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RESTRICTED LIE ALGEBRAS AND THEIR ENVELOPES

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ABSTRACT. Let L be a restricted Lie algebra over a field of characteristic p. Denote by $u(L)$ its restricted enveloping algebra and by $\omega u(L)$ the augmentation ideal of $u(L)$. We give an explicit description for the *dimension subalgebras* of L, namely those ideals of L defined by $D_n(L) = L \cap \omega u(L)^n$ for each $n \geq 1$. Using this expression we describe the nilpotence index of $\omega u(L)$. We also give a precise characterisation of those L for which $\omega u(L)$ is a residually nilpotent ideal. In this case we show that the minimal number of elements required to generate an arbitrary ideal of $u(L)$ is finitely bounded if and only if L contains a 1-generated restricted subalgebra of finite codimension. Subsequently we examine certain analogous aspects of the Lie structure of $u(L)$. In particular we characterise L for which $u(L)$ is residually nilpotent when considered as a Lie algebra, and give a formula for the Lie nilpotence index of $u(L)$. This formula is then used to describe the nilpotence class of the group of units of $u(L)$.

1. **Introduction.** Let L be a restricted Lie algebra over a field K of characteristic p with p-map $x \longmapsto x^{[p]}$. We shall write $u(L)$ for the restricted enveloping algebra of L, and denote by $\omega u(L)$ the *augmentation ideal* of $u(L)$. In other words, $\omega u(L)$ is the ideal of $u(L)$ generated by L. The general focus of this article is to study the ring-theoretic structure of $u(L)$ in terms of the structure of L. To do this, we first examine certain canonical series of restricted ideals of L. The first of these series is given by $D_n(L) = L \cap \omega u(L)^n$ for each positive integer n. We call these ideals the *dimension subalgebras* of L because of their similarity to the dimension subgroups $D_n(G) = G \cap \left(1 + \Delta(G)^n\right)$ of a group G. Here $\Delta(G)$ represents the augmentation ideal of the group ring KG. The application of dimension subgroups to the study of group rings has proven very fruitful, and it is our hope that dimension subalgebras will be similarly useful in the study of enveloping algebras. We are especially encouraged since the analogy between dimension subalgebras and dimension subgroups seems to bear very close scrutiny indeed. We begin in Section 2 by giving the following explicit description of the dimension subalgebras of L:

$$D_n(L) = \sum_{ip^j \geq n} \gamma_i(L)^{[p]^j},$$

where $\gamma_i(L)^{[p]^j}$ is the restricted subalgebra of L generated by the set of p^j-th powers of the i-th term of the lower central series of L. This is accomplished with the help of a modified form of the PBW theorem for restricted enveloping algebras. This description is then employed to describe what might naturally be called the *Loewy-series* of $u(L)$, namely the series $\{c_n\}$ defined by $c_n = \dim \omega u(L)^n / \omega u(L)^{n+1}$, in terms of the series $\{d_n\}$ given

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by $d_n = \dim D_n(L)/D_{n+1}(L)$. From this we deduce an exact expression for the nilpotence index $t\big(u(L)\big)$ of $\omega u(L)$:

$$t\big(u(L)\big) = 1 + (p-1)\sum_{n\geq 1} nd_n$$

It is well known that $\omega u(L)$ is nilpotent if and only if L is finite dimension and of finite exponent. Saying that L has *finite exponent* means that $L^{[p]^m} = 0$ for some m. Subsequently, we show that $\omega u(L)$ is residually nilpotent if and only if L is residually 'nilpotent of finite exponent.' In Section 3 we go on to prove that, in this case, the minimal number of elements required to generate an arbitrary ideal of $u(L)$ is finitely bounded if and only if L contains a 1-generated restricted subalgebra of finite codimension. All these results have analogues in the study of modular group rings and can be found in the work of Hartley [H], Jennings ([Je1], [Je3]), Lazard [L], and Shalev [Sh1]. Finally let us mention that dimension subalgebras have also been employed by Riley and Semple in [RS1] to study restricted Lie algebras in terms of their nilpotency coclass, and in [RS2] to study a class of compact Hausdorff topological restricted Lie algebras.

In the ensuing four sections we study the structure of $u(L)$ when considered as a Lie algebra via the Lie product $[x,y] = xy - yx$. The Lie structure of $u(L)$ was studied previously by the authors in [RSh] and [R2]. The primary result in [RSh] was a characterisation of L (p odd) for which $u(L)$ is soluble as a Lie algebra. Presently, we would like to study the 'Lie powers' of $u(L)$. The n-th *upper Lie power* of an associative algebra R is the ideal defined inductively by $R^{(1)} = R$ and $R^{(n)} = [R^{(n-1)}, R]R$. The least index n for which $R^{(n)} = 0$ is denoted by $t^{\mathrm{Lie}}(R)$. We begin in Section 4 by giving an explicit formula for the *upper Lie dimension subalgebras* of L:

$$D_{(n)}(L) = L \cap u(L)^{(n)} = \sum_{(i-1)p^j \geq n-1} \gamma_i(L)^{[p]^j}.$$

We also prove that if $u(L)$ is upper Lie nilpotent then

$$t^{\mathrm{Lie}}\big(u(L)\big) = 2 + (p-1)\sum_{n\geq 1} nd_{(n+1)},$$

where $d_{(n)} = \dim D_{(n)}(L)/D_{(n+1)}(L)$.

However, there is another likely candidate for the n-th 'Lie power' of an associative algebra R. Namely one could also consider the ideal $\gamma_n(R)R$ generated by the n-th term of the lower central series of R, which we call the n-th *lower Lie power* of R. The index of R in this sense is denoted by $t_{\mathrm{Lie}}(R)$. While it is clear that $t_{\mathrm{Lie}}(R) \leq t^{\mathrm{Lie}}(R)$, it is possible that $t_{\mathrm{Lie}}(R) = 3$ and yet $t^{\mathrm{Lie}}(R) = \infty$. See Gupta and Levin, [GL]. We prove that restricted enveloping algebras are more well behaved in this respect. Indeed, in Section 5 we first prove that the n-th *lower dimension subalgebra* of L, $D_{[n]}(L) = L \cap \Big(\gamma_n\big(u(L)\big)u(L)\Big)$, coincides with $D_{(n)}(L)$ provided that $p > 3$. In [RSh] it was shown that, for all restricted Lie algebras L, $t_{\mathrm{Lie}}\big(u(L)\big)$ is finite if and only if $t^{\mathrm{Lie}}\big(u(L)\big)$ is finite. Actually more was shown in [R2]; namely, only the assumption that $u(L)$ is hypercentral is sufficient to conclude that $u(L)$ is upper Lie nilpotent. We show in Section 7 that in fact

$t_{\mathrm{Lie}}\big(u(L)\big) = t^{\mathrm{Lie}}\big(u(L)\big)$, again under the assumption that the characteristic of L exceeds 3. Finally, this time for all $p > 0$, we are able to show that lower and upper residual Lie nilpotence are equivalent properties for a restricted enveloping algebra $u(L)$, and that they coincide with the condition that L is residually 'nilpotent with derived subalgebra of finite exponent.' These results have group ring analogues: see Bhandari and Passi ([BP1], [BP2]), Parmenter, Passi and Sehgal [PPS], Passi and Sehgal [PS], and Riley [R1].

In the final section we apply some of the results mentioned in the last paragraph to study the group of units of $u(L)$. Specifically, we prove that if $\omega u(L)$ is nilpotent and $p > 3$ then the nilpotence class of the group of units of $u(L)$ is precisely $t_{\mathrm{Lie}}\big(u(L)\big) - 1 = 1 + (p-1)\sum_{n\geq 1} nd_{(n+1)}$.

2. **Filtrations and dimension subalgebras.** Let L be any restricted Lie algebra over a field K of characteristic p. A *filtration* of $\omega u(L)$ is a descending sequence of ideals of $u(L)$

$$\omega u(L) = E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$$

satisfying $E_iE_j \subseteq E_{i+j}$ for all i, j. Such a sequence gives rise naturally to a sequence of subspaces

$$L = L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n \supseteq \cdots$$

by defining $L_n = L \cap E_n$. It is a routine matter to verify that this sequence satisfies the conditions $[L_i, L_j] \subseteq L_{i+j}$ and $L_i^{[p]} \subseteq L_{ip}$ for all i, j. For a subspace S of L, by $S^{[p]}$ we mean the restricted subalgebra of L generated by the elements $x^{[p]}$, $x \in S$. We call any such descending sequence of subspaces satisfying these two properties a *p-filtration* of L. Observe that each L_i is necessarily a restricted ideal of L.

The canonical filtration of $\omega u(L)$ given by the powers of $\omega u(L)$ determines the *dimension subalgebras* $D_n(L) = L \cap \omega u(L)^n$ of L. The primary aim of this section is to investigate some of the basic properties of dimension subalgebras and then to apply the results to the study of $\omega u(L)$. However, the techniques developed here will be sufficiently general to study a similar p-filtration in Section 4.

