Canad. J. Math. Vol. **52** (3), 2000 pp. 539–581

On Square-Integrable Representations of Classical p-adic Groups

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Abstract. In this paper, we use Jacquet module methods to study the problem of classifying discrete series for the classical p-adic groups $\mathrm{Sp}(2n, F)$ and $\mathrm{SO}(2n+1, F)$.

1 Introduction

1.1 Introduction

One of the central questions in the representation theory of p-adic groups is to determine the discrete series. This paper studies the problem of determining the noncuspidal discrete series for the classical groups $\mathrm{Sp}_{2n}(F)$ and $\mathrm{SO}_{2n+1}(F)$. Let $S_n(F)$ denote $\mathrm{Sp}_{2n}(F)$ or $\mathrm{SO}_{2n+1}(F)$ (we treat the two families simultaneously). Now, a noncuspidal discrete series representation occurs as a subquotient of a (parabolically) induced representation. Here, we constrain where one needs to look for such discrete series representations. Ultimately, we hope that such an analysis can be used to help prove exhaustion for the noncuspidal discrete series.

First, we reduce the problem of classifying the discrete series to classifying those square-integrable representations supported on sets of the form $\mathcal{S}\big((\rho,\beta);\sigma\big) = \{\nu^{\alpha}\rho\}_{\alpha\in\beta+\mathbb{Z}} \cup \{\sigma\}$, where $\rho \cong \tilde{\rho}$ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_n(F)$, $\nu = |\det|$, σ an irreducible supercuspidal representation of $S_r(F)$, and $\beta = 0$ or $\frac{1}{2}$. In general, if π is an irreducible representation (not necessarily square-integrable) supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ as above, we define $\chi_0(\pi)$. This is a subquotient of the (normalized) Jacquet module taken with respect to the smallest standard parabolic subgroup admitting a nonzero Jacquet module; it is minimal with respect to an appropriate ordering. This is used to produce $\delta_0(\pi)$, which has the form

$$\delta_0(\pi) = \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma,$$

where $a_1 \leq a_2 \leq \cdots \leq a_k$ ($\delta([\nu^{b}\rho, \nu^{a}\rho])$ denotes the generalized Steinberg representation of $\mathrm{GL}_{(a-b+1)n}(F)$ whose minimal Jacquet module is $\nu^{a}\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{b}\rho$). If $m = (a_1 - b_1 + 1)n + \cdots + (a_k - b_k + 1)n + r$, let $P = MN$ denote the standard parabolic subgroup of $S_m(F)$ with Levi factor

$$M = \mathrm{GL}_{(a_1-b_1+1)n}(F) \times \cdots \times \mathrm{GL}_{(a_k-b_k+1)n}(F) \times S_r(F).$$

Then, we show that

$$\pi \hookrightarrow \mathrm{Ind}_P^G\big(\delta_0(\pi)\big).$$

Received by the editors November 20, 1998.
AMS subject classification: 22E50.

https://doi.org/10.4153/CJM-2000-025-7 Published online by Cambridge University Press

Further, we show that π is square-integrable if and only if $a_i + b_i > 0$ for all i. Thus, every square-integrable representation supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ with ρ, β, σ as above, is a subrepresentation of an induced representation of this form. We note that not every δ_0 having the form described above occurs as $\delta_0(\pi)$ for a square-integrable π. However, we expect that with a couple of additional conditions on a_i, b_i, that will be the case.

We now discuss the contents section by section. The next section introduces notation and recalls some general results that will be needed later.

Section 2.1 reviews some results of Zelevinsky on induced representations for general linear groups. In Section 2.2, we define $\chi_0(\pi)$, $\delta_0(\pi)$ for π an irreducible representation of $\mathrm{GL}_m(F)$ supported on a set of the form $\{\nu^\alpha\rho\}_{\alpha\in\beta+\mathbb{Z}}$, where ρ is an irreducible unitary supercuspidal representation. We also establish some of the basic properties of $\chi_0(\pi)$, $\delta_0(\pi)$. In Section 2.3, we show how these can be used to show that the only irreducible square-integrable representations of $\mathrm{GL}_m(F)$ are the generalized Steinberg representations, a result originally due to Bernstein. The connection between $\delta_0(\pi)$ and the Langlands data (subrepresentation version of the Langlands classification) is established in Section 2.4.

We use Section 3.1 to review some background material on the representation theory of $S_n(F)$. In Section 3.2, we recall a result which allows us to reduce the problem of classifying the discrete series of $S_r(F)$ to that of classifying the square-integrable representations supported on sets of the form $\mathcal{S}\big((\rho,\beta);\sigma\big)$. In Section 3.3, we give a conjecture which, when coupled with recent work of Mœglin, leads to an expected parameterization of such square-integrable representations, at least for pairs (ρ,σ) with "generic reducibility".

The definitions and basic properties of $\chi_0(\pi)$, $\delta_0(\pi)$ mentioned above are discussed in detail in Section 4.1. In Section 4.2, we give the criterion for square-integrability mentioned above. In Section 4.3, we use this to determine which sets $\mathcal{S}\big((\rho,\beta);\sigma\big)$ support square-integrable representations. Section 4.4 gives some basic constraints on $\delta_0(\pi)$.

