquad B:=\{\tfrac{-1}{2r},\tfrac{-3}{2r},\dots\},\qquad C:=\{\tfrac{-1}{r},\tfrac{-2}{r},\dots\}.$$

Remark that A is the set of negative poles of $s\longmapsto t^{-s}\Gamma(s)q_{-rs}(x,x)$, and $A\backslash C$ is the set of poles of the function $s\longmapsto\frac{\Gamma(s)}{\Gamma(rs)}$. Clearly B and C are disjoint. Moreover, $A\cap C=\{-l\beta:\ l\in\mathbb{N}^*\}$. Furthermore, if β is odd, then $A\cap B=\varnothing$, and otherwise if β is even, then $A\cap B=\{-l\frac{\beta}{2}:\ l\in 2\mathbb{N}+1\}$. Such an $s=-\frac{2j+1}{2r}=l\frac{\beta}{2}\in A\cap B$ is a double pole for $\Gamma(s)q_{rs}(x)$.

7.2.3 Suppose that β is odd

Then A and B are disjoint. Thus, for $s=-\frac{2j+1}{2r}\in B$, $j\in\mathbb{N}$, the residue of $t^{-s}\Gamma(s)q_{rs}(x,x)$ is the one computed in (7.4).

For $s=-j\in A\backslash C$ (which means that $j\in\mathbb{N}^*,\beta\nmid j$), the residue of $t^{-s}\Gamma(s)q_{-rs}(x,x)$ in s is the one computed in (7.2).

If $s=-l\beta=-\frac{l\alpha}{r}\in A\cap C$ for some $l\in\mathbb{N}^*$, then $\Gamma(s)$ has a simple pole in s and by Proposition 3, (the meromorphic extension of) $q_{-rs}(x,x)$ vanishes at $s=-l\beta$. Hence, the product $t^{-s}\Gamma(s)q_{-rs}(x,x)$ is holomorphic in $s=-l\beta$ and $t^{l\beta}$, $l\in\mathbb{N}^*$, does not appear in the asymptotic expansion.

Therefore, if $r=\frac{\alpha}{\beta}$ is rational and β is odd, we obtain

$$h_t(x,x)\overset{t\searrow 0}{\sim}\sum_{j=0}^{n/2-1}t^{-\frac{n-2j}{2r}}\cdot\frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2}\right)}\cdot\frac{a_j(x,x)}{r}+\sum_{j=0}^{\infty}t^{\frac{2j+1}{2r}}\cdot\frac{\Gamma\left(-\frac{2j+1}{2r}\right)}{\Gamma\left(-\frac{2j+1}{2}\right)}\cdot\frac{a_{\frac{n+2j+1}{2}}(x,x)}{r}$$
$$+\sum_{\substack{j=1\\ \beta\nmid j}}^{\infty}t^j\frac{(-1)^j}{j!}\cdot q_{rj}(x,x).$$

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7.2.4 Assume now that β is even

For $s = -\frac{2j+1}{2r} \in B\backslash A$ ($j \in \mathbb{N}$ with $\alpha \nmid 2j+1$), the residue is computed as in (7.4). For $s = -j \in A\backslash(B \cup C)$ (namely $j \in \mathbb{N}^*$, $\frac{\beta}{2} \nmid j$), the residue is computed as in (7.2).

For $s \in C \cap A$ (namely $s = -l\beta$, $l \in \mathbb{N}^*$), the residue is again 0. Indeed, $\Gamma(s)$ has a simple pole in $-l\beta$ and by Proposition 3, (the meromorphic extension of) $q_{-rs}(x,x)$ vanishes in $-l\beta$, thus $t^{l\beta}$ does not appear in the asymptotic expansion of h_t.

Finally, if $s = -\frac{l\alpha}{2r} = -l\frac{\beta}{2} \in A \cap B$, $l \in 2\mathbb{N}+1$, then s is a double pole for $\Gamma(s)q_{-rs}(x,x)$. We write the Laurent expansions of the functions t^{-s}, $\frac{\Gamma(s)}{\Gamma(rs)}$, and $\Gamma(rs)q_{-rs}(x,x)$, respectively, in $s = -\frac{l\alpha}{2r} = -l\frac{\beta}{2} =: -k$:

$$t^{-s} = t^k - t^k \log t + \mathcal{O}(s+k)^2,$$

$$\frac{\Gamma(s)}{\Gamma(rs)} = \frac{(-1)^k}{k! \cdot \Gamma(-kr)}(s+k)^{-1} + \cdots,$$

$$\Gamma(rs)(q_{-rs}(x,x)) = \frac{1}{r} a_{\frac{n+l\alpha}{2}}(x,x)(s+k)^{-1} + \cdots.$$

Thus, we finally obtain that

$$\begin{aligned}\operatorname{Res}_{s=-k}\left(t^{-s} \cdot \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs)q_{-rs}(x,x)\right) &= t^k \cdot \frac{(-1)^k}{k!\Gamma(-kr)} \cdot \operatorname{FP}_{s=-k}\left(\Gamma(rs)q_{-rs}(x,x)\right)\\ &+ t^k \operatorname{FP}_{s=-k}\left(\frac{\Gamma(s)}{\Gamma(rs)}\right) \cdot \frac{a_{\frac{n+l\alpha}{2}}(x,x)}{r}\\ &+ t^k \log t \cdot \frac{(-1)^k}{k!\Gamma(-kr)} \frac{a_{\frac{n+l\alpha}{2}}(x,x)}{r}.\end{aligned}$$

