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Weak and semi-strong solutions of the Schneider-Tricomi problem in the euclidean plane

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Schneider (*Math. Nachr.* 60 (1974), 167-180) has established the following result. Consider the mixed type equation

(1) $$L[u] = k(y)\cdot u_{xx} + u_{yy} + \lambda(x, y)\cdot u = f(x, y)$$

in $G \subset R^2$ which is a simply connected region, bounded for $y > 0$ by a piece-wise smooth curve Γ_0 connecting the points $A(0, 0)$ and $B(1, 0)$, and for $y < 0$ by the solutions of $k(y)\cdot(dy)^2 + (dx)^2 = 0$ which meet at the point $G\left(\tfrac{1}{2}, y_c\right)$, such that $k(y) \gtrless 0$ for $y \gtrless 0$,

(2) $$\begin{cases} k(y) \in C^0(\bar{G}) \cap C^1\left(\bar{G}|\{(x, 0) \mid x \in [0, 1]\}\right) \cap C^2\left(\bar{G}_2\right) , \\ G_1 = G \cap \{y > 0\} , \quad G_2 = G \cap \{y < 0\} , \quad \lambda = \text{const.} < 0 , \\ \lambda \in C^1(\bar{G}) , \quad f \in L^2(\bar{G}) , \quad u \in C^0(\bar{G}) \cap C^1(\bar{G}) , \quad k'(y) > 0 \text{ in} \\ \qquad \bar{G} \cap \{y < 0\} , \quad \lim_{y \to 0-} \frac{k(y)}{k'(y)} = 0 , \end{cases}$$

$S(x, y) = F(y) + 8\lambda\cdot(k/k')^2 > 0$ in $\bar{G} \cap \{y < 0\}$, "Schneider's Condition", where $F(y) = 1 + 2(k/k')'$, and such that $S = S(x, y)$ is integrable in G_2, $\lim_{y \to 0-} F(y) > 0$ "Frankl's Condition". Then the Tricomi Problem (T): $L[u] = f$ with

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$u|_{\Gamma_0 \cup BC} = 0$ has *a weak solution* $u \in L^2(\overline{G})$, and the Adjoint Tricomi Problem (T^+): $L^+[w] = L[w] = f$ with $w|_{\Gamma_0 \cup AC} = 0$ has at most one *semistrong solution*.

In this present paper we get the above result of Schneider in a much more generalized way, so that here our uniqueness theorem and existence results include cases where $S(x, y)$ *may be negative* in G_2 .

Preliminary terminology

The Sobolev spaces $\tilde{W}^{2,2}(G)$ and $W^{2,2}(G)$ are defined as follows:

$$W^{2,2}(G) = \{u \mid u(x, y) \in L^2(G),\ D^\alpha u \in L^2(G) \text{ for } |\alpha| \le 2\}$$

with norm $\|\cdot\|_2$ and scalar product $(\cdot, \cdot)_2$;

$$\tilde{W}^{2,2}(G) = \left\{u \mid u(x, y) \in C^2(\overline{G}),\ u|_{\Gamma_0 \cup BC} = 0\right\} , [1];$$

$$C_0^\infty(G) \subsetneqq \tilde{W}^{2,2}(G) \subsetneqq C^2(\overline{G}) \ ;$$

$$W^{0,2}(G) = L^2(G) \ ; \quad \tilde{W}^{2,2}(G)^+ = \left\{w \mid w \in C^2(\overline{G}),\ w|_{\Gamma_0 \cup AC} = 0\right\} .$$

$W^{2,2}(G, \text{bd})$ is the Sobolev space with special boundary values,

$$W^{2,2}(G, \text{b}^+\text{d}) = \left\{w \in W^{2,2}(G) \mid (L[u], w)_0 = (w, L^+[w])_0,\ \forall u \in W^{2,2}(G, \text{bd})\right\}$$

$$= \overline{\left\{w \mid w(x, y) \in C^2(\overline{G}),\ w|_{\Gamma_0 \cup BC} = 0\right\}}^{\|\cdot\|_2} .$$

LEMMA 1 [1]. *For the existence of a semistrong solution of* (T) (*that is* $u \in L^2(G)$ *such that* $(u, L^+[w])_0 = (f, w)_0$, *for all* $w \in W^{2,2}(G, \text{b}^+\text{d})$) *it is necessary and sufficient that*

(3) $$\|w\|_0 \le C \cdot \|L^+[w]\|_0 ,$$

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where $C = \text{const.} > 0$ *for all* $w \in W^{2,2}(G, b^{+}d)$.