In the fifth chapter, we give an example to show how these results may be applied. We restrict our attention to the case where $\mathrm{Ind}_P^G(\nu^{\frac{1}{2}}\rho\otimes\sigma)$ is reducible, where P is the standard parabolic subgroup of $S_{n+r}(F)$ with Levi factor $M \cong \mathrm{GL}_n(F)\times S_r(F)$. By results in Section 4.3, only $\mathcal{S}\big((\rho,\frac{1}{2});\sigma\big)$ will support square-integrable representations, so we restrict our attention to representations supported on this set. The goal of this chapter is to classify those irreducible, square-integrable representations whose δ_0 has $k=2$. The case $k=1$ is already known (*cf.* [Tad5]); we discuss this case in Section 5.1. In Section 5.2, we show that if π is an irreducible, square-integrable representation and $k=2$, then $\delta_0(\pi)$ has one of the following forms:

1. $\delta_0(\pi) = \delta([\nu^{-d}\rho,\nu^c\rho])\otimes\delta([\nu^{-b}\rho,\nu^a\rho])\otimes\sigma$, or
2. $\delta_0(\pi) = \delta([\nu^{-c}\rho,\nu^b\rho])\otimes\delta([\nu^{-d}\rho,\nu^a\rho])\otimes\sigma$

for $a,b,c,d \in \frac{1}{2}+\mathbb{Z}$ with $a>b>c>d\geq -\frac{1}{2}$. Further, anything of form 1. or 2. above actually occurs as $\delta_0(\pi)$ for some irreducible, square-integrable representation π. This follows from the discussion in Section 5.3. We note that the irreducible, square-integrable representations appearing in $\mathrm{Ind}_P^G\big(\delta([\nu^{-d}\rho,\nu^c\rho])\otimes\delta([\nu^{-b}\rho,\nu^a\rho])\otimes\sigma\big)$ are classified by the results in [Tad5] (where P and G are clear from context). In Section 5.3, we do a corresponding analysis for $\mathrm{Ind}_P^G\big(\delta([\nu^{-c}\rho,\nu^b\rho])\otimes\delta([\nu^{-d}\rho,\nu^a\rho])\otimes\sigma\big)$, though our approach is somewhat different.

I would like to take this opportunity to thank a few people. In particular, conversations with Henry Kim, Alan Roche, Paul Sally, and Marko Tadić have been quite helpful during

various stages of the research for this paper. My thanks go out to all of them, and to the referee as well.

1.2 Notation and Preliminaries

In this section, we introduce notation and recall some results that will be needed in the rest of the paper. This largely follows the setup used in [Tad1].

Let F be a p-adic field with char $F = 0$. Let $|\cdot|$ denote the absolute value on F, normalized so that $|\varpi| = q^{-1}$, ϖ a uniformizer. As in [B-Z], we let $\nu = |\det|$ on $\mathrm{GL}_n(F)$ (with the value of n clear from context). Define $\times$ on $\mathrm{GL}(F)$ as in [B-Z]: if $\rho_1, \ldots, \rho_k$ are representations of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_k}(F)$, let $\rho_1 \times \cdots \times \rho_k$ denote the representation of $\mathrm{GL}_{n_1+\cdots+n_k}(F)$ obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k$ from the standard parabolic subgroup of $\mathrm{GL}_{n_1+\cdots+n_k}(F)$ with Levi factor $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F)$.

Frequently, we work in the Grothendieck group setting. That is, we work with the semisimplified representation. So, for any representation π and irreducible representation ρ, let $m(\rho, \pi)$ denote the multiplicity of ρ in π. We write $\pi = \pi_1 + \cdots + \pi_k$ if $m(\rho, \pi) = m(\rho, \pi_1) + \cdots + m(\rho, \pi_k)$ for every irreducible ρ. Similarly, we write $\pi \geq \pi_0$ if $m(\rho, \pi) \geq m(\rho, \pi_0)$ for every such ρ. For clarity, we use $=$ when defining something or working in the Grothendieck group; $\cong$ is used to denote an actual equivalence.

We now turn to symplectic and odd-orthogonal groups. Let

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

denote the $n \times n$ antidiagonal matrix above. Then,

$$\mathrm{SO}_{2n+1}(F) = \{X \in \mathrm{SL}_{2n+1}(F) \mid {}^{T}X J_{2n+1} X = J_{2n+1}\},$$
$$\mathrm{Sp}_{2n}(F) = \left\{X \in \mathrm{GL}_{2n}(F) \;\middle|\; {}^{T}X \begin{pmatrix} & -J \\ J & \end{pmatrix} X = \begin{pmatrix} & -J \\ J & \end{pmatrix}\right\}.$$

We use $S_n(F)$ to denote either $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. In either case, the Weyl group is $W = \{$permutations and sign changes on n letters$\}$.

We take as minimal parabolic subgroup in $S_n(F)$ the subgroup $P_\varnothing$ consisting of upper triangular matrices. Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of a nonnegative integer $m \leq n$ into positive integers. Let $M_\alpha \subset S_n(F)$ be the subgroup

$$M_\alpha = \left\{ \begin{pmatrix} X_1 & & & & & & \\ & \ddots & & & & & \\ & & X_k & & & & \\ & & & X & & & \\ & & & & {}^{\tau}X_k & & \\ & & & & & \ddots & \\ & & & & & & {}^{\tau}X_1 \end{pmatrix} \;\middle|\; X_i \in \mathrm{GL}_{n_i}(F), X \in S_{n-m}(F) \right\},$$

https://doi.org/10.4153/CJM-2000-025-7 Published online by Cambridge University Press

where ${}^{\tau}X = J^{T}X^{-1}J$. Then $P_\alpha = M_\alpha P_\varnothing$ is a parabolic subgroup of S_n and every parabolic subgroup is of this form (up to conjugation). For $\alpha = (n_1, \ldots, n_k)$, let $\rho_1, \ldots, \rho_k$ be representations of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_k}(F)$, respectively, and σ a representation of $S_{n-m}(F)$. Let $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ denote the representation of $S_n(F)$ obtained by inducing the representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ of M_α (extended trivially to P_α). If $m = n$, we write $\rho_1 \times \cdots \times \rho_k \rtimes 1$, where 1 denotes the trivial representation of $S_0(F)$.