8 Non-triviality of the coefficients

Let us prove Theorem 1.1. Recall the definition of the zeta function of a non-negative self-adjoint generalized Laplacian Δ:

$$\zeta_\Delta(s) := \sum_{\lambda \in \operatorname{Spec}\Delta\backslash\{0\}} \lambda^{-s} = \int_M q_{-s}(x,x)dg(x).$$

This series is absolutely convergent for $\Re s > \frac{n}{2}$ and extends meromorphically to $\mathbb{C}$ with possible simple poles in the set

$$\left\{\frac{n}{2} - j : j \in \mathbb{N}\backslash\left\{\frac{n}{2}\right\}\right\}$$

(see, for instance, [13]).

Consider the trivial bundle $\mathbb{C}$ over a compact Riemannian manifold M. As in [17], let $(\Delta+\xi)_{\xi>0}$ be a family of generalized Laplacians indexed by $\xi > 0$, and denote by q^{ξ}_{-s} the Schwartz kernels of the operators $(\Delta+\xi)^{-s}$. Note that for $\Re s > \frac{n}{2}$,

(8.1) $$\int_M q^{\xi}_{-s}(x,x)dx = \operatorname{Tr}(\Delta+\xi)^{-s} = \zeta_{\Delta+\xi}(s) = \sum_{\lambda_j \in \operatorname{Spec}\Delta} \left(\lambda_j + \xi\right)^{-s}.$$

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Since for $\Re s > \frac{n}{2}$ the sum is absolutely convergent, we obtain

$$\frac{d}{d\xi}\zeta_{\Delta+\xi}(s) = -s \cdot \sum_{\lambda_j \in \operatorname{Spec}\Delta} (\lambda_j + \xi)^{-s-1} = -s \cdot \zeta_{\Delta+\xi}(s+1).$$

By induction, it follows that for $\Re s > \frac{n}{2}$,

$$\frac{d}{d\xi^k}\zeta_{\Delta+\xi}(s) = (-1)^k s(s+1)\dots(s+k-1)\cdot\zeta_{\Delta+\xi}(s+k). \tag{8.2}$$

Using the identity theorem, (8.2) holds true on $\mathbb{C}$ as an equality of meromorphic functions. Consider $s \in \mathbb{R}\backslash(-\mathbb{N})$ and $k \in \mathbb{N}$ large enough such that $s + k > \frac{n}{2}$. Since $\zeta_{\Delta+\xi}(s+k)$ is a convergent sum of strictly positive numbers, the right-hand side is non-zero. Thus, for any fixed $s \in \mathbb{R}\backslash(-\mathbb{N})$, on any open set $U \subset (0,\infty)$, the function $\xi \longmapsto \zeta_{\Delta+\xi}(s)$ is not identically zero on U, and by (8.1), $q^{\xi}_{-s}(x,x)$ cannot be constant zero on M. Hence, for $s = -rj \notin -\mathbb{N}$, there exist $\xi_0 \in (0,\infty)$ and $x_0 \in M$ such that the coefficient $q^{\xi_0}_{rj}(x_0,x_0)$ of the asymptotic expansion of the Schwartz kernel h_t of $e^{-t(\Delta+\xi_0)^r}$ is non-zero.

Now suppose that $rj \in \mathbb{N}$. Then $r = \frac{\alpha}{\beta}$ is rational and j is a multiple of β, $j := l\beta$. If n is odd, we already proved in Theorem 7.1 that $t^{l\beta}$ does not appear in the asymptotic expansion of h_t as $t \searrow 0$. Furthermore, if n is even, by (7.3), $q_{rj}(x,x)$ is a non-zero multiple of the coefficient $a_{\frac{n}{2}+l\alpha}(x,x)$ in the asymptotic expansion (2.1) of the heat kernel p_t. It is well known that the heat coefficients in (2.1) are non-trivial (see, for instance, [13]). It follows that all coefficients obtained in Theorem 7.1 indeed appear in the asymptotic expansion, proving Theorem 1.1.

9 Non-locality of the coefficients $A_j(x)$ in the asymptotic expansions

Let us prove Theorem 1.3. We give an example of an n-dimensional manifold and a Laplacian for which the coefficients $A_j(x) = \frac{(-1)^j}{j!}q_{rj(x,x)}$, $j \in \mathbb{N}^*$, $rj \notin \mathbb{N}$ appearing in Theorem 7.1 are not locally computable in the sense of Definition 1.1 (i). Let $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be the n-dimensional torus from Example 2.2. Let Δ_g be the Laplacian on $\mathbb{T}^n$ given by the metric $g = d\theta_1^2 + \dots + d\theta_n^2$.

