ON A THEOREM ON ORDERED GROUPS

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1. Introduction. The following work establishes a new proof of the theorem: *Every archimedean ordered group is abelian.* This theorem has been proved differently by many authors. It was first proved by O. Hölder [**2**]. A second proof has been given by H. Cartan [**1**]: he uses the topology which is naturally introduced in the group by its order.

Later on F. W. Levi [**3**] gave a third proof; he showed that if in an ordered group we denote by A the subgroup of those elements which are either archimedean comparable (in a sense which will be defined later) with a particular element a or infinitely small relative to a, and by E the normal subgroup of the latter class of elements, then the factor group A/E is isomorphic to a module of real numbers. In fact, if the group is archimedean ordered, then A will consist of the whole group, E will contain the identity alone and the theorem follows.

In this paper an entirely different proof of the theorem is given by showing that in any ordered group the commutator of two elements is infinitely small relative to the greater of the two. An example in which the smaller of the two elements is infinitely small relative to their commutator is established, showing that the above result is the best possible.

We conclude by showing that in any finitely generated ordered group the commutator subgroup has infinite index, a result which had also been shown by F. W. Levi [**4**]. It follows that an ordered group which coincides with its commutator group cannot be finitely generated.

2. Definitions and Preliminaries.

DEFINITION 1. *A group G is called an ordered group, when in G a binary transitive relation $a < b$ can be defined such that, of the three alternatives $a < b$, $a = b$, $b < a$, one and only one takes place, and $a < b$ implies $at < bt$ and $ta < tb$ for all a, b and t in G.*

For $b < a$ we also write $a > b$; moreover $\leqslant$ and $\geqslant$ will be used in the customary sense. We shall denote by e the unit element of G.

The following simple results follow immediately from the definition:

(1) $a < b$ implies $b^{-1} < a^{-1}$.
(2) $a < b$ implies $t^{-1}at < t^{-1}bt$ for any t in G.
(3) $a < b$ and $c < d$ imply $ac < bd$.
(4) $a < b$ implies $a^m < b^m$ for any positive integer m.

DEFINITION 2. *An element a in an ordered group is called positive if $e < a$; it is called negative if $a < e$.*

DEFINITION 3. *The absolute value $|a|$ of a is defined by:*

$$|a| = a \text{ if } a \geqslant e, \; |a| = a^{-1} \text{ if } a \leqslant e.$$

Thus $|a|^n = |a^n|$ for any positive integer n.

LEMMA 1. *The following relations for the absolute value of the product of two elements hold:*

(1) $|ab| = |a|\,|b|$ *if* $a > e, b > e$.
(2) $|ab| = |b|\,|a|$ *if* $a < e, b < e$.
(3) $|ab| = |a|\,|b|^{-1}$ *if* $a > e, b < e$ *and* $|a| > |b|$;
$|ab| = |b|\,|a|^{-1}$ *if* $a > e, b < e$ *and* $|a| < |b|$.

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(4) $|ab| = |b|^{-1}\ |a|$ *if* $a > e, b < e$ *and* $|a| > |b|$;
$|ab| = |a|^{-1}\ |b|$ *if* $a < e, b > e$ *and* $|a| < |b|$.

Proof. $a > e$ and $b > e$ imply $ab > e$; thus

$$|ab| = ab = |a|\ |b|,$$

which proves (1). One proves (2) similarly. To prove the first part of (3) we have:

$$|a| > |b| \text{ implies } |a|\ |b|^{-1} > e, \quad \text{i.e. } ab > e;$$

thus $$|ab| = ab = |a|\ |b|^{-1}.$$

One proves the other relations similarly.

LEMMA 2. $|t^{-1}at| = t^{-1}|a|t$ *for all* t *in* G.

This follows by direct application of Lemma 1.

DEFINITION 4. $|a|$ *is said to be infinitely small relative to* $|b|$, *and we denote this by* $|a| \ll |b|$, *if* $|a|^m < |b|$ *for all positive integers* m. *If there exist two positive integers* m *and* n *such that*

$$|a| < |b|^m \quad \text{and} \quad |b| < |a|^n,$$

then a *and* b *are said to be archimedean comparable* (*or briefly*: *comparable*), *and we denote this by* $a \sim b$.