We recall some structures which will be useful later (*cf.* Section 1 of [Zel1] and Section 4 of [Tad3]). Let $R\big(\mathrm{GL}_n(F)\big)$ (resp., $R\big(S_n(F)\big)$) denote the Grothendieck group of the category of all smooth finite-length $\mathrm{GL}_n(F)$-modules (resp., $S_n(F)$-modules). Set $R = \bigoplus_{n\geq 0} R\big(\mathrm{GL}_n(F)\big)$ and $R[S] = \bigoplus_{n\geq 0} R\big(S_n(F)\big)$. The operators $\times$ and $\rtimes$ lift naturally to

$$\times : R \otimes R \longrightarrow R \quad \text{and} \quad \rtimes : R \otimes R[S] \longrightarrow R[S].$$

With these multiplications, R becomes an algebra and $R[S]$ a module over R.

Let π be an irreducible representation of $S_n(F)$. Then, there is a standard Levi M and an irreducible supercuspidal representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ of M (with ρ_i an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ and σ an irreducible supercuspidal representation of $S_{n-m}(F)$) such that π is a subquotient of $\rho_1 \times \cdots \times \rho_k \times \sigma$. We say that the multiset $\{\rho_1, \ldots, \rho_k; \sigma\}$ is in the support of π. Further, M and $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ are unique up to conjugation (*cf.* Theorem 2.9, [B-Z]). By Propositions 4.1 and 4.2 of [Tad3],

$$\begin{aligned}\rho_1 \times \cdots \times \rho_{i-1} \times \rho_i \times \rho_{i+1} \times \cdots \times \rho_k \rtimes \sigma \\ = \rho_1 \times \cdots \times \rho_{i-1} \times \tilde{\rho}_i \times \rho_{i+1} \times \cdots \times \rho_k \rtimes \sigma,\end{aligned}$$

where ˜ denotes contragredient. Thus, if $\{\rho_1, \ldots, \rho_{i-1}, \rho_i, \rho_{i+1}, \ldots, \rho_k; \sigma\}$ is in the support of π, so is $\{\rho_1, \ldots, \rho_{i-1}, \tilde{\rho}_i, \rho_{i+1}, \ldots, \rho_k; \sigma\}$. Therefore, every $\{\rho_1', \ldots, \rho_k'; \sigma\}$, with $\rho_i' = \rho_i$ or $\tilde{\rho}_i$, is in the support of π. Further, these exhaust the support of π. More generally, we extend the definition of support as in [Tad5]: If π is a finite-length representation and $\{\rho_1, \ldots, \rho_k; \sigma\}$ is in the support of π' for every irreducible subquotient π' of π, we say that $\{\rho_1, \ldots, \rho_k; \sigma\}$ is in the support of π.

We recall some notation of Bernstein-Zelevinsky [B-Z]. If $P = MU$ is a standard parabolic subgroup of G and ξ a representation of M, we let $i_{GM}(\xi)$ denote the representation obtained by (normalized) parabolic induction. Similarly, is π is a representation of G, we let $r_{MG}(\pi)$ denote the (normalized) Jacquet module of π with respect to P.

Next, we introduce some convenient shorthand for Jacquet modules (*cf.* [Tad3]). If π is a representation of some $S_n(F)$ and α is a partition of $m \leq n$, let $s_\alpha(\pi)$ denote the Jacquet module with respect to M_α. Note that, by abuse of notation, we also allow s_α to be applied to representations M_β when $M_\beta > M_\alpha$ (*cf.* Section 2.1, [B-Z]). Further, we define s_{GL} as in [Tad5]: for $\pi \leq \rho_1 \times \cdots \times \rho_k \rtimes \sigma$ with ρ_i a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ and σ a supercuspidal representation of $S_{n-m}(F)$, we set $s_{\mathrm{GL}}(\pi) = s_{(n_1+\cdots+n_k)}(\pi)$. We will occasionally use similar notation for representations of $\mathrm{GL}_n(F)$. If $\alpha = (n_1, \ldots, n_k)$ is a partition of $m \leq n$, $\mathrm{GL}_n(F)$ has a standard parabolic subgroup with Levi factor $L_\alpha \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F) \times \mathrm{GL}_{n-m}(F)$ (L_α consists of block-diagonal matrices; the corresponding parabolic subgroup of block upper triangular matrices). If π is a representation of $\mathrm{GL}_n(F)$, we let $r_\alpha(\pi)$ denote the Jacquet module of π with respect to L_α.

Finally, suppose π is a representation of $S_n(F)$. Consider

$$\mathcal{M}_{\min} = \{M \text{ standard Levi } \mid r_{MG}(\pi) \neq 0 \text{ but } r_{LG}(\pi) = 0\, \forall L < M\}.$$

Note that if π has supercuspidal support in the sense above, these are all conjugate. Then, formally set

$$s_{\min}(\pi) = \sum_{M \in \mathcal{M}_{\min}} r_{MG}(\pi).$$

If π has supercuspidal support of parabolic rank m, then $s_{\min}(\pi) \in \underbrace{R \otimes \cdots \otimes R}_{m} \otimes R[S]$. We may define $r_{\min}$ similarly for representations of $\mathrm{GL}_n(F)$.

2 The Case of $\mathrm{GL}_n(F)$

2.1 Background Material

We now review some results on induced representations for $\mathrm{GL}_n(F)$. This section is all based on the work of Zelevinsky [Zel1].

