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STRONG OSCILLATION OF ELLIPTIC EQUATIONS IN GENERAL DOMAINS

BY
C. A. SWANSON

Strong oscillation criteria will be obtained for the linear elliptic partial differential equation

$$Lv = \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jv]+B(x)v = 0 \tag{1}$$

$$x = (x^1, x^2, \ldots, x^n), \qquad D_i = \partial/\partial x^i$$

in unbounded domains R of general type in n-dimensional Euclidean space E^n. It will be assumed throughout that B and each A_{ij} are real-valued continuous functions in R, and that the matrix $(A_{ij}(x))$ is symmetric and positive definite in R. It is not required that R be quasiconical or quasicylindrical as in earlier work [3], [4], [5], [6], or even connected. For example, R is allowed to be a spiral domain with decreasing width as $|x|\to\infty$.

The following definition was first introduced by Glazman [2] in the case of the Schrödinger equation $\Delta v+B(x)v=0$ in all of E^n.

DEFINITION 1. The partial differential equation (1) is called *strongly oscillatory* in an unbounded domain R iff for arbitrary $r>0$ there exists a nontrivial solution v_r of (1) with a nodal domain contained in R_r, where

$$R_r = R \cap \{x \in E^n : |x| > r\}, \qquad 0 < r < \infty.$$

It follows from the n-dimensional version of Sturm's comparison theorem [7, p. 187] that every strongly oscillatory equation (1) is weakly oscillatory, i.e. every solution v of (1) has a zero in R_r for arbitrary $r>0$.

In the case that R is large enough at ∞ to contain a cone of the type

$$C_\theta = \{x \in E^n : x^i \geq |x| \cos \theta\}$$

for some θ, $0<\theta\leq\pi$, strong oscillation criteria for (1) were obtained by Headley and the author [3]. For example, (1) is strongly oscillatory in E^n if

$$\liminf_{r\to\infty} r^2 g(r) > \frac{(n-2)^2\lambda}{4} \tag{2}$$

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where λ is an upper bound on the largest eigenvalue $\wedge(x)$ of the matrix $(A_{ij}(x))$, and

$$g(r) = \min_{|x|=r} B(x)$$

If R is too "small" at ∞, it is clear that (2) is not sufficient for (1) to be strongly oscillatory in R. Consider the example that R is the strip

$$R = \left\{x \in E^2 : \frac{\pi}{4} < x^1 < \frac{3\pi}{4}\right\}.$$

Then the Schrödinger equation $\Delta v + B(x)v = 0$ is not strongly oscillatory in R even under the condition that $B(x)$ is uniformly bounded away from zero in R. In fact, if the equation $\Delta v + v = 0$ had a nontrivial solution v_r with any nodal domain N in R, then the solution $v(x) = \sin x^1$ would have a zero in N by the comparison theorem.

For a nonempty regular bounded domain $M \subset R$, let F be the functional defined by

$$F[u; M] = \int_M \left[\sum_{i,j=1}^{n} A_{ij}(x) D_i u D_j u - B(x)u^2\right] dx \tag{3}$$

with domain consisting of all real-valued piecewise C^1 functions u on $\bar{M}$.

THEOREM 1. *Equation* (1) *is strongly oscillatory in an unbounded domain $R \subset E^n$ if R contains a sequence of nonempty regular bounded domains M_k, $k = 1, 2, \ldots$, with piecewise C^1 boundaries, having the following properties:*

(i) *For arbitrary $r > 0$ there exists an integer $k_0(r)$ such that $M_k \subset R_r$ for all $k > k_0(r)$; and*

(ii) *There exists a piecewise C^1 function u_k on each M_k such that $u_k \equiv 0$ on ∂M_k and $F[u_k; M_k] \leq 0$ for all sufficiently large k.*

Proof. Let $\lambda(N)$ denote the smallest eigenvalue of the problem

$$Lu + \lambda u = 0 \quad \text{in } N, \qquad u = 0 \quad \text{on } \partial N$$

for an arbitrary subdomain N of M_k with piecewise C^1 boundary. Since $F[u_k; M_k] \leq 0$, Courant's minimum principle [1, p. 399] shows that $\lambda(M_k) \leq 0$. It is well known that $N \subset M$ implies that $\lambda(N) \geq \lambda(M)$ and that $\lambda(N) \to +\infty$ as N shrinks to a point, and hence there exists a subdomain N_k of M_k such that $\lambda(N_k) = 0$. Thus N_k is a nodal domain of a nontrivial solution of (1) for sufficiently large k. By assumption (i), for arbitrary $r > 0$ there exists an integer k such that $M_k \subset R_r$, and consequently $N_k \subset R_r$. This completes the proof of Theorem 1.