LEMMA 2 [1]. *For the existence of a semistrong solution of* (T) *(see* [1]) *it is necessary and sufficient that*

$$\|u\|_0 \le C\cdot\|L[u]\|_0 \,, \quad \|w\|_0 \le C\cdot\|L^{+}[w]\|_0 \,, \tag{4}$$

where $C = \text{const.} > 0$ *for all* $u \in W^{2,2}(G, bd)$ *, and for all* $w \in W^{2,2}(G, b^{+}d)$.

The Schneider-Tricomi problem

We investigate the expression

$$2(l[u], L[u]) = 2\cdot \iint_G l[u]\cdot L[u]\cdot dxdy \,, \tag{5}$$

where

$$\begin{cases} l[u] = a(x, y)\cdot u \quad \text{in } \overline{G}_1 \,, \\ \text{and} \\ l[u] = a(x, y)\cdot\left[u + 4\cdot\left(\sqrt{-k}\cdot e^{\beta\cdot x}\cdot u_x + u_y\right)\cdot(k/k')\right] \quad \text{in } \overline{G}_2 \,, \end{cases} \tag{6}$$

where

$$\begin{cases} a = a(x, y) = \exp\left(\int_0^y 4\lambda\,\frac{k(t)}{k'(t)}\cdot dt\right) \\ \qquad\cdot\left\{a_0 + \int_0^y \beta_0\cdot\left(t - y_c\right)\cdot\exp\left[-\int_0^t 4\lambda\,\frac{k(s)}{k'(s)}\cdot ds\right]\cdot dt\right\} \\ \qquad\qquad \text{in } \overline{G}_2 \quad \left(a_0 < 0,\ \beta_0 > 0\right) \,, \\ \text{and} \\ a = a(x, y) = a_0 - \left(\beta_0\cdot y_c\right)\cdot y \quad \text{in } \overline{G}_1 \,. \end{cases} \tag{7}$$

We apply Schneider's conditions:

$$a = a(x, y) \in C^2\left(\overline{G}_1\right) \cup C^2\left(\overline{G}_2\right) \,, \quad b = b(x, y) \,, \tag{8}$$

$$c = c(x, y) \in C^1\left(\overline{G}_1\right) \cup C^1\left(\overline{G}_2\right) \,,$$

$$\overline{G}_1 \cup \overline{G}_2 = \{(x, 0) \mid x \in [0, 1]\} \,, \quad 2|xy| = \rho\cdot x^2 + 1/\rho\cdot y^2 \quad (\rho > 0) \,,$$

(9) $a^+ - a^- = 0$, $b^+ - b^- = 0$, $c^+ - c^- \le 0$, $(a_y^+ - a_y^-) + (c^- - c^+)\cdot\lambda \ge 0$.

In $\bar{G}_i$:

$$(10)\quad \begin{cases} \tilde{A} = -k(y)\cdot(b_x - c_y) + c\cdot k\ (y) - 2k\cdot a - \rho_1\cdot b^2 = 0 , \\ \tilde{C} = (b_x - c_y) - 2a - \rho_1\cdot c^2 = 0 , \quad \tilde{A}\tilde{C} - \tilde{B}^2 = 0 , \\ \text{with} \\ \tilde{B} = -k(y)\cdot c_x - b_y - \rho_1\cdot bc , \\ \tilde{D} = k(y)\cdot a_{xx} + a_{yy} + 2\lambda a - \lambda(b_x + c_y) - \rho_2\cdot a^2 \ge d_0 > 0 , \end{cases}$$

where $\rho_i > 0$ $(i = 1, 2)$;

$$(11)\qquad (b\cdot dy - c\cdot dx)|_{\Gamma_0} \ge 0 , \quad (b\cdot dy + c\cdot dx)|_{\Gamma_1} \le 0 ,$$

$$[-d(a.\sqrt{-k}) + (b\cdot\lambda - k\cdot a_x)\cdot dy + (-c\cdot\lambda + a_y)\cdot dx]|_{\Gamma_1(=AG)} \ge 0 ,$$

where

$$(12)\qquad \Gamma_1 : x = -\int_0^y \sqrt{-k(t)}\cdot dt .$$

In G_1 :