We have just seen that a filtration of $\omega u(L)$ gives rise to a p-filtration of L. In fact the converse is also true. Suppose that $\{L_n\}$ is any p-filtration of L. For each $x \in L$ we define the *height* of x, which we shall denote by $\nu(x)$, to be the largest subscript n such that $x \in L_n$ if n exists, and to be ω if it does not. Now for each integer $m \geq 1$, let $E_m \subseteq u(L)$ be the K-linear span of all the products of the form

$$x_1 x_2 \cdots x_l$$

for some l, where $x_1, x_2, \ldots, x_l \in L$ and $\nu(x_1) + \cdots + \nu(x_l) \geq m$. It is easy to see that $\{E_n\}$ is a filtration of $\omega u(L)$. The close relationship between $\{L_n\}$ and $\{E_n\}$ is demonstrated in the following proposition.

THEOREM 2.1. *Let $\{L_n\}$ be a p-filtration of a restricted Lie algebra L, and let $\{x_i\}_{i\in I}$ be an ordered basis of L chosen so that $L_n = \langle x_i \mid \nu(x_i) \geq n\rangle_K$. Write $\{E_n\}$ for the filtration of $\omega u(L)$ induced by $\{L_n\}$. Then the following statements hold for each $n \geq 1$.*

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1. *$E_n = \langle \eta \mid \nu(\eta) \geq n \rangle_K$, where $\eta = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_l}^{a_l}$, $\nu(\eta) = \sum_{j=1}^{l} a_j \nu(x_{i_j})$, $i_1 < \cdots < i_l$, $l \geq 1$ and $0 \leq a_j \leq p-1$ for each j.*
2. *$\{\eta \mid \nu(\eta) = n\}$ is a K-basis of E_n modulo E_{n+1}.*
3. *$L_n = L \cap E_n$.*

PROOF. Let $E_{n,k}$ be the K-linear span of all the products of the form $y_1 y_2 \cdots y_l$, where $y_1, \ldots, y_l \in L$, $\sum_{j=1}^{l} \nu(y_j) \geq n$, and $l \leq k$. Let G be the subgroup of permutations σ of the permutation group $\mathcal{S}_k$ on $\{1, 2, \ldots, k\}$ that satisfy

$$y_1 y_2 \cdots y_k - y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(k)} \in E_{\sum_{j=1}^{k} \nu(y_j), k-1}$$

for all $y_1, \ldots, y_k \in L$.

CLAIM 1. $G = \mathcal{S}_k$.

Indeed, let $\tau \in \mathcal{S}_k$ satisfy $\tau(i) = i+1$, $\tau(i+1) = i$, and $\tau(j) = j$ for all $j \notin \{i, i+1\}$. Then

$$\begin{aligned} y_1 \cdots y_k - y_{\tau(1)} \cdots y_{\tau(k)} &= y_1 \cdots y_k - y_1 \cdots y_{i+1} y_i \cdots y_k \\ &= y_1 \cdots y_{i-1} [y_i, y_{i+1}] y_{i+2} \cdots y_k \\ &\in E_{\sum_{j=1}^{k} \nu(y_j), k-1} \end{aligned}$$

since $\nu([y_i, y_{i+1}]) \geq \nu(y_i) + \nu(y_{i+1})$. The claim now follows from the fact that $\mathcal{S}_k$ is generated by elementary transpositions.

CLAIM 2. $E_{n,k}$ is spanned by the set of monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_l}^{a_l}$, where $\sum_{j=1}^{l} a_j \nu(x_{i_j}) \geq n$, $i_1 < \cdots < i_l$, $0 \leq a_j \leq p-1$ for each j, and $\sum_{j=1}^{l} a_j \leq k$.

By definition these elements are contained in $E_{n,k}$. We prove that they span $E_{n,k}$ by induction on $k \geq 1$. The case $k = 1$ follows by our choice of basis:

$$E_{n,1} = \langle y \in L \mid \nu(y) \geq n \rangle_K = L_n.$$

Assume now that the claim holds for all $j < k$. Let $y_1, \ldots, y_k \in L$ satisfy $\sum_{j=1}^{k} \nu(y_j) \geq n$. Since $y_1 \cdots y_{k-1} \in E_{\sum_{j=1}^{k-1} \nu(y_j), k-1}$, by the induction hypothesis we find that $y_1 \cdots y_{k-1}$ is a linear combination of monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_l}^{a_l}$, where $i_1 < \cdots < i_l$, $0 \leq a_j \leq p-1$ for each j, $\sum_{j=1}^{l} a_j \nu(x_{i_j}) \geq \sum_{j=1}^{k-1} \nu(y_j)$, and $\sum_{j=1}^{l} a_j \leq k-1$. Also y_k is spanned by the x_{i_j} with $\nu(x_{i_j}) \geq \nu(y_k)$. However, by the first claim

$$x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_l}^{a_l} x_{i_j} - x_{i_1}^{a_1} \cdots x_{i_j}^{a_j+1} \cdots x_{i_l}^{a_l} \in E_{n,k-1}$$

Therefore we find that $y_1 \cdots y_k$ is spanned by the monomials $x_{i_1}^{a_1} \cdots x_{i_j}^{a_j+1} \cdots x_{i_l}^{a_l}$ modulo $E_{n,k-1}$. The claim now follows by induction unless $a_j = p-1$. In this case we may replace $x_{i_j}^{a_j+1} = x_{i_j}^{p}$ by $\sum \lambda_i x_i$, where $\lambda_i \in K$ and $\nu(x_i) \geq p\nu(x_{i_j})$. The resulting representation lies in $E_{n,k-p+1}$, and thus we are done by induction.

Part (1) now follows from the second claim since $E_n = \sum_{k\geq 1} E_{n,k}$. Parts (2) and (3) follow readily from part (1) and the PBW theorem for restricted enveloping algebras. See the monograph of Strade and Farnsteiner, [SF], for example. ∎

Close analogues of these facts are proved for ascending filtrations of (unrestricted) enveloping algebras in [SF; §1.1.9]. Notice that the generating elements of E_n in part (1) above actually form a basis of E_n by the PBW theorem.

Now set $E_0 = u(L)$ and put $l_n = \dim L_n/L_{n+1}$ and $e_n = \dim E_n/E_{n+1}$.

COROLLARY 2.2. *1. If each of the e_n is finite then the generating function $\sum_{n\geq 0} e_n\lambda^n$ in $Z[\lambda]$ is given by*

$$\sum_{n\geq 0} e_n\lambda^n = \prod_{j\geq 0}\Big(\frac{\lambda^{jp}-1}{\lambda^j-1}\Big)^{l_j}.$$

2. Suppose $E_r = 0$ for some r, and let t be the integer defined by $t = 1+\sum_{n\geq 1}(p-1)nl_n$. Then $E_t = 0$ but $E_{t-1}\neq 0$.

PROOF. Let $\{x_i\}_{i\in I}$ be a basis of L chosen as in Theorem 2.1. Then l_j is precisely the number of x_i's with height exactly j.

(1) Since the e_n are finite, so too must be the l_n. Thus, by part (2) of the theorem, we can choose $j\in I$ maximally so that $\nu(x_j)\leq n$ and rename $\{x_i \mid i\leq j\}$ simply as $\{x_1, x_2, \ldots, x_l\}$. Now any η with $\nu(\eta) = n$ can be written in the form $\eta = x_1^{a_1}\cdots x_l^{a_l}$. Then because e_n is the number of η with $\nu(\eta) = n$, we see that e_n is equal to the number of ways of choosing $0\leq a_i\leq p-1$ such that $\nu(\eta) = a_1\nu(x_1)+\cdots+a_l\nu(x_l) = n$. But it is not difficult to see that this number is the coefficient of λ^n in the product

$$\prod_{i=1}^{l}\Big(\frac{\lambda^{p\nu(x_i)}-1}{\lambda^{\nu(x_i)}-1}\Big).$$

But then

$$\sum_{n\geq 0} e_n\lambda^n = \prod_{j\geq 0}\Big(\frac{\lambda^{jp}-1}{\lambda^j-1}\Big)^{l_j}$$

since l_j is the number of x_i's with $\nu(x_i) = j$.