LEMMA 3. $|a| \ll |b|$ *if and only if* $a^n < |b|$ *for all integers* n (*negative, zero or positive*).

Proof. If $|a| \ll |b|$, then a^n, for any integer n, is trivially smaller than $|b|$. Conversely, $a^n < |b|$ for all integers n, implies that $|a|^m < |b|$ for all positive integers m; otherwise we would have $a^{\pm m} > |b|$ for some integer m.

LEMMA 4. *For any two elements* a, b, *different from* e, *of an ordered group, one and only one of the following relations holds*:

$$|a| \ll |b|, \quad a \sim b, \quad |b| \ll |a|.$$

Proof. If $|a| = |b|$, then $a \sim b$. If $|a| < |b|$, then either $|a|^n < |b|$ for all positive integers n and thus $|a| \ll |b|$, or there exists a positive integer m for which $|b| < |a|^m$ and thus $a \sim b$.

Similarly, if $|b| < |a|$, then either $|b| \ll |a|$ or $a \sim b$.

LEMMA 5. (1) $|a| \ll |b|$ *implies* $|t^{-1}at| \ll |t^{-1}bt|$ *for any* t *in* G.
(2) $|a| \ll |b|$ *and* $b \sim c$ *imply* $|a| \ll |c|$.
(3) $|a| \ll |b|$ *and* $d \sim a$ *imply* $|d| \ll |b|$.

Proof. (1) $(t^{-1}at)^n = t^{-1}a^n t < t^{-1}|b|t = |t^{-1}bt|$, by Lemma 2. Thus

$$|t^{-1}at| \ll |t^{-1}bt|,$$

by Lemma 3.

(2) $b \sim c$ implies $|c|^m > |b|$ for some positive integer m. If $|a|$ is not infinitely small relative to $|c|$, then there exists a positive integer n for which $|a|^n > |c|$. Thus

$$|a|^{nm} > |c|^m > |b|,$$

contrary to the assumption that $|a| \ll |b|$.

One proves (3) similarly.

LEMMA 6. *The relation* $\ll$ *is transitive.*

Proof. $|a| \ll |b|$ and $|b| \ll |c|$ imply $|a| \ll |c|$; for if not, there exists a positive integer n for which $|a|^n > |c|$, and since $|b| > |a|^n$ then $|b| > |c|$, contrary to the assumption that $|b| \ll |c|$.

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From Lemmas 5 and 6 follows immediately

LEMMA 7. *$|a'| \leqslant |a|$, $|a| \ll |b|$ and $|b| \leqslant |b'|$ imply $|a'| \ll |b'|$.*

LEMMA 8. *$a^n \sim a$ for all integers $n \neq 0$.*

This follows from the fact that

$$|a^n| < |a|^{m+1}, \quad m = |n| \quad \text{and} \quad |a| < |a^n|^2.$$

LEMMA 9. *Let μ be any positive integer and ν be any negative integer; then*

(1) $a^{-1}ua > u$ *implies* $a^{-\mu}ua^{\mu} > u$ *and* $a^{-\nu}ua^{\nu} < u$,

(2) $aua^{-1} > u$ *implies* $a^{\mu}ua^{-\mu} > u$ *and* $a^{\nu}ua^{-\nu} < u$.

The proof is immediate by induction.

We conclude this section with the following lemma.

LEMMA 10. *If we denote $a^{-1}b^{-1}ab$ by $[a, b]$, then the following identities hold in any group:*

(1) $$[a^m, b] = \prod_{\mu=m-1}^{0} (a^{-\mu}[a, b]a^{\mu}),$$

(2) $$[a^{-m}, b] = \prod_{\mu=m}^{1} (a^{\mu}[b, a]a^{-\mu}),$$

(3) $$[b, a^{-m}] = \prod_{\mu=1}^{m} (a^{\mu}[a, b]a^{-\mu}),$$

(4) $$[b, a^m] = \prod_{\mu=0}^{m-1} (a^{-\mu}[b, a]a^{\mu})$$

for all positive integers m.