First, if ρ is an irreducible supercuspidal representation of $\mathrm{GL}_r(F)$ and $m \equiv n \mod 1$, we define the segment

$$[\nu^m \rho, \nu^n \rho] = \{\nu^m \rho, \nu^{m+1} \rho, \ldots, \nu^n \rho\}.$$

We note that the induced representation $\nu^m \rho \times \nu^{m+1} \rho \times \cdots \times \nu^n \rho$ has a unique irreducible subrepresentation, which we denote by $\zeta([\nu^m \rho, \nu^n \rho])$, and a unique irreducible quotient, which we denote by $\delta([\nu^m \rho, \nu^n \rho])$.

Lemma 2.1.1 *Let ρ_1, ρ_2 be irreducible unitary supercuspidal representations of $\mathrm{GL}_{r_1}(F)$, $\mathrm{GL}_{r_2}(F)$. Suppose $m_1 \leq n_1$, $m_2 \leq n_2$ satisfy $m_1 \equiv n_1 \mod 1$, $m_2 \equiv n_2 \mod 1$. Then, $\delta([\nu^{m_1} \rho_1, \nu^{n_1} \rho_1]) \times \delta([\nu^{m_2} \rho, \nu^{n_2} \rho])$ is reducible if and only if all of the following hold:*

1. *$\rho_1 \cong \rho_2$*
2. *$m_1 \equiv n_1 \equiv m_2 \equiv n_2$*
3. *either (a) $m_1 < m_2$ and $m_2 - 1 \leq n_1 \leq n_2 - 1$, or (b) $m_1 > m_2$ and $m_1 - 1 \leq n_2 \leq n_1 - 1$.*

$\zeta([\nu^{m_1} \rho_1, \nu^{n_1} \rho_1]) \times \zeta([\nu^{m_2} \rho_2, \nu^{n_2} \rho_2])$ is irreducible if and only if the same conditions hold.

Proof This is a special case of Theorem 4.2 [Zel1]. ∎

Next, consider a representation of the form

$$\chi = (\rho_1^{(1)} \otimes \cdots \otimes \rho_1^{(k_1)}) \otimes (\rho_2^{(1)} \otimes \cdots \otimes \rho_2^{(k_2)}) \otimes \cdots \otimes (\rho_m^{(1)} \otimes \cdots \otimes \rho_m^{(k_m)})$$

with $\rho_1^{(j)}$ an irreducible representation of $\mathrm{GL}_{r_i^{(j)}}(F)$ for all i, j. By a shuffle of χ, we mean the usual: a permutation on χ such that for all i, $\rho_i^{(1)}, \ldots, \rho_i^{(k_i)}$ appear in that order. (That is, the relative orders in the parenthesized pieces are preserved.) Further, if χ is a representation of a standard Levi M of $\mathrm{GL}_n(F)$ and $\mathrm{sh}(\chi)$ is a shuffle of χ, then $\mathrm{sh}(\chi)$ is a representation of a standard Levi subgroup of $\mathrm{GL}_n(F)$ which we denote by $\mathrm{sh}(M)$. We have the following:

Lemma 2.1.2 **(shuffling)**

1. *Suppose* π *is an irreducible representation of* $\mathrm{GL}_n(F)$ *such that* $r_{MG}(\pi) \geq \chi$, *where* χ *has the form*

$$(\nu^{\alpha_{1,1}}\rho_1 \otimes \cdots \otimes \nu^{\alpha_{1,j_1}}\rho_1) \otimes (\nu^{\alpha_{2,1}}\rho_2 \otimes \cdots \otimes \nu^{\alpha_{2,j_2}}\rho_2) \otimes \cdots \otimes (\nu^{\alpha_{m,1}}\rho_m \otimes \cdots \otimes \nu^{\alpha_{m,j_m}}\rho_m),$$

where

(a) $\rho_1, \dots, \rho_m$ *are irreducible unitary supercuspidal representations of* $\mathrm{GL}_{r_1}(F), \dots, \mathrm{GL}_{r_m}(F)$,

(b) $\alpha_{i,k} \in \mathbb{R}$ *with* $\alpha_{i,1} \equiv \alpha_{i,2} \equiv \cdots \equiv \alpha_{i,j_i} \mod 1$ *for all i, and such that*

(c) *if* $\rho_i \cong \rho_k$, *then* $\alpha_{i,1} \not\equiv \alpha_{k,1} \mod 1$.

Then, for every shuffle $\mathrm{sh}(\chi)$ *of* χ, *we have* $r_{\mathrm{sh}(M)G}(\pi) \geq \mathrm{sh}(\chi)$. *Further, if* $r_{\mathrm{sh}(M)G}(\pi) \geq \mathrm{sh}(\chi)$ *for any such shuffle, we necessarily have* $r_{MG}(\pi) \geq \chi$, *and therefore* $r_{\mathrm{sh}(M)G}(\pi) \geq \mathrm{sh}(\chi)$ *for every such shuffle.*

2. $i_{GM}(\chi) \cong i_{Gsh(M)}\big(\mathrm{sh}(\chi)\big)$ *for any such shuffle.*

Proof See Lemma 5.4 and Section 10 of [Jan3]. ■

Lemma 2.1.3 *Let* $(\rho_1, \alpha_1), \dots, (\rho_m, \alpha_m)$ *be pairs with* $\rho_1, \dots, \rho_m$ *irreducible unitary supercuspidal representations of* $\mathrm{GL}_{r_1}(F), \dots, \mathrm{GL}_{r_m}(F)$; $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ *such that* $\rho_i \cong \rho_j$ *implies* $\alpha_i \not\equiv \alpha_j \mod 1$. *Let* $\tau(\rho_i, \alpha_i)$ *be an irreducible representation of a general linear group supported on* $\{\nu^\alpha \rho_i\}_{\alpha \in \alpha_i + \mathbb{Z}}$. *Let M be the standard Levi subgroup of* $G = \mathrm{GL}_n(F)$ *which admits* $\tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_m, \alpha_m)$ *as a representation. Then,*