(2) Since $E_r = 0$, necessarily $\dim L = m$ is finite. Hence $\eta = x_1^{a_1}\cdots x_m^{a_m}$ is a typical basis element of $E_0 = u(L)$, which has greatest possible height when all the a_i are $p-1$. In this case

$$\nu(\eta) = (p-1)\sum_{i=1}^{m}\nu(x_i) = (p-1)\sum_{n\geq 1} nl_n = t-1.$$

Therefore $E_{t-1}\neq 0$ but $E_t = 0$. ∎

We are now ready to apply these results to the dimension subalgebras of L, but first we require some more notation. Write $\{L_n\}\leq\{H_n\}$ for two filtrations of L if $L_n\subseteq H_n$ for all $n\geq 1$. Recall that $\gamma_i(L)$ is the i-th term of the lower central series of L. We use the left-normed convention for longer commutators, and by $[x,{}_ny]$ we mean $[x, y, y, \ldots, y]$, where the y appears in the latter expression n times. We write $\gamma_i(L)_p$ for the restricted

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subalgebra of L generated by $\gamma_i(L)$. Sometimes we use L' in place of $\gamma_2(L)$, and by L'_p we mean $(L')_p$. For any subspace S of L, let $S^{[p]^j}$ denote the restricted subalgebra of L generated by the set of elements $x^{[p]^j}$ with $x \in S$. Put $D_\omega(L) = \bigcap_{n\geq 1} D_n(L)$ and $\omega u(L)^\omega = \bigcap_{n\geq 1} \omega u(L)^n$. Observe that $D_\omega(L) \subseteq \omega u(L)^\omega$. Finally we define inductively a decreasing sequence of subspaces $\{M_n(L)\}$ of L by

$$M_1(L) = L \quad \text{and} \quad M_n(L) = M_{\lceil n/p\rceil}(L)^{[p]} + [L, M_{n-1}(L)].$$

Here $\lceil n/p\rceil$ denotes the least integer greater than or equal to n/p. This series is clearly the analogue of the Brauer-Jennings-Zassenhaus M-series of a group.

THEOREM 2.3. *Let L be any restricted Lie algebra. Then the following statements hold.*

1. *The p-filtration $\{D_n(L)\}$ is the unique smallest p-filtration of L.*
2. *The filtration of $\omega u(L)$ induced by $\{D_n(L)\}$ is precisely $\{\omega u(L)^n\}$.*
3. *$D_n(L) = M_n(L) = \sum_{ip^j\geq n} \gamma_i(L)^{[p]^j}$, for each integer $n \geq 1$.*
4. *$[D_m(L), D_n(L)] \subseteq \gamma_{m+n}(L)$, for each pair of integers $m, n \geq 1$.*

PROOF. (1) Assume that $\{L_n\} \leq \{D_n(L)\}$, and let $\{E_n\}$ be the filtration of $\omega u(L)$ induced by $\{L_n\}$. Then because $\omega u(L) = E_1$, we have $\omega u(L)^n = E_1^n \subseteq E_n$. Hence

$$D_n(L) \subseteq L \cap \omega u(L)^n \subseteq L \cap E_n = L_n$$

by part (3) of Theorem 2.1, and so $L_n = D_n(L)$.

(2) Let $\{E_n\}$ be the filtration of $\omega u(L)$ induced by $\{D_n(L)\}$. Then, as above, we have $\omega u(L)^n \subseteq E_n$. Conversely let $\{x_i\}_{i\in I}$ be a basis of L as in Theorem 2.1 with respect to the p-filtration $\{D_n(L)\}$, and let $x_{i_1}, \ldots, x_{i_l} \in L$ and $a_1, \ldots, a_l \in \{0, 1, \ldots, p-1\}$ be such that $i_1 < \cdots < i_l$ and $a_1\nu(x_{i_1}) + \cdots + a_l\nu(x_{i_l}) \geq n$. Then $\eta = x_{i_1}^{a_1} \cdots x_{i_l}^{a_l}$ is a typical basis element of E_n. But $x_{i_j} \in D_{\nu(x_{i_j})}(L) \subseteq \omega u(L)^{\nu(x_{i_j})}$, so that $\eta \in \omega u(L)^n$. Note that possibly $\nu(x_{i_j}) = \omega$. Hence $E_n = \omega u(L)^n$ as required.

(3) Set $L_n = \sum_{ip^j\geq n} \gamma_i(L)^{[p]^j}$. We first endeavour to show that $L_n \subseteq M_n(L) \subseteq D_n(L)$. Clearly $\gamma_i(L) \subseteq M_i(L)$ for each $i \geq 1$. Thus $\gamma_i(L)^{[p]} \subseteq M_i(L)^{[p]}$, and so by induction $\gamma_i(L)^{[p]^j} \subseteq \left(\gamma_i(L)^{[p]^{j-1}}\right)^{[p]} \subseteq M_{ip^{j-1}}(L)^{[p]} \subseteq M_{ip^j}(L)$. Thus $L_n \subseteq M_n(L)$. We show that $M_n(L) \subseteq D_n(L)$ by induction on n. When $n = 1$ this is trivial. Suppose then that $M_k(L) \subseteq D_k(L)$ for all $k < n$. Then

$$M_n(L) = M_{\lceil n/p\rceil}(L)^{[p]} + [L, M_{n-1}(L)] \subseteq D_{\lceil n/p\rceil}(L)^{[p]} + [L, D_{n-1}(L)] \subseteq D_n(L)$$

since $\{D_m(L)\}$ is a p-filtration of L.

To obtain the reverse inclusions, it suffices by part (1) to show only that $\{L_n\}$ is a p-filtration of L. The fact that $L = L_1 \supseteq L_2 \supseteq \cdots$ is clear. Fix positive integers a, b, i, j such that $ap^b \geq m$ and $ip^j \geq n$. Suppose that $x_a \in \gamma_a(L)$ and $y_i \in \gamma_i(L)$. Then

$$[x_a^{[p]^b}, y_i^{[p]^j}] = -[y_i, {}_{p^b}x_a, {}_{p^j-1}y_i] \in \gamma_{ap^b+ip^j}(L) \subseteq \gamma_{m+n}(L).$$

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It follows that $[L_m, L_n] \subseteq \gamma_{m+n}(L) \subseteq L_{m+n}$. Finally suppose that $x_{i_k} \in \gamma_{i_k}(L)$, $i_k p^{j_k} \geq n$ and $\alpha_k \in K$ for all $k \in \{1, \ldots, l\}$. Then $x = \sum_{k=1}^{l} \alpha_k x_{i_k}^{[p]^{j_k}}$ is a typical element of L_n. Observe that $x^{[p]} \equiv \sum_{k=1}^{l} \alpha_k^p x_{i_k}^{[p]^{j_k+1}}$ modulo $\gamma_p(L_n)$ by one of the defining properties of a p-map. But $\gamma_p(L_n) \subseteq \gamma_{pn}(L)$ as above. Hence $L_n^{[p]} \subseteq L_{np}$ for all n, and $\{L_n\}$ is a p-filtration of L.

(4) This follows from the proof of (3) above. ∎

Let us write $d_n = d_n(L)$ for the dimension of $D_n(L)/D_{n+1}(L)$ and c_n for the dimension of $\omega u(L)^n / \omega u(L)^{n+1}$. We call $\{c_n\}$ the *Loewy series* of $u(L)$. It is easy to see that $c_1 = d_1$ and that the c_n and d_n are all finite if and only if d_1 is finite. Applying Corollary 2.2 we obtain

COROLLARY 2.4. *1. If d_1 is finite and λ is an indeterminant, then*

$$\sum_{n\geq 0} c_n \lambda^n = \prod_{j\geq 0} \left(\frac{\lambda^{jp} - 1}{\lambda^j - 1} \right)^{d_j}.$$

2. Suppose that $\omega u(L)$ is a nilpotent ideal of $u(L)$. Then the integer $t = 1 + (p-1)\sum_{n\geq 1} n d_n$ is the least index such that $\omega u(L)^t = 0$.

Let us denote the integer t in the corollary by $t\big(u(L)\big)$ or just $t(L)$.

We remark that another immediate consequence of the above theorem is that $D_n(L) = D_{\lceil n/p \rceil}(L)^{[p]} + \gamma_n(L)$ for each $n \geq 1$.

LEMMA 2.5. *A restricted Lie algebra is nilpotent with finite exponent if and only if $D_n(L) = 0$ for sufficiently large n.*

PROOF. Sufficiency is clear from Theorem 2.3, so choose m so large that both $\gamma_m(L) = 0$ and $L^{[p]^l} = 0$ whenever $p^l \geq m$. Then from the remark above we know that

$$D_{mp}(L) = D_m(L)^{[p]} + \gamma_{mp}(L) = D_m(L)^{[p]}.$$

By induction we obtain $D_{mp^j}(L) = D_m(L)^{[p]^j}$ for all $j \geq 1$. Now take $n = mp^l$ and notice that $D_n(L) = 0$. ∎

As another application of Theorem 2.3, we are now ready to classify all restricted enveloping algebras $u(L)$ for which $\omega u(L)$ is a residually nilpotent ideal.

THEOREM 2.6. *Let L be any restricted Lie algebra. The following statements are equivalent:*

(i) $\omega u(L)^\omega = 0$;

(ii) $D_\omega(L) = 0$; and

(iii) L is residually 'nilpotent and of finite exponent'.