Remark that the eigenvalues of Δ_g are $\{k_1^2 + \dots + k_n^2 : k_1,\dots,k_n \in \mathbb{Z}\}$. Let $\varphi_l(t) = \frac{1}{\sqrt{2\pi}}e^{ilt}$ be the standard orthonormal basis of eigenfunctions of each Δ_{S^1}. Then, for $\Re s > \frac{n}{2}$ and $\theta = (\theta_1,\dots,\theta_n) \in \mathbb{T}^n$, the Schwartz kernel of Δ_g^{-s} is given by

$$q^{\Delta_g}_{-s}(\theta,\theta) = \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n\backslash\{0\}} \left(k_1^2 + \dots + k_n^2\right)^{-s} \varphi_{k_1}(\theta_1)\overline{\varphi_{k_1}(\theta_1)}\dots\varphi_{k_n}(\theta_n)\overline{\varphi_{k_n}(\theta_n)}.$$

Consider the n-dimensional zeta function

$$\zeta_n(s) := \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n\backslash\{0\}} \left(k_1^2 + \dots + k_n^2\right)^{-s} = \sum_{k\in\mathbb{N}^*} k^{-s}R_n(k),$$

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where $R_n(k)$ is the number of representations of k as a sum of n squares. Since $\varphi_l(t)\overline{\varphi_l(t)} = \frac{1}{2\pi}$ for any $t \in S^1$, it follows that

$$q_{-s}^{\Delta_g}(\theta,\theta) = \frac{1}{(2\pi)^n}\zeta_n(s), \tag{9.1}$$

for any $\Re s > \frac{n}{2}$, and clearly $q_{-s}^{\Delta_g}$ is independent of θ.

Now let us change the metric locally on each component S^1. Let U be an open interval in S^1, and $\psi : S^1 \longrightarrow [0,\infty)$ a smooth function with $\operatorname{supp}\psi \subset U$. Consider the new metric $(1+\psi(\theta))\,d\theta^2$ on each S^1. Then there exist $p > 0$ and an isometry $\Phi : \left(S^1, (1+\psi(\theta))\,d\theta^2\right) \longrightarrow \left(S^1, p^2 d\theta^2\right)$. Remark that the Laplacian on S^1 given by the metric $p^2 d\theta^2$ corresponds under this isometry to p^{-2} times the Laplacian for the metric $d\theta^2$. Let

$$\tilde{g} = \sum_{j=1}^{n}\left(1+\psi(\theta_j)\right)d\theta_j^2 \qquad\qquad g_p = \sum_{j=1}^{n} p^2 d\theta_j^2 = p^2 g.$$

Then clearly $\Phi\times\cdots\times\Phi : (\mathbb{T}^n,\tilde{g}) \longmapsto (\mathbb{T}^n, g_p)$ is an isometry, and let $\tilde{\Delta}$, Δ_p be the corresponding Laplacians on $\mathbb{T}^n$. Denote by $q_{-s}^{\tilde{\Delta}}$ and $q_{-s}^{\Delta_p}$ the Schwartz kernels of the complex powers $\tilde{\Delta}^{-s}$ and Δ_p^{-s}. We have for $\Re s > \frac{n}{2}$,

(9.2)
$$q_{-s}^{\Delta_p}(\theta,\theta) = \frac{1}{(2\pi p)^n}\sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n\setminus\{0\}}\left(p^{-2}k_1^2+\cdots+p^{-2}k_n^2\right)^{-s} = \frac{p^{2s}}{(2\pi p)^n}\zeta_n(s).$$

Remark that

$$q_{-s}^{\Delta_p}(\theta,\theta) = q_{-s}^{\tilde{\Delta}}(\Phi(\theta),\Phi(\theta)),$$

and both of them are independent of θ. By (9.2), for $\Re s > \frac{n}{2}$, we obtain

$$q_{-s}^{\tilde{\Delta}}(\theta,\theta) = \frac{p^{2s-n}}{(2\pi)^n}\zeta_n(s). \tag{9.3}$$

Now we prove that $\zeta_n(s)$ has a meromorphic extension on $\mathbb{C}$ with so-called trivial zeros at $s = -1,-2,\dots$. By Proposition 1, for $\Re s > \frac{n}{2}$, we have

$$\zeta_n(s)\Gamma(s) = \int_0^\infty t^{s-1}\sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n\setminus\{0\}} e^{-t(k_1^2+\cdots+k_n^2)}\,dt = \int_0^\infty t^{s-1}F(t)\,dt,$$

where $F(t) := \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n\setminus\{0\}} e^{-t(k_1^2+\cdots+k_n^2)}$. Using the multidimensional Poisson formula (see, for instance, [3]), it follows that

$$1+F(t) = \sum_{k\in\mathbb{Z}^n} f_t(k) = \sum_{k\in\mathbb{Z}^n}\hat{f}_t(2\pi k) = \pi^{n/2}t^{-n/2}\left(1+F\left(\frac{\pi^2}{t}\right)\right),$$

and therefore

$$F(t) = -1 + \pi^{n/2}t^{-n/2} + \pi^{n/2}t^{-n/2}F\left(\frac{\pi^2}{t}\right).$$

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Since $F(t)$ goes to 0 rapidly as $t \to \infty$, the function $A(s) = \int_1^\infty t^{s-1}F(\pi t)dt$ is entire. Remark that

$$\zeta_n(s)\Gamma(s) = \int_0^\pi t^{-s}F(t)dt + \int_\pi^\infty t^{s-1}F(t)dt$$
$$= \pi^s\left(-\frac{1}{s} + \frac{1}{s-\frac{n}{2}} + A\left(\frac{n}{2}-s\right) + A(s)\right),$$

so

$$\pi^{-s}\zeta_n(s)\Gamma(s) = -\frac{1}{s} + \frac{1}{s-\frac{n}{2}} + A\left(\frac{n}{2}-s\right) + A(s). \tag{9.4}$$