1. $\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_m, \alpha_m)$ *is irreducible.*
2. $$\mathrm{mult}\Big(\tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_m, \alpha_m), r_{MG}\big(\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_m, \alpha_m)\big)\Big) = 1.$$

   Further, if $\tau'(\rho_i, \alpha_i)$ *is an irreducible representation of a general linear group supported on* $\{\nu^\alpha \rho_i\}_{\alpha \in \alpha_i + \mathbb{Z}}$, *then*

$$\mathrm{mult}\Big(\tau'(\rho_1, \alpha_1) \otimes \cdots \otimes \tau'(\rho_m, \alpha_m), r_{MG}\big(\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_m, \alpha_m)\big)\Big) = 0$$

   unless $\tau'(\rho_i, \alpha_i) \cong \tau(\rho_i, \alpha_i)$ *for all i.*

3. *If* π *is an irreducible representation of* $\mathrm{GL}_n(F)$ *and* $r_{MG}(\pi) \geq \tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_m, \alpha_m)$, *then*

$$\pi = \tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_m, \alpha_m).$$

Proof The first claim is an immediate consequence of [Zel1, Proposition 8.5]. Claims 2. and 3. follow fairly easily—see Corollary 5.6 and [Jan3, Section 10] for details. ■

2.2 A Basic Lemma for $GL_n(F)$

Let ρ be an irreducible unitary supercuspidal representation of $GL_n(F)$, $0 \leq \alpha_0 < 1$. Suppose π is a representation of $GL_{mn}(F)$ of finite length, supported on $\{\nu^\alpha \rho\}_{\alpha \in \alpha_0 + \mathbb{Z}}$. Then, we make the following definition:

Definition 2.2.1 Let $\chi_0(\pi)$ denote the lowest element of $r_{\min}(\pi)$ with respect to the lexicographic order. $\chi_0(\pi)$ is unique up to multiplicity.

Lemma 2.2.2 *$\chi_0(\pi)$ has the form*

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1 - 1}\rho \otimes \cdots \otimes \nu^{b_1}) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k - 1}\rho \otimes \cdots \otimes \nu^{b_k}\rho),$$

with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $a_i \in \alpha_0 + \mathbb{Z}$ for all i.

Proof Write

$$\chi_0 = \nu^{\alpha_1}\rho \otimes \nu^{\alpha_2}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho.$$

Clearly, $\alpha_i \in \alpha_0 + \mathbb{Z}$ for all i. Let $j \geq 1$ be the smallest integer such that $\alpha_{j+1} \geq \alpha_1$.

Suppose $j > 1$. Then, we claim that $\alpha_2 = \alpha_1 - 1$. To see this, observe that if $\alpha_2 < \alpha_1 - 1$

$$\chi_0 = (\nu^{\alpha_1}\rho \otimes \nu^{\alpha_2}\rho) \otimes \nu^{\alpha_3}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{\min}(\pi)$$
$$\Downarrow$$
$$(\nu^{\alpha_1}\rho \times \nu^{\alpha_2}\rho) \otimes \nu^{\alpha_3}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{(2n,n,\ldots,n)}(\pi)$$
$$\Downarrow$$
$$\chi_0' = (\nu^{\alpha_2}\rho \otimes \nu^{\alpha_1}\rho) \otimes \nu^{\alpha_3}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{\min}(\pi),$$

since $\nu^{\alpha_1}\rho \times \nu^{\alpha_2}\rho \cong \nu^{\alpha_2}\rho \times \nu^{\alpha_1}\rho$ is irreducible. However, $\chi_0 < \chi_0'$ in the lexicographic order, contradicting the definition of $\chi_0(\pi)$. Thus, $\alpha_2 = \alpha_1 - 1$.

Next, suppose $j > 2$. Then, we claim that $\alpha_3 = \alpha_1 - 2 = \alpha_2 - 1$. First, if $\alpha_3 < \alpha_2 - 1$, then the same argument as above tells us that

$$\chi_0' = \nu^{\alpha_1}\rho \otimes \nu^{\alpha_3}\rho \otimes \nu^{\alpha_2}\rho \otimes \nu^{\alpha_4}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{\min}(\pi).$$

Again, $\chi_0' < \chi_0$ lexicographically, contradicting the definition of χ_0. Thus, $\alpha_3 = \alpha_1 - 1 = \alpha_2$ or $\alpha_3 = \alpha_1 - 2 = \alpha_2 - 1$. However, if $\alpha_3 = \alpha_1 - 1 = \alpha_2$, we have

$$\chi_0 = \nu^{\alpha_1}\rho \otimes \nu^{\alpha_1 - 1}\rho \otimes \nu^{\alpha_1 - 1}\rho \otimes \nu^{\alpha_4}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{\min}(\pi)$$
$$\Downarrow$$
$$\left(\delta([\nu^{\alpha_1 - 1}\rho, \nu^{\alpha_1}\rho]) \times \nu^{\alpha_1 - 1}\rho\right) \otimes \nu^{\alpha_4}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{(3n,n,\ldots,n)}(\pi)$$
$$\Downarrow$$
$$\chi_0' = \nu^{\alpha_1 - 1}\rho \otimes \nu^{\alpha_1}\rho \otimes \nu^{\alpha_1 - 1}\rho \otimes \nu^{\alpha_4}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \leq r_{\min}(\pi),$$

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(noting that $\delta([\nu^{\alpha_1-1}\rho,\nu^{\alpha_1}\rho])\times\nu^{\alpha_1-1}\rho$ is the only irreducible representation of $GL_{3n}(F)$ containing $\nu^{\alpha_1}\rho\otimes\nu^{\alpha_1-1}\rho\otimes\nu^{\alpha_1-1}\rho$ in its minimal Jacquet module) and again, we have $\chi_0'<\chi_0$ lexicographically, a contradiction. Thus, we are left with $\alpha_3=\alpha_1-2$, as claimed.