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PROOF. The implication (i) $\Rightarrow$ (ii) is clear. To prove (ii) $\Rightarrow$ (iii) suppose that $D_\omega(L) = 0$. Let x be a nontrivial element of L. Then $x \notin D_n(L)$ for some n. By Lemma 2.5 it is clear that $L/D_n(L)$ is nilpotent and of finite exponent. Hence the implication holds.

Finally consider (iii) $\Rightarrow$ (i). Suppose to the contrary that L is residually 'nilpotent and of finite exponent' but that y is a nontrivial element of $\omega u(L)^\omega$. Let $\{x_i\}_{i\in I}$ be a basis of L as in Theorem 2.1 with respect to $\{D_n(L)\}$. By the PBW theorem for restricted enveloping algebras we can write y uniquely as the linear combination of monomials of the form $x_{i_1}^{a_1}\cdots x_{i_l}^{a_l}$, where $i_1 < \cdots < i_l$ and the a_j lie in $\{0,1,\ldots,p-1\}$. But then, by part (2) of the Theorem 2.3, the height of each of these monomials must be infinite for otherwise $y \notin \omega u(L)^{n+1}$, where n is the least of these finite heights. Notice that $\nu(x_{i_j}) = \omega$ is equivalent to the fact that $x_{i_j} \in D_\omega(L)$. Thus it suffices to show only that $D_\omega(L) = 0$ in order to obtain the desired contradiction. To this end, assume that $0 \neq y \in D_\omega(L)$. Then by assumption there exists a restricted ideal I of L such that $y \notin I$ and L/I is nilpotent and of finite exponent. Therefore we have $0 \neq y + I \in D_n(L) + I/I = D_n(L/I)$ for all n. This contradicts Lemma 2.5. ∎

3. **Nilpotency indices and numbers of generators of ideals.** Let $\mathcal{F}_p$ denote the class of finite dimensional restricted Lie algebras with finite exponent; in other words, the class of restricted Lie algebras L with $\omega u(L)$ nilpotent. It is well known that L lies in the class $\mathcal{F}_p$ precisely when $\omega u(L)$ is nilpotent. See [RSh], for example. Throughout this section we shall assume that $L \in \mathcal{F}_p$ unless stated specifically otherwise.

We shall write $I \lhd_p L$ to indicate that I is a restricted ideal of L. In this case, we put $\Phi_L(I) = I^{[p]} + [I, L]$. The proof of the next lemma is identical to that of the analogous statement for p-groups.

LEMMA 3.1. *Suppose $I \lhd_p L$ is nontrivial. Then the following statements hold.*
1. $\Phi_L(I) = \bigcap\{J \mid J \lhd_p L, J \subset I, \dim I/J = 1\}$.
2. *If S is a subset of I whose image in $I/\Phi_L(I)$ spans $I/\Phi_L(I)$ then S generates I as a restricted ideal.*
3. $\Phi_L(I) \neq I$.

COROLLARY 3.2. *If $D_n(L) = D_{pn}(L)$ then $D_n(L) = 0$.*

PROOF. We have

$$D_{pn}(L) = D_n(L)^{[p]} + \gamma_{pn}(L) \subseteq D_n(L)^{[p]} + [D_n(L), L] = \Phi_L\big(D_n(L)\big),$$

so that $D_n(L) = D_{pn}(L) = \Phi_L\big(D_n(L)\big)$. Thus $D_n(L) = 0$ by Lemma 3.1. ∎

Recall that $d_n = \dim D_n(L)/D_{n+1}(L)$. Observe that Corollary 3.2 implies that, if $d_n = d_{n+1} = \cdots = d_{pn-1} = 0$ then $d_m = 0$ for all $m \geq n$.

LEMMA 3.3. *Suppose that $d_n = 0$. Then*
1. $\gamma_{n+1}(L) = 0$,
2. $d_m = 0$ *for all $m > n$ such that m is prime to p,*
3. $D_{pm}(L) = D_m(L)^{[p]}$ *provided $pm > n$, and*
4. $d_{pm} \leq d_m$ *provided $pm > n$ and K is finite.*

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PROOF. (1) We have $\gamma_n(L) \subseteq D_n(L) = D_{n+1}(L)$, so that

$$\gamma_{n+1}(L) \subseteq [D_n(L), L] = \gamma_{n+2}(L).$$

Hence $\gamma_{n+1}(L) = 0$ since L is nilpotent.

(2) Since $D_m(L) = D_{p\lceil \frac{m}{p} \rceil}(L) + \gamma_m(L)$, the result follows from part (1).

(3) By part (1) we have $D_{pm}(L) = D_m(L)^{[p]} + \gamma_{pm}(L) = D_m(L)^{[p]}$.

(4) Consider the map $\phi\colon D_m(L)/D_{m+1}(L) \longrightarrow D_{pm}(L)/D_{pm+1}(L)$ induced by the p-map. If $x, y \in D_m(L)$ then $\gamma_p(\langle x, y\rangle) \subseteq \gamma_{pm}(L) \subseteq \gamma_{n+1}(L) = 0$, and thus $x^{[p]} + y^{[p]} = (x+y)^{[p]}$. Therefore ϕ is a surjection because, by part (3), $D_{pm}(L)$ is spanned by elements of the form $x^{[p]}$, $x \in D_m(L)$. Since K is finite, we can conclude that $d_{pm} \leq d_m$. ∎

Similar as well as additional constraints for p-groups were studied by the second author in [Sh1]. However, the analogy here between p-groups and restricted Lie algebras in the class $\mathcal{F}_p$ is not perfect. Indeed, for a p-group G define integers $d_i(G)$ by $p^{d_i(G)} = |D_i(G)/D_{i+1}(G)|$, where $D_i(G)$ is the i-th dimension subgroup of G over K. In [Sh1], it was shown that $d_{p^i}(G) = 0$ implies $D_{p^i}(G) = 1$. The analogous statement is not true in $\mathcal{F}_p$. In fact, $d_p(L) = 0 \not\Rightarrow d_n(L) = 0$ for all $n > p$.

EXAMPLE. Let L be the 4-dimensional restricted Lie algebra over F_p generated by x, y, z, w subject to the relations $x^{[p]} = y^{[p]} = w^{[p]} = 0$, $z^{[p]} = w$ and $[x, y] = z$ is central. Then $L^{[p]} = F_p w$, $L' = F_p z$ and $\gamma_3(L) = 0$. When p is odd this yields

1. $D_p(L) = L^{[p]} + \gamma_p(L) = F_p w$,
2. $D_{2p}(L) = L^{[p]^2} + \gamma_2(L)^{[p]} + \gamma_{2p}(L) = (F_p z)^{[p]} = F_p w$, and
3. $D_{2p+1}(L) = L^{[p]^2} + \gamma_3(L)^{[p]} + \gamma_{2p+1}(L) = 0$.

Therefore $d_{2p} = 1$ despite the fact that $d_p = 0$. In particular notice that $D_p(L) \neq 0$.

It is likely that a closer look at arithmetical properties of the series $\{d_n\}$ in the Lie algebra case will yield further constraints.

We can now study the nilpotency index of $\omega u(L)$, denoted by $t(L)$. We say that L is *cyclic* if it is generated as a restricted Lie algebra by a single element. This means that $L = \langle x^{[p]^i} \mid i \geq 0\rangle_K$ for some x. Recall that a restricted Lie algebra L has finite exponent if $L^{[p]^k} = 0$ for some positive integer k. If e is the least integer such that $L^{[p]^e} = 0$ then we put $\exp(L) = e$.

PROPOSITION 3.4. *We have $t(L) \leq p^{\dim L}$, with equality if and only if L is cyclic.*

PROOF. Given $\{d_n\}$, let $\{n_k\}_{k\geq 0}$ be a monotonically increasing series of positive integers with the property that each $n \in N$ occurs in $\{n_k\}$ exactly d_n times. Then $\sum n_k = \sum n d_n$, and so by Corollary 2.4 we have

$$t(L) = 1 + (p-1)\sum_{k\geq 0} n_k.$$

It is clear that $d_1 \geq 1$, so $n_0 = 1$. By the remark following Corollary 3.2 we have $n_{k+1} \leq p n_k$. Therefore by induction we obtain $n_k \leq p^k$. Let $d = \dim L = \sum d_n$. Then the length of the series $\{n_k\}$ is exactly d. Combining our deductions to this point we obtain

$$t(L) \leq 1 + (p-1) \sum_{0 \leq k < d} p^k = p^d.$$

Now, equality implies $n_k = p^k$ for all $0 \leq k < d$. In particular, $n_1 = p$, so 1 occurs in $\{n_k\}$ exactly once. This means that $d_1 = 1$. It follows that

$$\dim L/D_p(L) = 1.$$

But $D_p(L) = L^{[p]} + \gamma_p(L) \subseteq \Phi_L(L)$, so we may conclude that L is cyclic by Lemma 3.1. ∎

As for lowers bounds, we remark that

$$t(L) \geq 1 + (p-1)\dim L,$$

with equality precisely when $L' = L^{[p]} = 0$. The easy proof is omitted.