Therefore, ζ_n extends meromorphically to $\mathbb{C}$ with a simple pole in $s = \frac{n}{2}$ and zeros at $s = -1, -2, \ldots$. Furthermore, since the RHS is invariant through the involution $s \mapsto \frac{n}{2} - s$, it follows that $\zeta_n(s)$ does not have any other zeros for $s \in (-\infty, 0)$. We obtain the well-known functional equation of the Epstein zeta function

$$\pi^{-s}\zeta_n(s)\Gamma(s) = \pi^{s-n/2}\zeta_n\left(\frac{n}{2}-s\right)\Gamma\left(\frac{n}{2}-s\right)$$

(see, for instance, [9, equation (63)]). Remark that for $r \in (0,1)$ and $j \in \mathbb{N}^*$ with $rj \notin \mathbb{N}$, $\zeta_n(-rj)$ is not zero.

Using the identity theorem, it follows that (9.1) and (9.3) hold true as an equality of meromorphic functions on $\mathbb{C}$, and furthermore, we get

$$q_{rj}^{\Delta_g}(\theta,\theta) \neq q_{rj}^{\tilde{\Delta}}(\theta,\theta),$$

for $rj \notin \mathbb{N}$. Since we modified the metric locally in $U^n \subset \mathbb{T}^n$ and the corresponding kernel $q_{rj}^{\tilde{\Delta}}$ changed its behavior globally, it follows that it is not locally computable in the sense of Definition 1.1 (i).

Furthermore, let us see that the heat coefficients $A_j(x) = \frac{(-1)^j}{j!}q_{rj}(x,x)$ for $j = \mathbb{N}^*$, $rj \notin \mathbb{N}$ are not cohomologically local in the sense of Definition 1.1 (iii). We argue by contradiction. Let j be fixed. Suppose that there exists a function C, locally computable in the sense of Definition 1.1 (i), such that

$$\int_{\mathbb{T}^n} q_{rj}^{\Delta_g}\,\mathrm{dvol}_g = \int_{\mathbb{T}^n} C(g)\,\mathrm{dvol}_g, \qquad \int_{\mathbb{T}^n} q_{rj}^{\tilde{\Delta}}\,\mathrm{dvol}_{\tilde{g}} = \int_{\mathbb{T}^n} C(\tilde{g})\,\mathrm{dvol}_{\tilde{g}}. \tag{9.5}$$

Using (9.1) and (9.3), it follows that

$$(2\pi)^n\zeta_n(-rj) = \int_{\mathbb{T}^n} C(g)\,\mathrm{dvol}_g, \qquad (2\pi p)^n p^{-2rj}\zeta_n(-rj) = \int_{\mathbb{T}^n} C(\tilde{g})\,\mathrm{dvol}_{\tilde{g}}.$$

Remark that in the case of the trivial bundle with the trivial connection over a locally homogeneous Riemannian manifold (M,h) (i.e., such that every two points have isometric neighborhoods), the function $C(M,h) \in \mathcal{C}^\infty(M)$ is constant on M. This follows directly from Definition 1.1 (i). Therefore, $C(g)$, $C(\tilde{g})$, and $C(g_p)$ are constant functions.

Since $(\mathbb{T}^n, \tilde{g})$ is (globally) isometric to $(\mathbb{T}^n, g_p)$, it follows that $C(\tilde{g}) = C(g_p)$. Furthermore, since $(\mathbb{T}^n, g_p)$ is locally isometric to $(\mathbb{T}^n, g)$ and $C(g_p)$, $C(g)$ are constant functions, it also follows that they are equal: $C(g_p) = C(g)$. Hence we

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conclude that $C(\tilde{g}) = C(g_p) = C(g) =: C$, for some $C \in \mathbb{C}$, and thus we have

$$\int_{\mathbb{T}^n} C \,\mathrm{dvol}_{\tilde{g}} = \int_{\mathbb{T}^n} C \,\mathrm{dvol}_{g_p}. \tag{9.6}$$

Since $g_p = p^2 g$, we obtain that

$$\int_{\mathbb{T}^n} C \,\mathrm{dvol}_{g_p} = p^n \int_{\mathbb{T}^n} C \,\mathrm{dvol}_g, \tag{9.7}$$

and then using (9.5)–(9.7), we get

$$(2\pi p)^n p^{-2rj} \zeta_n(-rj) = p^n \cdot (2\pi)^n \zeta_n(-rj).$$

But, we proved above that $\zeta_n(-rj)$ does not vanish for $rj \notin \mathbb{N}$. We obtain a contradiction because $p^{-2rj} \neq 1$ for $r \in (0,1)$, $j = 1, 2, \ldots$.

10 Interpretation of h_t on the heat space for $r = 1/2$

In Theorems 6.1 and 7.1, we studied the asymptotic behavior of the heat kernel h_t of Δ^r, $r \in (0,1)$ for small-time t in two distinct cases: when we approach $t = 0$ along the diagonal in $M \times M$, and when we approach a compact set away from the diagonal. We now give a simultaneous asymptotic expansion formula for both cases when $r = \frac{1}{2}$. Furthermore, in order to understand the asymptotic behavior as t goes to zero in *any* direction (not just the case when t goes to 0 in the vertical one),we will pull-back the formula on a certain *linear* heat space $\mathrm{M_{heat}}$.