We now move to the more general step. Suppose $j>i$. Inductively, we may assume

$$\chi_0(\pi)=\nu^{\alpha_1}\rho\otimes\nu^{\alpha_1-1}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_i}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho.$$

We want to show that $\alpha_i=\alpha_1-i+1$. First, if $\alpha_i<\alpha_1-i+1$, we can use the same argument that we used to show $\alpha_2=\alpha_1-1$ to get

$$\chi_0'=\nu^{\alpha_1}\rho\otimes\nu^{\alpha_1-1}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+3}\rho\otimes(\nu^{\alpha_i}\rho\otimes\nu^{\alpha_1-i+2}\rho)\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho \\ \leq r_{\min}(\pi).$$

However, $\chi_0'<\chi_0$ lexicographically, contradicting the definition of χ_0. Similarly, if $\alpha_i=\alpha_1-i+2$, we can use the same argument we used to show $\alpha_3=\alpha_2-1$ to get

$$\chi_0'=\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+4}\rho\otimes(\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_1-i+3}\rho\otimes\nu^{\alpha_1-i+2}\rho)\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho \\ \leq r_{\min}(\pi).$$

Again, $\chi_0'<\chi_0$ lexicographically, a contradiction. Thus, $\alpha_i\in\{\alpha_1-1,\alpha_1-2,\dots,\alpha_1-i+3\}\cup\{\alpha_1-i+1\}$.

Now, suppose $\alpha_i=\alpha_1-k$ with $k\leq i-3$. Then,

$$\chi_0=\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k-2}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+3}\rho \\ \otimes(\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi)$$

$$\Downarrow$$

$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k-2}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+3}\rho \\ \otimes(\nu^{\alpha_1-i+2}\rho\times\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{(n,\dots,n,2n,n,\dots,n)}(\pi)$$

$$\Downarrow$$

$$\chi_0=\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k-2}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+3}\rho \\ \otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi)$$

$$\Downarrow$$

(similarly commuting $\nu^{\alpha_1-k}\rho$ around $\nu^{\alpha_1-i+3}\rho,\dots,\nu^{\alpha_1-k-2}\rho$)

$$\Downarrow$$

$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-2}\rho\otimes\cdots\otimes\nu^{\alpha_1-i+3}\rho \\ \otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi).$$

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Next,

$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+1}\rho\otimes(\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_1-k-2}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi)$$
$$\Downarrow$$
$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+1}\rho\otimes\big(\delta([\nu^{\alpha_1-k-1}\rho,\nu^{\alpha_1-k}\rho])\times\nu^{\alpha_1-k}\rho\big)\otimes\nu^{\alpha_1-k-2}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{(n,\dots,n,3n,n,\dots,n)}(\pi)$$
$$\text{or}$$
$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+1}\rho\otimes\big(\zeta([\nu^{\alpha_1-k-1}\rho,\nu^{\alpha_1-k}\rho])\times\nu^{\alpha_1-k}\rho\big)\otimes\nu^{\alpha_1-k-2}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{(n,\dots,n,3n,n,\dots,n)}(\pi)$$
$$\Downarrow$$
$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+1}\rho\otimes(\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k-1}\rho)\otimes\nu^{\alpha_1-k-2}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi)$$
$$\text{or}$$
$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+1}\rho\otimes(\nu^{\alpha_1-k-1}\rho\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_1-k-2}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi).$$

We can rule out the second of these possibilities since it is lexicographically lower than χ_0. Thus, we have (shifting parentheses)

$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+2}\rho\otimes(\nu^{\alpha_1-k+1}\rho\otimes\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_1-k-1}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi)$$
$$\Downarrow$$
$$\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+2}\rho\otimes\big(\delta([\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k+1}\rho])\times\nu^{\alpha_1-k}\rho\big)\otimes\nu^{\alpha_1-k-1}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{n,\dots,n,3n,n,\dots,n}(\pi)$$
$$\Downarrow$$
$$\chi_0'=\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{\alpha_1-k+2}\rho\otimes(\nu^{\alpha_1-k}\rho\otimes\nu^{\alpha_1-k+1}\rho\otimes\nu^{\alpha_1-k}\rho)\otimes\nu^{\alpha_1-k-1}\rho$$
$$\otimes\cdots\otimes\nu^{\alpha_1-i+2}\rho\otimes\nu^{\alpha_{i+1}}\rho\otimes\cdots\otimes\nu^{\alpha_m}\rho\leq r_{\min}(\pi).$$

However, $\chi_0'<\chi_0$ lexicographically, contradicting the definition of χ_0. Thus, we cannot have $\alpha_i=\alpha_1-k$ with $k\leq i-3$. The only possibility remaining is $\alpha_i=\alpha_1-i+1$, as needed. Thus, by induction, we have $\alpha_i=\alpha_1-i+1$ for $1\leq i\leq j$.