Put $p^e = \exp(L)$. If $x \in L$ has exponent p^e then we have

$$t(L) \geq t(\langle x \rangle) \geq p^e.$$

Therefore $t(L)$ is large if $\exp(L)$ is large. The next result establishes a partial converse to this phenomenon.

PROPOSITION 3.5. *Let $d = \dim L$, $p^e = \exp(L)$, and suppose that $t(L) \geq p^{d-c}$ for a fixed constant c. Then $e \geq d - c'$, where c' depends only on c.*

PROOF. Choose positive integers c_1, c_2 depending on c such that $c_2 > 2c_1$ and

$$(2c_1 - 1)p^{c_1+1} + p^{d-c_1+2} < p^{d-c}$$

for all $d \geq c_2$. Suppose that $d \geq c_2$.

CLAIM. $\dim L/D_{p^{c_1}}(L) < 2c_1$.

Suppose to the contrary, and let $\{n_k\}$ be as in the proof of Proposition 3.4. Then $n_0, n_1, \ldots, n_{2c_1-1} < p^{c_1}$. Since $n_{k+1} \leq pn_k$, we also have

$$n_{2c_1} < p^{c_1+1}, n_{2c_1+1} < p^{c_1+2}, \ldots, n_d < p^{d-c_1+1}.$$

It follows that

$$\sum_{k=0}^{d-1} n_k < 2c_1p^{c_1} + (p^{c_1+1} + \cdots + p^{d-c_1+1}) = 2c_1p^{c_1} + \frac{p^{d-c_1+2} - p^{c_1+1}}{p-1}.$$

Hence

$$\begin{aligned} t(L) = 1 + (p-1)\sum_{k=0}^{d-1} n_k &\leq (p-1)\cdot 2c_1p^{c_1} + p^{d-c_1+2} - p^{c_1+1} \\ &\leq (2c_1 - 1)p^{c_1+1} + p^{d-c_1+2} < p^{d-c}. \end{aligned}$$

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by the choice of c_1. This contradicts the basic assumption.

Having proved the claim, it follows that for some $0 \leq i < c_1$ we have $\dim D_{p^i}(L)/D_{p^{i+1}}(L) = 1$. This implies that $d_n = 0$ for some $p^i \leq n < p^{i+1}$, so $D_{p^{i+1}}(L) = D_{p^i}(L)^{[p]}$ by Lemma 3.3. Setting $H = D_{p^i}(L)$ we see that $H/H^{[p]}$ is 1-dimensional, and hence cyclic. We also have $\dim L/H \leq \dim L/D_{p^{c_1}}(L) < 2c_1$. Recall that we have assumed that $d \geq c_2$. We see that, in any case, the proposition holds with $c' = \max\{2c_1 - 1, c_2\} = c_2$ ∎

We can now prove the main results of this section. For an ideal I in $u(L)$, let $\nu(I)$ denote the minimal number of generators required to generate I as an ideal, and let

$$\bar{\nu}\big(u(L)\big) = \sup\{\nu(I) \mid I \lhd u(L)\}.$$

We shall show that, in a sense, $\bar{\nu}\big(u(L)\big)$ is small if and only if L has a cyclic restricted subalgebra of small codimension. First note that, if $H \leq_p L$ is cyclic of codimension c, then $\bar{\nu}\big(u(H)\big) = 1$ and $\bar{\nu}\big(u(L)\big) \leq p^c$, as $u(L)$ is a free $u(H)$-module of rank p^c. The next result concerns the converse.

THEOREM 3.6. *Suppose $\bar{\nu}\big(u(L)\big) \leq p^c$. Then L has a cyclic restricted subalgebra H such that* $\dim L/H \leq c'$, *where c' depends only on c.*

PROOF. Let $c_i = \dim \omega u(L)^i/\omega u(L)^{i+1}$ for $i \geq 0$. It is clear that $c_i = \nu\big(\omega u(L)^i\big)$ by Nakayama's Lemma, so that in particular $c_i \leq \bar{\nu}\big(u(L)\big) \leq p^c$ for all i. Let $t = t(L)$ and $d = \dim L$. Then

$$p^d = \dim u(L) = \sum_{0 \leq i < t} c_i \leq tp^c.$$

Thus $t \geq p^{d-c}$. The result now follows by applying Proposition 3.5. ∎

It is now a straightforward matter to deduce the following result.

THEOREM 3.7. *Let L be a restricted Lie algebra that is residually 'nilpotent of finite exponent'. Then $\bar{\nu}\big(u(L)\big)$ is finite if and only if L has a cyclic restricted subalgebra of finite codimension.*

PROOF. Sufficiency is clear, so let us prove necessity. From Theorem 2.6 we know that $D_\omega(L) = 0$. Suppose that $\bar{\nu}\big(u(L)\big) \leq p^c$. Then $\dim L/D_n(L) \in \mathcal{F}_p$ since $\dim \omega u(L)/\omega u(L)^n = \sum_{i=1}^{n-1} c_i \leq (n-1)p^c$. Because $\bar{\nu}\Big(\big(L/D_n(L)\big)\Big) \leq p^c$ for all n, applying Theorem 3.6 yields restricted subalgebras H_n such that $H_n \supseteq D_n(L)$, $\dim L/H_n \leq c'$ and $H_n/D_n(L)$ is cyclic for all n, where c' is as in the theorem. Let $H = \bigcap_{n \geq 0} H_n$. We claim that L/H is finite dimensional. Indeed $L/\Phi_L(L)$ is finite dimensional for otherwise $u\big(L/\Phi_L(L)\big)$ has no bound on the number of generators of ideals. It follows that $L/\Phi_L\Big(\Phi_L\big(\cdots\big(\Phi_L(L)\big)\big)\Big)$ is also finite dimensional. Therefore because $H \supseteq \Phi_L\Big(\Phi_L\big(\cdots\big(\Phi_L(L)\big)\big)\Big)$, where the iteration is carried out c' times, the claim is verified. Since $H + D_n(L)/D_n(L)$ is cyclic for all n, it follows that H is cyclic, as required. ∎

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4. **Upper Lie dimension subalgebras.** To make the notation less cumbersome, henceforth $u(L)$ will be denoted by R. Recall from the Section 1 the definitions of the upper and lower Lie powers of R. To study these two series of ideals of R it is helpful to first study what we shall term the Lie dimension subalgebras of L; namely,

$$D_{[m]}(L) = L \cap \big(\gamma_m(R)R\big) \quad \text{and} \quad D_{(m)}(L) = L \cap R^{(m)}.$$

More precisely, $D_{(m)}(L)$ is the m-th *upper Lie dimension subalgebra* of L, while $D_{[m]}(L)$ is called the m-th *lower Lie dimension subalgebra* of L. They will play roles similar to that played by the dimension subalgebras in Section 2. The analogy is closest when considering upper Lie dimension subalgebras, which we shall be our sole area of concentration in this present section. Although it is not true that $\{R^{(m)}\}$ is always a filtration of $\omega u(L)$, if we define $A_m = u(L'_p) \cap R^{(m+1)}$ for each $m \geq 1$, it turns out that $\{A_m\}$ is a filtration of $\omega u(L'_p)$.

The second part of the next lemma is reminiscent of a theorem due to Sandling, [S].

LEMMA 4.1. *Let L be an arbitrary restricted Lie algebra.*

1. $R^{(m)}R^{(n)} \subseteq R^{(m+n-1)}$ *for each pair of integers* $m, n \geq 1$.
2. $R^{(m+1)} = \sum\big(\prod_j \gamma_{m_j}(L)\big)R$, *the ideal generated by* $\sum\big(\prod_j \gamma_{m_j}(L)\big)$, *for all integers* $m \geq 1$, *where the sum is over all finite sequences* $\{m_j\}$ *for which* $m_j \geq 2$ *and* $\sum(m_j - 1) = m$.

PROOF. (1) This is well known and true for all rings R. It follows from the Jacobi identity and the identity $[ab, c] = a[b, c] + [a, c]b$.