These relations may be verified by direct multiplication.

3. The Main Result.

THEOREM 1. *The following relation holds in any ordered group:*

$$|[a, b]| \ll \max(|a|, |b|),$$

for any elements a and b of the group.

Proof. To prove this theorem we have to consider three possibilities.

(I) Both a and b are positive:

Assume that $a > b > e$. We distinguish four cases:

(i) $a^{-1}[a, b]a > [a, b] > e$,

(ii) $e > a^{-1}[a, b]a > [a, b]$,

(iii) $e < a^{-1}[a, b]a \leqslant [a, b]$,

(iv) $a^{-1}[a, b]a \leqslant [a, b] < e$.

We notice that e cannot fall between $a^{-1}[a, b]a$ and $[a, b]$, since $x \gtrless e$ implies $t^{-1}xt \gtrless e$.

Case (*i*). Relations (i) together with Lemma 9 give

$$a^{-\mu}[a, b]a^{\mu} > [a, b] > e$$

for all positive integers μ. For all positive integers m we have

$$[a^m, b] = \prod_{\mu=m-1}^{0} (a^{-\mu}[a, b]a^{\mu}) > [a, b]^m > e.$$

Thus $$|[a, b]|^m = |[a, b]^m| < |[a^m, b]| = a^{-m}b^{-1}a^m b < a^{-m}ea^m a = a.$$

Case (*ii*). Relations (ii) give

$$e < a^{-1}[b, a]a < [b, a] < a[b, a]a^{-1}.$$

Thus, by Lemma 9,

$$a^{\mu}[b, a]a^{-\mu} > [b, a] > e,$$

for all positive integers μ; and, for all positive integers m, by Lemma 10, (2),

$$[a^{-m}, b] = \prod_{\mu=m}^{1} (a^{\mu}[b, a]a^{-\mu}) > [b, a]^{m} > e.$$

Thus $$|\,[a, b]\,|^{m} = |\,[b, a]\,|^{m} < |\,[a^{-m}, b]\,| = a^{m}b^{-1}a^{-m}b < a^{m}ea^{-m}a = a.$$

Case (*iii*). Relations (iii) give

$$e < a^{-1}[a, b]a \leqslant [a, b] \leqslant a[a, b]a^{-1}.$$

Thus, by Lemma 9,

$$a^{\mu}[a, b]a^{-\mu} \geqslant [a, b] > e,$$

for all positive integers μ; and, for all positive integers m, by Lemma 10, (3),

$$[b, a^{-m}] = \prod_{\mu=1}^{m} (a^{\mu}[a, b]a^{-\mu}) \geqslant [a, b]^{m} > e.$$

Thus $$|\,[a, b]\,|^{m} = |\,[a, b]^{m}\,| \leqslant |\,[b, a^{-m}]\,| = b^{-1}a^{m}ba^{-m} < ea^{m}aa^{-m} = a.$$

Case (*iv*). Relations (iv) give

$$a^{-1}[b, a]a \geqslant [b, a] > e.$$

Thus, by Lemma 9,

$$a^{-\mu}[b, a]a^{\mu} \geqslant [b, a] > e$$

for all positive integers μ; and, for all positive integers m, by Lemma 10, (4),

$$[b, a^{m}] = \prod_{\mu=0}^{m-1} (a^{-\mu}[b, a]a^{\mu}) \geqslant [b, a]^{m} > e.$$

Thus $$|\,[a, b]\,|^{m} = |\,[b, a]^{m}\,| \leqslant |\,[b, a^{m}]\,| = b^{-1}a^{-m}ba^{m} < ea^{-m}aa^{m} = a.$$

Thus the theorem is true for positive elements.