For the first application of Theorem 1, let M_k be specialized to the open disk defined by

$$M_k(x_k; a_k) = \{x \in E^n : |x - x_k| < a_k\}$$
$$x \in R, \qquad a_k > 0, \qquad k = 1, 2, \ldots$$

THEOREM 2. *Equation* (1) *is strongly oscillatory in an unbounded domain* $R\subset E^n$ *if* R *contains a sequence of open disks* $\{M_k(x_k; a_k)\}$ *with the following properties:*

(i) $\lim_{k\to\infty}(|x_k|-a_k)=+\infty$;

(ii) $(A_{ij}(x))$ *is bounded* (*as a form*) *in* $\bigcup_k M_k(x_k; a_k)$;

(iii) $B(x)$ *is positive in each* $M_k(x_k; a_k)$, *and*

$$\lim_{k\to\infty} a_k^{2-n}\int_{M_k(x_k;a_k/2)} B(x)\,dx=+\infty$$

Proof. It will be shown that hypothesis (ii) of Theorem 1 is satisfied if u_k is taken to be the piecewise C^1 function on $\bar{M}_k(x_k; a_k)$ defined by

$$\begin{aligned} u_k(x) &= 1 && \text{if } 0\le r\le a_k/2\\ &= 2(a_k-r)/a_k && \text{if } a_k/2<r\le a_k\\ &= 0 && \text{if } a_k<r \end{aligned}$$

where $r=|x-x_k|$. If α is an upper bound on $(A_{ij}(x))$ and ω_n denotes the area of the unit $(n-1)$-sphere in $E^n(\omega_1=1$ by convention), then

$$\begin{aligned}\int_{M_k(x_k;a_k)}\sum_{i,j}A_{ij}(x)D_iu_kD_ju_k\,dx &\le \alpha\int_{M_k(x_k;a_k)}|\nabla u_k|^2\,dx\\ &= \frac{4\alpha\omega_n}{a_k^2}\int_{a_k/2}^{a_k} r^{n-1}\,dr\\ &= \frac{4\alpha\omega_n}{n}(1-2^{-n})a_k^{n-2}\end{aligned}$$

Substitution into (3) yields the inequality

$$a_k^{2-n}F[u_k; M_k(x_k; a_k)]\le 4\alpha\omega_n(1-2^{-n})-a_k^{2-n}\int_{M_k(x_k;a_k/2)}B(x)\,dx$$

Hypothesis (iii) then shows that $F[u_k; M_k(x_k; a_k)]<0$ for all sufficiently large integers k, and therefore hypothesis (ii) of Theorem 1 is satisfied.

By (i), there exists an integer $k_0(r)$ corresponding to each $r>0$ such that $|x_k|-a_k>r$ whenever $k>k_0(r)$. Then $x\in M_k(x_k; a_k)$ implies that

$$|x|\ge|x_k|-|x-x_k|>|x_k|-a_k>r$$

so that $x\in R_r$, and $M_k(x_k; a_k)\subset R_r$ for all $k>k_0(r)$. Hence (1) is strongly oscillatory by Theorem 1.

COROLLARY 3. *Equation* (1) *is strongly oscillatory in an unbounded domain* $R\subset E^n$ *under hypotheses* (*i*) *and* (*ii*) *of Theorem* 2 *and*

(iii′) $B(x)\ge b_k>0$ *in each* $M_k(x_k; a_k)$ *where* $\lim_{k\to\infty} b_ka_k^2=+\infty$.

COROLLARY 4. *Equation* (1) *is strongly oscillatory in* R *if* R *contains each open disk* $M_k(x_k; a)$, $k=1, 2, \ldots, a>0$, *with the following properties:*

(i) $\lim_{k\to\infty}|x_k|=+\infty$

(ii) $(A_{ij}(x))$ *is bounded in* $\bigcup_k M_k(x_k; a)$*; and*

(iii) $\lim_{|x|\to\infty} B(x)=+\infty$ *uniformly in* R.

Another application of Theorem 1 arises in the case that the eigenvalue problem

$$(4) \qquad \begin{cases} \alpha \sum_{j=1}^{n} D_j^2 u_k + B_k u_k = 0 & \text{in } M_k \\ u_k \equiv 0 & \text{on } \partial M_k \end{cases}$$

has a nontrivial eigenfunction u_k and associated eigenvalue B_k (necessarily $B_k>0$) such that

$$(5) \qquad \int_{M_k} u_k^2(x)B(x)\,dx > B_k \int_{M_k} u_k^2(x)\,dx$$

for all sufficiently large integers k, where α denotes an upper bound for $(A_{ij}(x))$.

THEOREM 5. *Equation* (1) *is strongly oscillatory in an unbounded domain R if R contains a sequence of nonempty regular bounded domains M_k, $k=1, 2, \ldots$, with piecewise C^1 boundaries, on which* (4) *has an eigenfunction u_k and associated eigenvalue B_k satisfying the inequality* (5).

Proof. Since (5) holds as well as the inequality

$$\sum_{i,j=1}^{n} A_{ij}(x)z^i z^j \le \alpha\,|z|^2$$

for all $x \in R$ and all $z \in E^n$, where α is an upper bound on $(A_{ij}(x))$, it follows from (3) that

$$F[u_k; M_k] \le \int_{M_k} (\alpha\,|\nabla u_k|^2 - B_k u_k^2)\,dx = 0$$

upon integration by parts and use of (4). Hence (1) is strongly oscillatory by Theorem 1.