(2) The fact that the right hand side is contained in the left follows from part (1). To prove the reverse inclusion we use induction on $m \geq 1$. The case $m = 1$ follows from the identity above. Using the same identity we also find that

$$\begin{aligned} R^{(m+1)} &= [R^{(m)}, R]R \\ &\subseteq \Big[\sum\Big(\prod_j \gamma_{m_j}(L)\Big)R, R\Big]R \\ &\subseteq \sum\Big(\prod_j \gamma_{m_j}(L)\Big)[R, R]R + \sum\Big[\prod_j \gamma_{m_j}(L), R\Big]R \\ &\subseteq \sum\Big(\prod_j \gamma_{m_j}(L)\Big)\gamma_2(L)R + \sum\Big(\prod_j \gamma_{n_j}(L)\Big)R, \end{aligned}$$

where $n_j = m_j$ for all but one $j = j'$ for which $n_{j'} = m_{j'} + 1$. Therefore $\sum(n_j - 1) = m + 1$, as required. ■

LEMMA 4.2. *For each positive integer m the following statements hold.*

1. $A_m = \sum\big(\prod_j \gamma_{m_j}(L)\big)u(L'_p)$, *where the sum is taken over all finite sequences* $\{m_j\}$ *for which* $\sum(m_j - 1) = m$.
2. $D_{(m+1)}(L) = L \cap A_m$.

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PROOF. (1) The inclusion ($\supseteq$) follows from part (2) of the previous lemma. Let C be a vector space complement of L'_p in L. Fix an ordered basis of L by extending an ordered basis of L'_p by an ordered basis of C. For each $y \in L$ write $y = a + c$, where $a \in L'_p$ and $c \in C$. With respect to the fixed basis of L define the K-linear map $\theta\colon u(L) \longrightarrow u(L'_p)$ induced by $y \longmapsto a$. The fact that θ is well defined follows from the PBW theorem. Now let $x \in R^{(m+1)}$. Then, by part (2) of Lemma 4.1, $x = \sum(\prod_j g_{m_j})y$, where $g_{m_j} \in \gamma_{m_j}(L)$, $y \in u(L)$ and $\sum(m_j - 1) = m$ in each summand. Then because $\theta(x) = x$, we see that in the expression for x we may replace the y's by elements lying in $u(L'_p)$.

(2) This can be deduced from part (1) and the fact that

$$D_{(m+1)}(L) \subseteq D_{(2)}(L) = L'_p. \qquad \blacksquare$$

For each integer $m \geq 1$ put $L_m = \sum_{(i-1)p^j \geq m} \gamma_i(L)^{[p]^j}$.

LEMMA 4.3. *1.* $\{L_m\}$ *is a p-filtration of* L'_p.
2. $L_m \subseteq D_{(m+1)}(L)$ *for each integer* $m \geq 1$.

PROOF. (1) Imitating the proof of part (4) of Theorem 2.3 we easily obtain that actually $[L_m, L_n] \subseteq \gamma_{m+n+2}(L) \subseteq L_{m+n+1}$. To see that $L_m^{[p]} \subseteq L_{mp}$, suppose that $x_{i_k} \in \gamma_{i_k}(L)$, $(i_k - 1)p^{j_k} \geq m$ and $\alpha_k \in K$. Then $x = \sum_k \alpha_k x_{i_k}^{[p]^{j_k}}$ is a typical element of L_m. Observe that

$$x^{[p]} \equiv \sum_k \alpha_k x_{i_k}^{[p]^{j_k+1}} \quad \text{modulo} \quad \gamma_p(L_m).$$

Now since $\gamma_p(L_m) \subseteq \gamma_{mp+p}(L) \subseteq L_{mp}$, we can conclude $x^{[p]} \in L_{mp}$ as desired.

(2) Suppose that i and j are such that $(i-1)p^j \geq m$. Then, by part (1) of Lemma 4.1, if $x \in \gamma_i(L)$ then

$$x^{[p]^j} \in (R^{(i)})^{p^j} \subseteq R^{((i-1)p^j+1)} \subseteq R^{(m+1)}.$$

The result follows. $\blacksquare$

Define a sequence $\{M_{(n)}(L)\}$ of subspaces of L by

$$M_{(1)}(L) = L,\ M_{(2)}(L) = L'_p \quad \text{and} \quad M_{(n+1)}(L) = M_{(\lceil \frac{n+p}{p} \rceil)}(L)^{[p]} + [L, M_{(n)}(L)].$$

We are now ready for the primary result of this section.

THEOREM 4.4. *Let L be any restricted Lie algebra of characteristic p.*
1. The filtration of $\omega u(L'_p)$ *induced by* $\{L_m\}$ *is* $\{A_m\}$.
2. $D_{(m)}(L) = M_{(m)}(L) = \sum_{(i-1)p^j \geq m-1} \gamma_i(L)^{[p]^j}$ *for each integer* $m \geq 1$.
3. $[D_{(m)}(L), D_{(n)}(L)] \subseteq \gamma_{m+n}(L)$ *for each pair of integers* $m, n \geq 1$.

PROOF. (1) Choose arbitrary elements x_j from L_{m_j} $(1 \leq j \leq l)$ such that $\sum_{j=1}^{l} m_j \geq m$. Then $x_1 x_2 \cdots x_l$ is a typical generator of the m-th term of the filtration of $\omega u(L'_p)$ induced by $\{L_m\}$, which we shall call B_m. Then by Lemmas 4.1 and 4.3,

$$x_1 x_2 \cdots x_l \in \prod_{j=1}^{l} R^{(m_j+1)} \subseteq R^{(1+\sum_{j=1}^{l} m_j)} \subseteq R^{(m+1)}.$$

Hence $B_m \subseteq A_m$. For the reverse inclusion, choose $x \in A_m$. Then according to Lemma 4.2 we can write $x = \sum(\prod_j x_{m_j})y$, where each $x_{m_j} \in \gamma_{m_j}(L)$, $y \in u(L'_p)$ and $\sum(m_j - 1) = m$. But then $x \in \sum(\prod_j L_{m_j-1})u(L'_p) \subseteq B_m$, and therefore $B_m = A_m$.

(2) As in part (3) of Theorem 2.3 we find easily that $L_m \subseteq M_{(m+1)}(L) \subseteq D_{(m+1)}(L)$. For the inclusion $D_{(m+1)}(L) \subseteq L_m$ use part (3) of Theorem 2.1 together with part (2) of Lemma 4.2.

(3) See the proof of Lemma 4.3. ∎

Let us write $d_{(m)}$ for the dimension of $D_{(m)}(L)/D_{(m+1)}(L)$.

COROLLARY 4.5. *If $R^{(n)} = 0$ for some n, then integer the t defined by $t = 2 + \sum_{m\geq 1}(p-1)md_{(m+1)}$ is the least index for which $R^{(t)} = 0$. In other words, $t = t^{\mathrm{Lie}}(R)$.*

PROOF. Since $R^{(m+1)}$ and A_m are generated by the same sets by Lemmas 4.1 and 4.2, it is clear that $R^{(m+1)} = 0$ if and only if $A_m = 0$. Now appeal to Corollary 2.2. ∎

Let us just remark that we could define a 'Lie-Loewy' series of $u(L)$ with a generating function similar to the one given in Corollary 2.4.

5. **Lower Lie dimension subalgebras.** The primary result of this section is that for most characteristics upper and lower Lie dimension subalgebras coincide.

THEOREM 5.1. *If L is any restricted Lie algebra of characteristic $p > 3$, then for each integer $m \geq 1$*

$$D_{[m]}(L) = D_{(m)}(L) = \sum_{(i-1)p^j \geq m-1} \gamma_i(L)^{[p]^j}.$$

The following simple lemma will play a key role throughout the remainder of this article.

LEMMA 5.2. *For each integer $j \geq 1$ and for each pair of elements x, y in an arbitrary restricted Lie algebra L, we have the following:*

$$(x+y)^{[p]^j} \equiv x^{[p]^j} + y^{[p]^j} \quad \textit{modulo} \quad \sum_{l=1}^{j} \gamma_{p^l}(H)^{[p]^{j-l}},$$

where $H = \langle x, y\rangle$ is the subalgebra of L generated by x and y.