$$\tilde{A} = -2k\cdot a \ge 0 , \quad \tilde{C} = -2\cdot a \ge 0 , \quad \tilde{B} = 0 , \quad \tilde{D} = 2\lambda\cdot a - \rho_2\cdot a^2 \ge 0 ,$$

because $\lambda < 0$ by hypothesis, $(b\cdot dy - c\cdot dx)|_{\Gamma_0} = 0$. In G_2 :

$$(b\cdot dy + c\cdot dx)|_{\Gamma_1} = \left[b\left(\frac{-1}{\sqrt{-k}}\right) + c\right]\cdot dx\Big|_{\Gamma_1} = -(-k)^{-\frac{1}{2}}\cdot c\cdot[-k\cdot R(x)]\cdot dx|_{\Gamma_1}$$

$$= -c\cdot R(x)dx|_{\Gamma_1} = 0 .$$

Assume

$$(13)\qquad \lim_{y\to 0-} \frac{k(y)}{k'(y)} = 0 ,$$

and choose

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(14) $$b = c\cdot\sqrt{-k}\cdot e^{\beta\cdot x} \ , \quad c = \frac{4ak}{k} \ \text{ in } \ \overline{G}_2 \ ,$$

where $a = a(x, y)$ is defined by (7), and β is a given positive constant such that

(15) $$R(x) = e^{\beta\cdot x} - 1 \geq 0 \ .$$

$\tilde{A} \geq 0$ and $\tilde{B} \geq 0$ if (in $\overline{G}_2$)

(16) $$R(x, y) = F(y) + 8\lambda\cdot(k/k')^2 + 2\beta\cdot\left((-k)^{3/2}/k'\right)\cdot e^{\beta\cdot x}$$
$$= S(x, y) + 2\beta\cdot\left((-k)^{3/2}/k'\right)\cdot e^{\beta\cdot x} > 0$$

in $\overline{G}_2$. On the other hand, $\tilde{A}\tilde{C} - \tilde{B}^2 \geq 0$ in $\overline{G}_2$ if

(17) $$V(x, y) = A\cdot F^2 + B\cdot F + C < 0 \ ,$$

where $A = a^2\cdot R^+(x)$,

$$R^+(x) = (-a)\cdot\left[4\cdot\lambda\left((-k)/k'\right)+\beta(\sqrt{-k})\cdot e^{\beta\cdot x}\right] + \beta_0\cdot\left(y-y_c\right) > 0 \ ,$$

$$B = 4\cdot\left[R^+(x)\cdot a_y+\beta\cdot e^{\beta\cdot x}\cdot a\cdot\sqrt{-k}\right]\cdot a\cdot(k/k') \ , \quad a_y = 4\lambda(k/k')\cdot a + \beta_0\cdot\left(y-y_c\right) \ ,$$

$$C = 4\left[-(\beta\cdot a)^2\cdot e^{2\beta\cdot x}\cdot k+2\beta\cdot e^{\beta\cdot x}\cdot a\cdot a_y(\sqrt{-k} + R^+(x)\cdot\left(a_y\right)^2\right]\cdot(k/k')^2 \ .$$

The *Schneider-Tricomi problem*, or *problem* (T_s) consists of finding a solution $u \in C^0(\overline{G}) \cap C^1(\overline{G})$ assuming prescribed values on $\Gamma_0 \cup \Gamma_2$; that is

(18) $$u\big|_{\Gamma_0\cup\Gamma_2} = 0 \ .$$

THEOREM. *Assume conditions* (2), (15), (16), (17), *and that* $\left[-d(a\sqrt{-k})+b\cdot\lambda\cdot dy+\left(a_y-c\cdot\lambda\right)\cdot dx\right]\big|_{\Gamma_1} \geq 0$, $\lambda < 0$ *in* $\overline{G}$, *and* $S(x, y) = d$ *such that* $d_0 \leq d \leq d^0$ *in* $\overline{G}_2$; *and that* $R(x, y)$ *is integrable in* $\overline{G}_2$, $\lim_{y\to 0-} F(y) > 0$. *Then the Tricomi problem* (T_s) *has a weak solution* $u \in L^2(\overline{G})$, *and the adjoint Tricomi problem* (T_s^+) *has at most one semi-strong solution* $(d_0 = \text{const.} < 0 \ , \ d^0 = \text{const.} > 0)$.

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