In [19], Melrose used his blow-up techniques to give a conceptual interpretation for the asymptotic of the heat kernel p_t. Recall that the heat space M_H^2 is obtained by performing a parabolic blow-up of $\{t = 0\} \times \mathrm{Diag}$ in $[0,\infty) \times M \times M$. The heat space M_H^2 is a manifold with corners with boundary hypersurfaces given by the boundary defining functions ρ and ω_0. The heat kernel p_t belongs to $\rho^{-n}\mathcal{C}^\infty(M_H^2)$, and vanishes rapidly at the boundary hypersurface $\{\omega_0 = 0\}$ (see [19, Theorem 7.12]).

In order to study the Schwartz kernel h_t of $e^{-t\Delta^{1/2}}$, we introduce the *linear heat space* $\mathrm{M_{heat}}$, which is just the standard blow-up of $\{0\} \times \mathrm{Diag}$ in $[0,\infty) \times M \times M$ (see [20] for details regarding the blow-up of a submanifold). Let ff be the *front face*, i.e., the newly added face, and denote by lb the *lateral boundary* which is the lift of the old boundary $\{0\} \times M \times M$. The blow down map is given locally by

$$\beta_H : \mathrm{M_{heat}} \longrightarrow [0,\infty) \times M \times M \qquad \beta_H(\rho, \omega, x') = (\rho\omega_0, \rho\omega' + x', x'),$$

where

$$\omega \in \mathbb{S}_H^n = \{\omega = (\omega_0, \omega') \in \mathbb{R}^{n+1} : \ \omega_0 \geq 0,\ \omega_0^2 + |\omega'|^2 = 1\}.$$

Proof of Theorem 1.4 We want to show that $h_t \in \rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M_{heat}}) + \rho\log\rho \cdot \omega_0 \cdot \mathcal{C}^\infty(\mathrm{M_{heat}})$, and in fact, the second (logarithmic) term does not occur when n is even. First, we deduce the unified formula for h_t as $t \searrow 0$ both on the diagonal and away from it. By Mellin formula 1 and inverse Mellin formula 6, for $\tau > n$, we get

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$$h_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y) = \tfrac{1}{2\pi i}\int_{\Re s=\tau} t^{-s}\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)q_{-s/2}(x,y)ds$$
$$= \tfrac{1}{2\pi i}\int_{\Re s=\tau} t^{-s}\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty T^{\frac{s}{2}-1}\left(p_T(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)\right)dTds.$$

We use the Legendre duplication formula as in [2] (see, for instance, [22]):

$$\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{\sqrt{2\pi}}2^{s-\frac{1}{2}}\Gamma\left(\frac{s+1}{2}\right),$$

obtaining that $h_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ is equal to

$$\tfrac{1}{\sqrt{4\pi}}\tfrac{1}{2\pi i}\int_{\Re s=\tau}\int_0^\infty\left(\frac{2\sqrt{T}}{t}\right)^s\Gamma\left(\frac{s+1}{2}\right)\left(p_T(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)\right)dTds.$$

Set $X := \frac{2\sqrt{T}}{t}$. Using Propositions 4, 5, and Fubini, we first compute the integral in s. Changing the variable $S = \frac{s+1}{2}$ and applying the residue theorem, we get

$$\tfrac{1}{2\pi i}\int_{\Re s=\tau} X^s\Gamma\left(\frac{s+1}{2}\right)ds = \tfrac{2}{2\pi i}\int_{\Re S=\frac{\tau+1}{2}} X^{2S-1}\Gamma(S)dS = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}X^{-2k-1}$$
$$= 2X^{-1}e^{-X^{-2}} = \frac{t}{\sqrt{T}}e^{-\frac{t^2}{4T}}.$$

Thus, we obtain

$$h_t(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y) = \tfrac{t}{2\sqrt{\pi}}\int_0^\infty T^{-3/2}e^{-\frac{t^2}{4T}}\left(p_T(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)\right)dT. \tag{10.1}$$

Since $p_T(x,y) - \mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ decays exponentially as T goes to infinity, it follows that the integral from 1 to ∞ in the right-hand side of equation (10.1) is of the form $t\cdot\mathcal{C}^\infty_{t,x,y}\left([0,\infty)\times M^2\right)$. Furthermore, by the change of variable $u = \frac{t}{2\sqrt{T}}$, we have

$$-\tfrac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}dT\cdot\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y) = -\tfrac{2}{\sqrt{\pi}}\int_{t/2}^\infty e^{-u^2}du\cdot\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y).$$

Since $\int_{t/2}^\infty e^{-u^2}du$ tends to $\frac{\sqrt{\pi}}{2}$ as $t\searrow 0$, the term $-\frac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}dT\,\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ will cancel in the limit as $t\to 0$ with $-\mathrm{P}_{\mathrm{Ker}\,\Delta}(x,y)$ from the left-hand side of (10.1).