In particular, consider the case that R contains each of the congruent rectangles M_k defined by

$$(6) \qquad M_k = \{x \in E^n : c_k^j \le x^j \le c_k^j + 4t^j, j = 1, 2, \ldots, n\}$$

for a sequence of points $\{c_k\}$ with $\lim_{k\to\infty} |c_k| = \infty$, where $t^j>0$ for $j=1, 2, \ldots, n$. Let N_k denote the subrectangle given by

$$N_k = \{x \in M_k : c_k^j + t^j < x^j < c_k^j + 3t^j, j = 1, 2, \ldots, n\}$$

THEOREM 6. *Equation* (1) *is strongly oscillatory in R if*

(i) $M_k \subset R$ *for* $k \ge k_0$, *where M_k is the rectangle given by* (6);

(ii) $B(x)\ge 0$ *on each* M_k*; and*

$$(7) \qquad \int_{N_k} B(x)\,dx > \alpha\pi^2 4^{n-2} t^1 t^2 \ldots t^n \sum_{j=1}^{n} (t^j)^{-2}, \qquad k \ge k_0.$$

Proof. The eigenvalues B_k and associated eigenfunctions u_k for (4) in this case are given explicitly by

$$B_k = \alpha\pi^2 \sum_{j=1}^{n} (4t^j)^{-2}$$

$$u_k(x) = \prod_{j=1}^{n} \sin\left[\frac{\pi(x^j - c_k^j)}{4t^j}\right], \qquad x \in M_k.$$

Since $B(x) \geq 0$ on M_k and $|u_k(x)|^2 > 2^{-n}$ in N_k, a routine check shows that (7) implies (5), and Theorem 5 applies.

Example. Suppose that R contains arbitrarily long parallelepipeds in the x^1-direction, i.e. for arbitrary $t^1 = k > 0$, there exists a point $c_k \in R$ and numbers $t^j > 0$ for $j = 2, \ldots, n$ such that $M_k \subset R$.

Evidently this condition is fulfilled if R contains an infinite cylinder parallel to the x^1-axis, and also for a class of "spiral" domains containing no infinite ray. If there exists a nonnegative function g on $[c_k^1 + k, c_k^1 + 3k]$ such that $B(x) \geq g(x^1)$ for $x \in N_k$, the criterion (7) of Theorem 6 is implied by the condition

$$\int_{c_k^1+k}^{c_k^1+3k} g(t)\, dt > \alpha\pi^2 2^{n-3} k\alpha_k,$$

where

$$\alpha_k = k^{-2} + \sum_{j=2}^{n} (t^j)^{-2}$$

This holds, for example, if $\lim_{t\to\infty} g(t) = +\infty$.

Theorem 6 generates a hierarchy of analogous oscillation criteria, which generally become stronger as the measure of N_k becomes smaller.

Theorem 1 is particularly easy to apply when R is quasiconical or quasicylindrical since u_k can be chosen as the same function u for all k. As an example, a recent oscillation criterion of Kreith and Travis [5] in E^n will be derived from Theorem 1. The matrix $(A_{ij}(x))$ is not required to be bounded above for this result.

Let $A(r, \theta)$ denote the largest eigenvalue of the matrix $(A_{ij}(x))$, where (r, θ) denote hyperspherical coordinates in E^n, $r = |x|$, $\theta = \theta_1, \ldots, \theta_{n-1}$. Let Ω designate the unit $(n-1)$-sphere in E^n and define

$$a(r) = \int_\Omega A(r, \theta)\, d\theta$$

$$b(r) = \int_\Omega B(r, \theta)\, d\theta, \qquad 0 < r < \infty.$$

Theorem 7 (Kreith and Travis). *Equation* (1) *is strongly oscillatory in* E^n *if the following ordinary differential equation is oscillatory at* $r = \infty$:

$$[r^{n-1} a(r) z']' + r^{n-1} b(r) z = 0. \tag{8}$$

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Proof. Let $z(r)$ be a nontrivial solution of (8) with zeros r_k, $r_1 < r_2 < \cdots$, $\lim r_k = +\infty$. Take M_k in Theorem 1 to be the annular domain defined by

$$M_k = \{x \in E^n : r_k < |x| < r_{k+1}\}, \qquad k = 1, 2, \ldots$$

Then $u(x) = z(|x|) \equiv 0$ on ∂M_k for all k, and it is easily checked that

$$F[u; M_k] \leq \int_{r_k}^{r_{k+1}} \left[a(r)\left(\frac{dz}{dr}\right)^2 - b(r)z^2\right] r^{n-1}\, dr = 0$$

upon integration by parts and use of (8).

As pointed out by Kreith and Travis, it is a corollary of Theorem 7 that the Schrödinger equation $\Delta v + B(x)v = 0$ is strongly oscillatory in E^2 if

$$\int_{E^2} B(x)\, dx = +\infty$$

on account of Leighton's oscillation theorem [7, p. 70]. Corresponding to every classical oscillation criterion for (8) there is an obvious strong oscillation criterion for (1) in E^n. Analogues of Theorem 7 for quasiconical or quasicylindrical domains in E^n can be established by similar analysis.

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University of British Columbia,
Vancouver, British Columbia