Now, in the statement of the lemma, we have $a_1=\alpha_1$ and $b_1=\alpha_1-j+1$. Repeating this argument to deal with a_2, b_2 through a_k, b_k finishes the lemma. ■

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Definition 2.2.3 With notation as above, if

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k-1}\rho \otimes \cdots \otimes \nu^{b_k}\rho),$$

set

$$\delta_0(\pi) = \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]).$$

Corollary 2.2.4 *Let π be an irreducible representation with $\chi_0(\pi)$, $\delta_0(\pi)$ be as above. Let $M = \mathrm{GL}_{(a_1-b_1+1)n}(F) \times \cdots \times \mathrm{GL}_{(a_k-b_k+1)n}(F)$. Then,*

$$\pi \hookrightarrow i_{GM}\big(\delta_0(\pi)\big).$$

Proof First, observe that by central character considerations, there is a direct summand V_0 of the space of $r_{M_{\min}G}(\pi)$ such that the semisimplification of V_0 consists of copies of $\chi_0(\pi)$. By Frobenius reciprocity, this implies $\pi \hookrightarrow i_{GM_{\min}}\big(\chi_0(\pi)\big)$. By Lemma 5.5 of [Jan3], there is an irreducible subquotient θ of $i_{MM_{\min}}\big(\chi_0(\pi)\big)$ such that $\pi \hookrightarrow i_{GM}(\theta)$. We claim that $\theta = \delta_0(\pi)$. Suppose not. Consider any $\chi \leq r_{M_{\min}M}(\theta)$. Then, $\chi < \chi_0(\pi)$ lexicographically since $\chi_0(\pi)$ is the highest term in $r_{M_{\min}M}\Big(i_{MM_{\min}}\big(\chi_0(\pi)\big)\Big)$. However, by Frobenius reciprocity, $\theta \leq r_{MG}(\pi)$. This implies $\chi \leq r_{M_{\min}G}(\pi)$, contradicting the definition of $\chi_0(\pi)$. Thus, $\theta = \delta_0(\pi)$. ■

2.3 A Result of Bernstein

We now give an application of the results of the previous section. A well-known theorem of Bernstein (*cf.* [Zel1, Theorem 9.3]) says that an irreducible representation of a p-adic general linear group is essentially square-integrable if and only if it has the form $\delta([\nu^{\alpha}\rho, \nu^{\alpha+k}\rho])$ for some irreducible unitary supercuspidal representation ρ and some non-negative integer k. A proof of this fact may be obtained fairly easily at this point.

We first recall the Casselman criterion. Let π be an irreducible square-integrable representation of $\mathrm{GL}_n(F)$. Suppose

$$\chi = \nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \leq r_{\min}(\pi),$$

with ρ_i an irreducible unitary supercuspidal representation of $\mathrm{GL}_{n_i}(F)$, $\alpha_i \in \mathbb{R}$, and $n_1 + \cdots + n_k = n$. Then,

$$\begin{gathered} n_1\alpha_1 > 0 \\ n_1\alpha_1 + n_2\alpha_2 > 0 \\ \vdots \\ n_1\alpha_1 + n_2\alpha_2 + \cdots + n_{k-1}\alpha_{k-1} > 0 \\ n_1\alpha_1 + n_2\alpha_2 + \cdots + n_{k-1}\alpha_{k-1} + n_k\alpha_k = 0. \end{gathered}$$

Conversely, if π is an irreducible representation such that the inequalities above hold for every $\chi \leq r_{\min}(\pi)$, then π is square-integrable.

https://doi.org/10.4153/CJM-2000-025-7 Published online by Cambridge University Press

Theorem 2.3.1 **(Bernstein)** *π is an irreducible square-integrable representation of* $\mathrm{GL}_n(F)$ *if and only if π has the form $\pi = \delta([\nu^{-m}\rho, \nu^m\rho])$, where ρ is an irreducible unitary supercuspidal representation of* $\mathrm{GL}_k(F)$*, $m \in \frac{1}{2}\mathbb{Z}$ with $m \geq 0$, and $n = (2m+1)k$.*

Proof As noted in [Zel1, Theorem 9.3], the Casselman criterion implies $\delta([\nu^{-m}\rho, \nu^m\rho])$ is square-integrable.

In the other direction, we first claim that if π is square-integrable, $\mathrm{supp}(\pi)$ must be contained in a set of the form $\{\nu^\alpha\rho\}_{\alpha\in\alpha_0+\mathbb{Z}}$ for some irreducible unitary supercuspidal ρ and some $\alpha_0 \in \mathbb{R}$. This follows easily from Lemma 2.1.2. For example, suppose $\mathrm{supp}(\pi) \subset \{\nu^\alpha\rho_1\}_{\alpha\in\alpha_0+\mathbb{Z}} \cup \{\nu^\beta\rho_2\}_{\beta\in\beta_0+\mathbb{Z}}$, but not completely in either. (Here, we allow the possibility that $\rho_1 \cong \rho_2$ or $\alpha_0 \equiv \beta_0 \mod 1$, but not both.) By Lemma 2.1.2, we could obtain

$$\chi_1 = \nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_1 \otimes \nu^{\beta_1}\rho_2 \otimes \cdots \otimes \nu^{\beta_\ell}\rho_2 \leq r_{\min}(\pi)$$

and

$$\chi_2 = \nu^{\beta_1}\rho_2 \otimes \cdots \otimes \nu^{\beta_\ell}\rho_2 \otimes \nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_1 \leq r_{\min}(\pi)$$

for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$. One of the Casselman criterion inequalities for χ_1 gives $n_1\alpha_1 + \cdots + n_1\alpha_k > 0$. Similarly, one of the Casselman criterion inequalities for χ_2 gives $n_2\beta_1+\cdots+n_2\beta_\ell > 0$. Adding these gives $n_1\alpha_1+\cdots+n_1\alpha_k+n_2\beta_1+\cdots+n_2\beta_\ell > 0$. However, by the Casselman criterion, we must have $n_1\alpha_1 + \cdots + n_1\alpha_k + n_2\beta_1 + \cdots + n_2\beta_\ell = 0$, a contradiction. Thus, $\mathrm{supp}(\pi)$ must be contained in a set of the form $\{\nu^\alpha\}_{\alpha\in\alpha_0+\mathbb{Z}}$, as claimed.