Let us study the remaining integral term $\frac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}p_T(x,y)dT$. By Theorem 2.1,

$$p_T(x,y) = T^{-n/2}e^{-\frac{d(x,y)^2}{4T}}\sum_{j=0}^N T^j a_j(x,y) + R_{N+1}(T,x,y),$$

where the remainder $R_{N+1}(T,x,y)$ is of order $\mathcal{O}(T^{N+1})$; therefore,

$$\tfrac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}p_T(x,y)dT = \tfrac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}R_{N+1}(T,x,y)dT$$
$$+\tfrac{t}{2\sqrt{\pi}}\int_0^1 T^{-3/2}e^{-\frac{t^2}{4T}}T^{-n/2}e^{-\frac{d(x,y)^2}{4T}}\sum_{j=0}^N T^j a_j(x,y)dT.$$

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Since $R_{N+1}(T,x,y)$ is of order $\mathcal{O}(T^{N+1})$, the first integral is again of type $t\cdot\mathcal{C}^{\infty}_{t,x,y}$. By changing the variable $u = \frac{t^2+d(x,y)^2}{4T}$ in the second integral, we get

$$\begin{aligned}
&\frac{t}{2\sqrt{\pi}}\sum_{j=0}^{N} a_j(x,y)\int_0^1 T^{-\frac{n+3}{2}+j}e^{-\frac{t^2+d(x,y)^2}{4T}}\,dT\\
&= \frac{t}{2\sqrt{\pi}}\sum_{j=0}^{N} a_j(x,y)\left(\frac{t^2+d(x,y)^2}{4}\right)^{-\frac{n+1}{2}+j}\int_{\frac{t^2+d(x,y)^2}{4}}^{\infty} u^{\frac{n+1}{2}-j-1}e^{-u}\,du\\
&= \frac{t}{2\sqrt{\pi}}\sum_{j=0}^{N} a_j(x,y)\Gamma\left(\frac{n+1}{2}-j,\frac{t^2+d(x,y)^2}{4}\right)\left(\frac{t^2+d(x,y)^2}{4}\right)^{-\frac{n+1}{2}+j},
\end{aligned}$$

where $\Gamma(z,\xi) := \int_{\xi}^{\infty} u^{z-1}e^{-u}\,du$ is the upper incomplete Gamma function. We conclude that $h_t(x,y)$ is equal to

(10.2)

$$t\cdot\mathcal{C}^{\infty}_{t,x,y} + \frac{t}{2\sqrt{\pi}}\sum_{j=0}^{N} a_j(x,y)\Gamma\left(\frac{n+1}{2}-j,\frac{t^2+d(x,y)^2}{4}\right)\left(\frac{t^2+d(x,y)^2}{4}\right)^{-\frac{n+1}{2}+j}.$$

■

10.1 The case when *n* is even

If $z>0$, then one can easily check that $\Gamma(z,\xi)\in \xi^z\mathcal{C}^{\infty}_{\xi}[0,\varepsilon)+\Gamma(z)$, for some $\varepsilon>0$. Furthermore, for $z\in(-\infty,0]\backslash\{0,-1,-2,\dots\}$,

$$\begin{aligned}
\Gamma(z,\xi) &= -\frac{1}{z}\xi^z e^{-\xi}+\frac{1}{z}\Gamma(z+1,\xi)\\
&= \xi^z e^{-\xi}\sum_{k=0}^{a-1}\frac{-1}{z(z+1)\dots(z+k)}\xi^k+\frac{1}{z(z+1)\dots(z+a)}\Gamma(z+a,\xi)\\
&= \xi^z\mathcal{C}^{\infty}_{\xi}[0,\varepsilon)+\frac{1}{z(z+1)\dots(z+a-1)}\Gamma(z+a,\xi),
\end{aligned}$$

where a is a positive integer such that $z+a>0$. Thus, for a non-integer $z<0$, we have

$$\Gamma(z,\xi)=\xi^z\mathcal{C}^{\infty}_{\xi}[0,\varepsilon)+\frac{1}{z(z+1)\dots(z+a-1)}\Gamma(z+a).$$

We want to interpret equation (10.2) on the heat space $\mathrm{M}_{\mathrm{heat}}$; thus, we pull back (10.2) through β_H:

$$\begin{aligned}
\beta_H^*h &= \rho\omega_0\beta_H^*\mathcal{C}^{\infty}_{t,x,y}+\frac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{j=0}^{N}\left(\frac{\rho^2}{4}\right)^{-\frac{n+1}{2}+j}\beta_H^*a_j(x,y)\Gamma\left(\frac{n+1}{2}-j,\frac{\rho^2}{4}\right)\\
&= \rho\omega_0\beta_H^*\mathcal{C}^{\infty}_{t,x,y}+\frac{1}{2\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=0}^{n/2}\rho^{2j}2^{n+1-2j}\beta_H^*a_j(x,y)\Gamma\left(\frac{n+1}{2}-j\right)\\
&\quad+\frac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{j=0}^{n/2}\beta_H^*a_j(x,y)\mathcal{C}^{\infty}_{\rho^2}[0,\varepsilon)+\frac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{j=n/2+1}^{N}\beta_H^*a_j(x,y)\mathcal{C}^{\infty}_{\rho^2}[0,\varepsilon)
\end{aligned}$$

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$$+\frac{1}{2\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=n/2+1}^{N}\rho^{2j}2^{n+1-2j}\beta_H^*a_j(x,y)\frac{2^{-n/2+j}}{(n+1-2j)(n+3-2j)\dots(-1)}\Gamma\left(\frac{1}{2}\right).$$