(II) One element, a say, is negative and b is positive:

Assume that $a < e < b < a^{-1}$. By (I), we have

$$|\,[a^{-1}, b]\,| = |\,a[a, b]^{-1}a^{-1}\,| \ll |\,a\,|.$$

Thus, by Lemma 5, (1), we get

$$|\,[a, b]^{-1}\,| \ll |\,a\,|,$$

and hence $$|\,[a, b]\,| \ll |\,a\,|.$$

(III) Both a and b are negative:

Assume that $a < b < e$. By (I) we have

$$|\,[a^{-1}, b^{-1}]\,| = |\,ab[a, b]b^{-1}a^{-1}\,| \ll |\,a\,|.$$

By Lemma 5, (1) we get

$$|\,[a, b]\,| \ll |\,b^{-1}ab\,|.$$

But $$|b^{-1}ab| = b^{-1}|a|b < |a|\ |a|e = |a|^2.$$
Thus $$|[a, b]| \ll |a|^2.$$
Since $|a|^2 \sim |a|$ (Lemma 8), then, by Lemma 5,
$$|[a, b]| \ll |a|.$$
(I), (II) and (III) together show that Theorem 1 is generally true.

Definition 5. *An ordered group is called an archimedean ordered group if, for any $b \neq e$, $|a| \ll |b|$ implies $a = e$.*

Thus an ordered group is archimedean ordered if any two elements not equal to e are archimedean comparable.

Theorem 1 together with Definition 5 give

Theorem 2. *Every archimedean ordered group is abelian.*

4. Example. The result which we obtained in Theorem 1 is the best possible result in the sense that $|[a, b]|$ need not be small compared to both a and b; on the contrary here is an example in which
$$\min(|a|, |b|) \ll |[a, b]|.$$
Let H be the free abelian group generated by
$$\dots, b_{-1}, b_0, b_1, b_2, \dots;$$
we define an order in H such that
$$e < \dots \ll b_{-1} \ll b_0 \ll b_1 \ll \dots .$$
An element of H is thus to be positive if the generator b_μ with the highest suffix in it appears with a positive exponent. The mapping $b_\mu \to b_{\mu+1}$ defines an automorphism of H which leaves the order of H invariant. Now form
$$G = \{H, a\}$$
with $$a^{-1}b_\mu a = b_{\mu+1} \quad (\mu = 0, \pm 1, \pm 2, \dots).$$
Order G by making $a > e$ and $h \ll a$ for all h in H. Now in G we have
$$[b_0, a] = b_0^{-1}a^{-1}b_0 a = b_0^{-1}b_1,$$
and, since $b_1 > b_0^{m+1}$ for all m, then
$$b_0^m < b_0^{-1}b_1 = [b_0, a] \quad \text{for all } m.$$
Thus $$|b_0| = \min(b_0, a) \ll |[b_0, a]|.$$

5. Finally we conclude with the following theorem, a result which was obtained differently by F. W. Levi:

Theorem 3. *Let $a_1, a_2, \dots, a_n$ be n elements of an ordered group G not all equal to e, let B be the subgroup generated by them and $C(B)$ its commutator group; then $B/C(B)$ contains elements of infinite order.*

Proof. Let $a = \max(|a_1|, |a_2|, \dots, |a_n|)$, let A_1 be the set of elements which are comparable with a or which are in modulus infinitely small relative to a, and let A_0 be the set of elements which are in modulus infinitely small relative to a. Then clearly $A_1 \supseteq B$ and $a \in B$. One easily shows that A_1 is a subgroup of G and A_0 a subgroup of A_1.

If $b, c \in A_1$, with $|c| > |b|$, say, then by Theorem 1,

$$|[b, c]| \ll |c|,$$

and, since c is either comparable to a or $|c| \ll |a|$, then

$$|[b, c]| \ll |a|.$$

Thus $$[b, c] \in A_0,$$

and hence $$C(A_1) \subseteq A_0.$$

In particular $$C(B) \subseteq A_0.$$

Now, since $a^{\pm 1}, a^{\pm 2}, \dots$, are all comparable with a (Lemma 8), none of them is situated in $C(B)$. Hence the element $aC(B)$ of $B/C(B)$ is of infinite order.

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