PROOF. We use induction on j. If $j = 1$, then $(x+y)^{[p]} \equiv x^{[p]} + y^{[p]}$ modulo $\gamma_p(H)$. Assume then that

$$(x+y)^{[p]^j} = x^{[p]^j} + y^{[p]^j} + \sum \alpha h_{p^l}^{[p]^{j-l}}$$

where each $h_{p^l} \in \gamma_{p^l}(H)$ and $\alpha \in K$ depends on h_{p^l}. Then by the initial step of the induction we have

$$(x+y)^{[p]^{j+1}} \equiv x^{[p]^{j+1}} + y^{[p]^{j+1}} + \sum \alpha^p h_{p^l}^{[p]^{j+1-l}} \quad \text{modulo} \quad \gamma_p(H_1),$$

where H_1 is generated by $x^{[p]^j}, y^{[p]^j}$ and the $h_{p^l}^{[p]^{j-l}}$ appearing in the sum above. But it is easy to verify that $\gamma_p(H_1) \subseteq \gamma_{p^{j+1}}(H)$. This completes the proof. ∎

LEMMA 5.3. *If $[x_1, \ldots, x_i]^{[p]^j} \in \gamma_{(i-1)p^j+1}(R)R$ for all $i \geq 2$, $j \geq 0$ and $x_1, \ldots, x_i \in L$, then $D_{(m)}(L) \subseteq D_{[m]}(L)$ for all $m \geq 1$.*

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PROOF. We may assume that $m \geq 2$. For each $i \geq 2$ fix $j(i) \geq 0$ minimally so that $(i-1)p^{j(i)} \geq m-1$. It suffices by part (2) of Theorem 4.4 to prove that $\gamma_i(L)^{[p]^{j(i)}} \subseteq \gamma_m(R)R$. To see this, we use reverse induction on i ($2 \leq i \leq m$). Indeed, if $i = m$ this is obvious. Now assume that $\gamma_k(L)^{[p]^{j(k)}} \subseteq \gamma_m(R)R$ for all $k > i$. Because any $x \in \gamma_i(L)$ is the sum of monomials $\alpha[x_1, \dots, x_i]$, with $\alpha \in K$ and each $x_l \in L$, using Lemma 5.2 we find that

$$x^{[p]^{j(i)}} \equiv \sum \alpha^{p^{j(i)}}[x_1, \dots, x_i]^{[p]^{j(i)}} \quad \text{modulo} \quad \sum_{l=1}^{j(i)} \gamma_{p^l}(H)^{[p]^{j(i)-l}},$$

where $H = \gamma_i(L)$. But observe that $(ip^l - 1)p^{j(i)-l} \geq (i-1)p^{j(i)} \geq m-1$, so that $j(i) - l \geq j(ip^l)$. Now since $ip^l \geq ip > i$, the induction hypothesis implies that

$$\gamma_{p^l}(H)^{[p]^{j(i)-l}} \subseteq \gamma_{ip^l}(L)^{[p]^{j(ip^l)}} \subseteq \gamma_m(R)R$$

for each l ($1 \leq l \leq j(i)$). It now follows that $x^{[p]^{j(i)}} \in \gamma_m(R)R$, as required. ∎

LEMMA 5.4. *Let R be any associative ring in which both 2 and 3 are units.*

1. *If m and n are positive integers one of which is odd, then*

$$\big(\gamma_m(R)R\big)\big(\gamma_n(R)R\big) \subseteq \gamma_{m+n-1}(R)R.$$

2. *For all positive integers m, n and $x_1, x_2, \dots, x_m \in R$ we have*

$$[x_1, x_2, \dots, x_m]^n \in \gamma_{(m-1)n+1}(R)R.$$

Part (1) of the lemma is Theorem 2.8 of Sharma and Srivastava's [SS], while part (2) is implicit in the proofs of their Lemmas 2.10 and 2.11.

Since the inclusion $D_{[m]}(L) \subseteq D_{(m)}(L)$ is obvious, Theorem 5.1 now follows from a combination of Lemmas 5.3 and 5.4.

It is not clear whether the statement of Theorem 5.1 remains true for the exceptional characteristics 2 and 3. However, we do offer the following proposition which will be of use in the next section where we deal with the residual Lie nilpotence of $u(L)$.

PROPOSITION 5.5. *Let the L be a restricted Lie algebra with characteristic $p > 0$. Then $D_{[ap^n+2]}(L) = D_{(ap^n+2)}(L)$ for every $n \geq 0$ and every $a \in \{0, 1, 2\}$.*

Before proving the proposition, we first state the following lemma. The proof of parts (1) and (2) can be found in [GL], while part (3) is proved by Levin and Sehgal in [LS].

LEMMA 5.6. *For any associative ring R the following statements hold.*

1. $\big(\gamma_m(R)R\big)\big(\gamma_n(R)R\big) \subseteq \gamma_{m+n-2}(R)R$ *for every $m, n \geq 2$.*
2. $\big(\gamma_m(R)R\big)^n \subseteq \gamma_{n(m-2)+2}(R)R$ *for every $m \geq 2$ and $n \geq 1$.*
3. $[x, y]^n \in \gamma_{n+1}(R)R$ *for every $x, y \in R$ and $n \geq 1$.*

PROOF OF PROPOSITION 5.5. By Theorem 4.4 we know that $D_{(m)}(L) = D_{(i)}(L)^{[p]} + \gamma_m(L)$ for all $m \geq 2$, where $i = \lceil \frac{m-1+p}{p} \rceil$. Put $m = ap^n + 2$ where $0 \leq a \leq p-1$. Then $i = ap^{n-1} + 2$, and thus $D_{(ap^n+2)}(L) = D_{(ap^{n-1}+2)}(L)^{[p]} + \gamma_{ap^n+2}(L)$. Notice also by Theorem 4.4 that

$$D_{(ap^{n-1}+2)}(L)^{[p]} \equiv \sum_{(i-1)p^j \geq ap^{n-1}+1} \gamma_i(L)^{[p]^{j+1}} \quad \text{modulo} \quad \gamma_{ap^n+2}(L)$$

since

$$\gamma_p\Big(D_{(ap^{n-1}+2)}(L)\Big) \subseteq \gamma_{ap^n+2p}(L) \subseteq \gamma_{ap^n+2}(L).$$

For each j $(0 \leq j \leq n)$ choose $i(j)$ minimally such that $(i(j)-1)p^j \geq ap^{n-1}+1$. It is easy to verify that actually

$$i(j) = \begin{cases} ap^{n-j-1}+2, & \text{if } 0 \leq j \leq n-1; \text{ and} \\ 2, & \text{if } j = n. \end{cases}$$

Therefore if $0 \leq j \leq n-1$, then

$$\gamma_{i(j)}(L)^{[p]^{j+1}} \subseteq \Big(\gamma_{ap^{n-j-1}+2}(R)R\Big)^{p^{j+1}} \subseteq \gamma_{ap^n+2}(R)R$$

by part (2) of Lemma 5.6. Now suppose that $j = n$. Then by part (3) of Lemma 5.6 we see

$$[x, y]^{[p]^{n+1}} \subseteq \gamma_{p^{n+1}+1}(R)R \subseteq \gamma_{ap^n+2}(R)R.$$

Hence by Lemma 5.2 we also have

$$\gamma_2(L)^{[p]^{n+1}} \subseteq \gamma_{ap^n+2}(R)R + \sum_{l=1}^{n+1} \gamma_{p^l}(H)^{[p]^{n+1-l}} = \gamma_{ap^n+2}(R)R,$$

where $H = \gamma_2(L)$. Indeed, by part (1) of Lemma 5.6 we have

$$\gamma_{p^l}(H)^{[p]^{n+1-l}} \subseteq \Big(\gamma_{2p^l}(R)R\Big)^{p^{n+1-l}} \subseteq \gamma_{p^{n+1-l}(2p^l-2)+2}(R)R \subseteq \gamma_{ap^n+2}(R)R.$$

We have just shown that $D_{(ap^n+2)}(L) \subseteq \gamma_{ap^n+2}(R)R$, which is tantamount to proving the proposition. ∎

6. **Residual Lie nilpotence.** We shall see momentarily that the two possible notions of residual Lie nilpotence of $R = u(L)$, lower and upper, coincide. To prove our claim, we give a characterisation of these properties in terms of L.

THEOREM 6.1. *The following statements are equivalent for any restricted Lie algebra L of characteristic $p > 0$.*

(i) $\bigcap_{n\geq 1} R^{(n)} = 0$;
(ii) $\bigcap_{n\geq 1} \gamma_n(R)R = 0$;
(iii) $\bigcap_{n\geq 1} D_{[n]}(L) = 0$;
(iv) $\bigcap_{n\geq 1} D_{(n)}(L) = 0$; *and*
(v) *L is residually 'nilpotent with derived subalgebra of finite exponent'.*

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PROOF. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial given Theorem 5.1 and Proposition 5.5. By part (2) of Theorem 4.4, $L/D_{(n)}(L)$ is nilpotent with L' of finite exponent. Thus (v) follows from (iv). It remains only to prove that (v) implies (i). We may assume that $K = F_p$, the field of p elements, since $u(L \otimes_{F_p} K) \cong u(L) \otimes_{F_p} K$. Let $\mathcal{X}$ be the class of all nilpotent restricted Lie algebras such that L' has finite exponent. Suppose first that $L \in \mathcal{X}$. Then because L is nilpotent and $u(L)$ is a free $u(L_p')$-module, it is not hard to see using Lemma 4.1 that

$$\bigcap_{n\geq 1} R^{(n)} \subseteq \bigcap_{n\geq 1} \left(\gamma_2(L)^n R\right) = \omega u(L_p')^{\omega} R.$$