Since $\Gamma\left(\frac{n+1}{2}-j\right)=\frac{\sqrt{\pi}(n-2j-1)!!}{2^{n/2-j}}$ for $j\in\{0,1,\dots,n/2\}$, it follows that

(10.3)

$$\beta_H^*h=\rho\omega_0\beta_H^*\mathcal{C}^\infty_{t,x,y}+\omega_0\rho\mathcal{C}^\infty_{\rho^2}[0,\varepsilon)+\rho^{-n}\omega_0\sum_{j=0}^{n/2}\rho^{2j}2^{n/2-j}(n-2j-1)!!\beta_H^*a_j(x,y)$$
$$+\rho^{-n}\omega_0\sum_{j=n/2+1}^{N}\rho^{2j}\frac{(-1)^{j-n/2}2^{n/2-j}}{(2j-n-1)!!}\beta_H^*a_j(x,y).$$

The case $\rho\neq 0$ and $\omega_0\to 0$ corresponds to $x\neq y$ and $t\searrow 0$ before the pull-back. We obtain that β_H^*h is in $\mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$ and it vanishes at first order on lb, which is compatible with Theorem 6.1.

If $\rho\to 0$ and $\omega_0=1$, which corresponds to $x=y$ and $t\searrow 0$, then $\beta_H^*h=\rho^{-n}\omega_0\sum_{j=0}^{N}\rho^{2j}A_j(x)$, where we denoted by $A_j(x)$ the coefficients appearing in (10.3). Again, this result is compatible with Theorem 7.1, and moreover, the coefficients are precisely the ones from [2, Theorem 3.1].

Remark that formula (10.3) is stronger than Theorems 6.1 and 7.1. If both ρ and ω_0 tend to 0 (with different speeds), it describes the behavior of h_t as t goes to zero from any positive direction (not only the vertical one).

10.2 The case when *n* is odd

Remark that for small ξ, we have

$$\Gamma(0,\xi)=\int_\xi^\infty t^{-1}e^{-t}dt=\int_\xi^1\frac{e^{-t}-1}{t}dt+\int_\xi^1 t^{-1}dt+\int_1^\infty t^{-1}e^{-t}dt$$
$$=-\log\xi+\mathcal{C}^\infty_\xi[0,\varepsilon).$$

Furthermore, if p is a negative integer, inductively we obtain

$$\Gamma(-p,\xi)=\frac{e^{-\xi}\xi^{-p}}{p!}\sum_{k=0}^{p-1}(-1)^k(p-k-1)!\xi^k+\frac{(-1)^p}{p!}\Gamma(0,\xi)$$
$$=\xi^{-p}\mathcal{C}^\infty_\xi[0,\varepsilon)-\frac{(-1)^p}{p!}\log\xi+\mathcal{C}^\infty_\xi[0,\varepsilon).$$

We pull-back equation (10.2) on the heat space $\mathrm{M}_{\mathrm{heat}}$:

$$\beta_H^*h=\rho\omega_0\beta_H^*\mathcal{C}^\infty_{t,x,y}+\frac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{j=0}^{N}\left(\frac{\rho^2}{4}\right)^{-\frac{n+1}{2}+j}\beta_H^*a_j(x,y)\Gamma\left(\frac{n+1}{2}-j,\frac{\rho^2}{4}\right)$$
$$=\rho\omega_0\beta_H^*a_j(x,y)+\frac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{l=0}^{(n-1)/2}\beta_H^*a_j(x,y)\mathcal{C}^\infty_{\rho^2}[0,\varepsilon)$$
$$+\frac{1}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=0}^{(n-1)/2}\rho^{2j}\beta_H^*a_j(x,y)2^{n-2j}\Gamma\left(\frac{n+1}{2}-j\right)$$

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$$+\frac{2}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=(n+1)/2}^{N}\rho^{2j}\log\rho\,\beta_H^*a_j(x,y)2^{n-2j}\frac{(-1)^{j-\frac{n+1}{2}+1}}{\left(j-\frac{n+1}{2}\right)!}$$
$$+\frac{2}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=(n+1)/2}^{N}\rho^{2j}\beta_H^*a_j(x,y)2^{n-2j}\frac{(-1)^{j-\frac{n+1}{2}}}{\left(j-\frac{n+1}{2}\right)!}\log 2$$
$$+\tfrac{1}{2\sqrt{\pi}}\rho\omega_0\sum_{j=(n+1)/2}^{N}\beta_H^*a_j(x.y)\mathcal{C}^\infty_{\rho^2}[0,\varepsilon)$$
$$+\tfrac{1}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=(n+1)/2}^{N}\rho^{2j}\beta_H^*a_j(x,y)2^{n-2j}\frac{(-1)^{j-\frac{n+1}{2}}}{\left(j-\frac{n+1}{2}\right)!}\mathcal{C}^\infty_{\rho^2}[0,\varepsilon).$$

Therefore, we obtain

$$\begin{aligned}\beta_H^*h &= \rho\omega_0\beta_H^*\mathcal{C}^\infty_{t,x,y}+\omega_0\rho\,\mathcal{C}^\infty_{\rho^2}[0,\varepsilon)+\omega_0\rho^{-n}\mathcal{C}^\infty_{\rho^2}[0,\varepsilon)\\ &+\frac{1}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=0}^{(n-1)/2}\rho^{2j}\beta_H^*a_j(x,y)2^{n-2j}\left(\frac{n+1}{2}-j\right)!\\ &+\frac{2}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=(n+1)/2}^{N}\rho^{2j}\log\rho\,\beta_H^*a_j(x,y)2^{n-2j}\frac{(-1)^{j-\frac{n+1}{2}+1}}{\left(j-\frac{n+1}{2}\right)!}\\ &+\frac{2}{\sqrt{\pi}}\rho^{-n}\omega_0\sum_{j=(n+1)/2}^{N}\rho^{2j}\beta_H^*a_j(x,y)2^{n-2j}\frac{(-1)^{j-\frac{n+1}{2}}}{\left(j-\frac{n+1}{2}\right)!}\log 2.\end{aligned} \tag{10.4}$$

If $\rho \neq 0$ and $\omega_0 \to 0$ (corresponding to $x \neq y$ and $t \searrow 0$ before the pull-back on M_{heat}), we obtain that $\beta_H^*h \in \mathcal{C}^\infty(M_{heat})$ and it vanishes at order 1 at lb, which is compatible with the result of Theorem 6.1.

In the case $\rho \to 0$ and $\omega_0 = 1$ which corresponds to $x = y$ and $t \searrow 0$, we obtain $\beta_H^*h = \rho^{-n}\mathcal{C}^\infty_{\rho^2} + \rho^{-n}\sum_{j=0}^{N}\rho^{2j}A_j(x) + \rho^{-n}\sum_{j=(n+1)/2}^{N}\rho^{2j}\log\rho B_j(x)$, where we denoted by A_j and B_j the coefficients appearing in (10.4). This result is compatible with Theorem 7.1 and again, we find some of the coefficients appearing in [2, Theorem 3.1].

11 The heat kernel as a polyhomogeneous conormal section

Let us recall the notions of index family and polyhomogeneous conormal functions on a manifold with corners with two boundary hypersurfaces. (For an accessible introduction, see [15], and for full details of the theory, see [18].) A discrete subset $F \in \mathbb{C}\times\mathbb{N}$ is called an *index set* if the following conditions are satisfied:

1) For any $N \in \mathbb{R}$, the set $F \cap \{(z,p) : \Re z < N\}$ is finite.
2) If $p > p_0$ and $(z,p) \in F$, then $(z,p_0) \in F$.

If X is a manifold with corners with two boundary hypersurfaces B_1 and B_2 given by the boundary defining functions x and y, a smooth function f on $\mathring{X}$ is said to be *polyhomogeneous conormal* with index sets E and F, respectively, if in a small neighborhood $[0,\varepsilon)\times B_1$, f has the asymptotic expansion

$$f(x,y) \overset{x\searrow 0}{\sim} \sum_{(z,p)\in F} a_{z,p}(y)\cdot x^z\log^p x,$$

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where $a_{z,p}$ are smooth coefficients on B_2, and for each $a_{z,p}$ there exists a sequence of real numbers $b_{w,q}$, such that

$$a_{z,p}(y) \overset{y\searrow 0}{\sim} \sum_{(w,q)\in E} b_{w,q} \cdot y^w \log^q y.$$

One can prove that f is a polyhomogeneous conormal function on X with index sets $F_p = \{(k,0) : k \in \mathbb{Z}, k \geq -p\}$ and $F_0 = \{(n,0) : n \in \mathbb{N}\}$ if and only if $f \in y^{-p}\mathcal{C}^\infty(X)$. Furthermore, f is a polyhomogeneous conormal function on X with index sets $F' = \{(n,1) : n \in \mathbb{N}^*\}$ and F_0 if and only if $f \in \mathcal{C}^\infty(X) + \log y \cdot \mathcal{C}^\infty(X)$. Therefore, we can restate Theorem 1.4 as follows:

Theorem 11.1 *For $r = \frac{1}{2}$, the heat kernel h_t of the operator $e^{-t\Delta^{1/2}}$ is a polyhomogeneous conormal section on the linear heat space $\mathrm{M}_{\mathrm{heat}}$ with values in $\mathcal{E} \boxtimes \mathcal{E}^*$. The index set for the lateral boundary is*

$$F_{\mathrm{lb}} = \{(k,0) : k \in \mathbb{N}^*\}.$$

If n is even, the index set of the front face is

$$F_{\mathrm{ff}} = \{(-n+k,0) : k \in \mathbb{N}\},$$

whereas for n odd, the index set toward ff is given by

$$F_{\mathrm{ff}} = \{(-n+k,0) : k \in \mathbb{N}\} \cup \{(k,1) : k \in \mathbb{N}^*\}.$$

It seems reasonable to expect that the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$ for $r \in (0,1)$ can be lifted to a polyhomogeneous conormal section in a certain "transcendental" heat space M^r_{Heat} depending on r with values in $\mathcal{E} \boxtimes \mathcal{E}^*$. However, already in the case $r = 1/3$, our method leads to complicated computations involving Bessel modified functions. We therefore leave this investigation open for a future project.

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Institute of Mathematics of the Romanian Academy, Bucharest, Romania
e-mail: cianghel@imar.ro