But, by Theorem 2.6, $\omega u(L_p')^{\omega} = 0$. Now consider the general case of L residually $\mathcal{X}$. Suppose, to the contrary, that there is some $x \in \bigcap_{n\geq 1} R^{(n)}$ and $x \neq 0$. Let $\{x_i\}_{i\in I}$ be an ordered basis of L. By the PBW theorem put $x = \sum \alpha x_{i_1}^{a_1} \cdots x_{i_n}^{a_n}$, where $i_1 < \cdots < i_n$ and $\alpha, a_1, \ldots a_n \in F_p$. Then, by assumption, for each of the finite number of F_p-linear combinations y of the x_{i_j} appearing in the expression for x there exists a restricted ideal J_y of L with $y \notin J_y$ and $L/J_y \in \mathcal{X}$. Set $J = \bigcap_y J_y$. Then L/J lies in $\mathcal{X}$ since L/J can be embedded in $\prod_y L/J_y$. Now consider the canonical map $^{-}\colon u(L) \longrightarrow u(L/J)$. By construction the x_{i_j}'s are linearly independent modulo J, and so by the PBW theorem $\bar{x} \neq 0$. However, $\bar{x} \in \bigcap_{n\geq 1} u(L/J)^{(n)}$, contradicting the case above. ∎

7. **Indices of Lie nilpotence.** Suppose that $R = u(L)$ is Lie nilpotent. Recall that $t_{\mathrm{Lie}}(R)$ is the least index m for which $\gamma_m(R) = 0$, and similarly that $t^{\mathrm{Lie}}(R)$ be the least index n for which $R^{(n)} = 0$. It is clear that $t_{\mathrm{Lie}}(R) \leq t^{\mathrm{Lie}}(R)$. The main result of this section is that these two indices are in fact equal when the characteristic of R is greater than 3.

As in Section 4, $d_{(m)}$ denotes the dimension of $D_{(m)}(L)/D_{(m+1)}(L)$.

THEOREM 7.1. *Let L be a restricted Lie algebra of characteristic $p > 3$ with R Lie nilpotent. Then $t_{\mathrm{Lie}}(R) = t^{\mathrm{Lie}}(R) = 2 + (p-1)\sum_{m\geq 1} m d_{(m+1)}$.*

PROOF. By Corollary 4.5 and the remarks above, it remains to show only that $t_{\mathrm{Lie}}(R) \geq t^{\mathrm{Lie}}(R)$. We claim that $D_{(m)}(L)$ is spanned by elements of the form $[x_1, x_2, \ldots, x_i]^{[p]^j}$, where the elements x_k lie in L and $(i-1)p^j \geq m-1$. Indeed, since $D_{(m)}(L) = \sum_{(i-1)p^j \geq m-1} \gamma_i(L)^{[p]^j}$ by Theorem 4.4, we are required to prove only that if $x = \sum \alpha[x_1, \ldots, x_i] \in \gamma_i(L)$ and $(i-1)p^j \geq m-1$, then $x^{[p]^j} \equiv \sum \alpha^{p^j}[x_1, \ldots, x_i]^{[p]^j}$ modulo $D_{(m+1)}(L)$. But by Lemma 5.2

$$x^{[p]^j} \equiv \sum \alpha^{p^j}[x_1, \ldots, x_i]^{[p]^j} \quad \text{modulo} \quad \sum_{l=1}^{j} \gamma_{p^l}(H)^{[p]^{j-l}},$$

where $H = \gamma_i(L)$. Also observe that $\gamma_{p^l}(H)^{[p]^{j-l}} \subseteq \gamma_{ip^l}(L)^{[p]^{j-l}} \subseteq D_{(m+1)}(L)$ since $(ip^l - 1)p^{j-l} \geq m$. Thus the claim follows.

Write $L_m = D_{(m+1)}(L)$ and $e_m = d_{(m+1)}$ for each $m \geq 1$. Then $\{L_m\}$ is a p-filtration of L_p' by Lemma 4.3. Since R is Lie nilpotent, choose k minimally so that $L_{k+1} = 0$. By the claim we can choose a basis

$$y_{11}, \ldots, y_{1e_1}, y_{21}, \ldots, y_{2e_2}, \ldots, y_{ke_k}$$

of L'_p such that each y_{mn} has the form $[x_1, \ldots, x_i]^{[p]^j}$ where $(i-1)p^j \geq m$. Put $y = y_{11}^{p-1} y_{12}^{p-1} \cdots y_{1e_1}^{p-1} \cdots y_{ke_k}^{p-1}$. By the PBW theorem, necessarily $y \neq 0$. Now notice by part (2) of Lemma 5.4 that

$$y_{mn}^{p-1} = [x_1, \ldots, x_i]^{p^j(p-1)} \in \gamma_{(i-1)p^j(p-1)+1}(R)R \subseteq \gamma_{(p-1)m+1}(R)R.$$

Because $(p-1)m+1$ and $(p-1)me_m + 1$ are odd, by part (1) of the same lemma we find that

$$y \in \prod_{m=1}^{k} (\gamma_{(p-1)m+1}(R)R)^{e_m} \subseteq \prod_{m=1}^{k} \gamma_{(p-1)me_m+1}(R)R \subseteq \gamma_{1+\sum_{m=1}^{k}(p-1)me_m}(R)R.$$

Since $y \neq 0$, we must have $t_{\mathrm{Lie}}(R) \geq 2 + \sum_{m=1}^{k}(p-1)md_{(m+1)} = t^{\mathrm{Lie}}(R)$. ∎

8. **The nilpotence class of the unit group.** In this final section we use the preceding result to deduce a concrete formula for the nilpotence class of the group of units of $R = u(L)$ when L lies in the class $\mathcal{F}_p$. Denote the group of units of R by $\mathcal{U}(R)$, and write $\mathrm{cl}\big(\mathcal{U}(R)\big)$ for its nilpotence class. Recall that if L lies in $\mathcal{F}_p$ then $\omega u(L)$ is nilpotent as an ideal. In this case $\mathcal{U}(R) = K^* \times \mathcal{V}(R)$, where $\mathcal{V}(R) = 1 + \omega u(L)$. Obviously, $\mathcal{U}(R)$ and the group $\mathcal{V}(R)$ have the same nilpotence class. Therefore it suffices to consider $\mathcal{V}(R)$.

Recently Du has given a proof, [D], of Jennings' conjecture on radical rings, [Je4]. According to Jennings' conjecture, if R is any radical algebra satisfying $t_{\mathrm{Lie}}(R) < \infty$ then the nilpotence class of the circle group $(R, \circ)$, where $x \circ y = x + y + xy$, is exactly $t_{\mathrm{Lie}}(R) - 1$. Consequently, in our situation the class of $\mathcal{V}(R)$ is exactly $t_{\mathrm{Lie}}(R) - 1$. Now using Theorem 7.1 we may infer the following result.

THEOREM 8.1. *Let L be a restricted Lie algebra lying in the class $\mathcal{F}_p$, where $p > 3$. Then* $\mathrm{cl}\big(\mathcal{U}(R)\big) = 1 + (p-1)\sum_{m \geq 1} md_{(m+1)}$.

This theorem provides a systematic method of studying the class of $\mathcal{U}(R)$. The following results are some straightforward consequences.

COROLLARY 8.2. *Let $L \in \mathcal{F}_p$ with $p > 3$. Then* $\mathrm{cl}\big(\mathcal{U}(R)\big) \equiv 1 \bmod (p-1)$.

COROLLARY 8.3. *Let $L \in \mathcal{F}_p$ with $p > 3$. Then the following statements hold.*

1. *If L is not abelian then* $\mathrm{cl}\big(\mathcal{U}(R)\big) \geq p$, *with equality if and only if L' is 1-dimensional of exponent p.*
2. *If* $\mathrm{cl}\big(\mathcal{U}(R)\big) > p$ *then* $\mathrm{cl}\big(\mathcal{U}(R)\big) \geq 2p-1$, *with equality if and only if L' is 2-dimensional of exponent p and $\gamma_3(L) = 0$.*
3. *If* $\mathrm{cl}\big(\mathcal{U}(R)\big) > 2p-1$ *then* $\mathrm{cl}\big(\mathcal{U}(R)\big) \geq 3p-2$, *with equality if and only if L' is 3-dimensional of exponent p and $\gamma_3(L) = 0$ or L' is 2-dimensional of exponent p and $\gamma_4(L) = 0$.*

PROOF. Let us only prove part (2): the proof of the other parts are similar. From the theorem the next possible value of $\mathrm{cl}\big(\mathcal{U}(R)\big)$ greater than p is $2p-1$. There is only one possible case for equality: $d_{(2)} = 2$ and $d_{(n)} = 0$ for all $n \geq 3$. Indeed, otherwise Theorem 8.1 forces $d_{(2)} = 0$. But then $D_{(3)}(L) = \gamma_2(L)_p = D_{(4)}(L)$ since $p \geq 5$, so that $d_{(3)} = 0$, an impossibility. Thus we have $d_{(2)} = 2$ and $d_{(n)} = 0$ for all $n \geq 3$. In other words, $\gamma_2(L)^{[p]} + \gamma_3(L) = D_{(3)}(L) = 0$. The result follows. ∎

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