Since $\mathrm{supp}(\pi) \subset \{\nu^\alpha\}_{\alpha\in\alpha_0+\mathbb{Z}}$, we can apply Lemma 2.2.2. Let χ_0 be as in Lemma 2.2.2. If $k = 1$ (notation as in Lemma 2.2.2), then the Casselman criterion inequalities require $a_1 = -b_1$. By Corollary 2.2.4, *e.g.*, we then get $\pi = \delta([\nu^{-a_1}\rho, \nu^{a_1}\rho])$, as needed. Suppose $k \geq 2$. The Casselman criterion inequalities require $a_1 > -b_1$. So, suppose $a_i > -b_i$ for $i = 1, \ldots, j-1$ and $a_j \leq -b_j$ (note that if such a j did not exist, we would have $\big(a_1+(a_1-1)+\cdots+b_1\big)+\cdots+\big(a_k+(a_k-1)+\cdots+b_k\big) > 0$, contradicting the Casselman criterion). Since $a_j \geq a_1, \ldots, a_{j-1}$ and $b_j \leq b_1, \ldots, b_{j-1}$, we have

$$\delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \cong \delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho])$$

is irreducible for all $i \leq j-1$. From this fact and Corollary 2.2.4, we have

$$\begin{aligned}
\pi \hookrightarrow\ & \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{j-2}}\rho, \nu^{a_{j-2}}\rho]) \\
& \times \{\delta([\nu^{b_{j-1}}\rho, \nu^{a_{j-1}}\rho]) \times \delta([\nu^{b_j}\rho, \nu^{a_j}\rho])\} \\
& \times \delta([\nu^{b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \\
\cong\ & \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{j-2}}\rho, \nu^{a_{j-2}}\rho]) \\
& \times \{\delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{b_{j-1}}\rho, \nu^{a_{j-1}}\rho])\} \\
& \times \delta([\nu^{b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])
\end{aligned}$$

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$$\vdots$$
$$\cong \delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{j-2}}\rho, \nu^{a_{j-2}}\rho])$$
$$\times \delta([\nu^{b_{j-1}}\rho, \nu^{a_{j-1}}\rho]) \times \delta([\nu^{b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]).$$

Therefore, by Frobenius reciprocity,

$$r_{\min}(\pi) \geq (\nu^{a_j}\rho \otimes \nu^{a_j-1}\rho \otimes \cdots \otimes \nu^{b_j}\rho) \otimes \nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho \otimes \cdots \otimes \nu^{b_k}\rho_k.$$

In particular, since $a_j \leq -b_j$, we see that

$$na_j + n(a_j - 1) + \cdots + nb_j \leq 0,$$

contradicting the Casselman criterion. Thus, if π is square-integrable, we must have $k = 1$, as needed. ■

2.4 Connection with the Langlands classification

In this section, we establish the connection between $\delta_0(\pi)$ and the Langlands data for π. We shall also give a lemma which will be needed later.

Let us briefly review the Langlands classification for general linear groups. First, if δ is an essentially square-integrable representation of $GL_n(F)$, then there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon\delta}\delta$ is unitarizable. Suppose $\delta_1, \ldots, \delta_k$ are irreducible, essentially square-integrable representations of $GL_{n_1}(F), \ldots, GL_{n_k}(F)$ with $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k)$. (We allow weak inequalities since we are assuming δ_i is essentially square-integrable; if we allowed δ_i essentially tempered, we would have strict inequalities. The formulations are equivalent.) Then, $\delta_1 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation (Langlands subrepresentation). Further, any irreducible representation of a general linear group may be realized this way. We favor the subrepresentation version of the Langlands classification over the quotient version since $\pi \hookrightarrow \delta_1 \times \cdots \times \delta_k$ tells us $\delta_1 \otimes \cdots \otimes \delta_k$ appears in the (appropriate) Jacquet module for π.

For notational convenience, let $\delta(\rho, m) = \delta([\nu^{\frac{-m+1}{2}}\rho, \nu^{\frac{m+1}{2}}\rho])$ for $m \in \mathbb{Z}$ with $m \geq 0$. Then, $\nu^{\alpha}\delta(\rho, m) = \delta([\nu^{\frac{-m+1}{2}+\alpha}\rho, \nu^{\frac{m+1}{2}+\alpha}\rho])$. Write

$$\delta_0(\pi) = \nu^{\alpha_1}\delta(\rho, m_1) \otimes \nu^{\alpha_2}\delta(\rho, m_2) \otimes \cdots \otimes \nu^{\alpha_k}\delta(\rho, m_k).$$

In particular, $\alpha_i = \frac{a_i+b_i}{2}$ and $m_i = (a_i - b_i + 1)$. We then have the following:

Proposition 2.4.1 *Let $\delta_0(\pi)$ be as above and $\nu^{\alpha'_1}\delta(\rho, m'_1), \nu^{\alpha'_2}\delta(\rho, m'_2), \ldots, \nu^{\alpha'_k}\delta(\rho, m'_k)$ be a permutation of $\nu^{\alpha_1}\delta(\rho, m_1), \nu^{\alpha_2}\delta(\rho, m_2), \ldots, \nu^{\alpha_k}\delta(\rho, m_k)$ with $\alpha'_1 \leq \alpha'_2 \leq \cdots \leq \alpha'_k$. Then,*

$$\nu^{\alpha'_1}\delta(\rho, m'_1) \times \nu^{\alpha'_2}\delta(\rho, m'_2) \times \cdots \times \nu^{\alpha'_k}\delta(\rho, m'_k)$$
$$\cong \nu^{\alpha_1}\delta(\rho, m_1) \times \nu^{\alpha_2}\delta(\rho, m_2) \times \cdots \times \nu^{\alpha_k}\delta(\rho, m_k).$$

Further,

$$\delta'_0(\pi) = \nu^{\alpha'_1}\delta(\rho, m'_1) \otimes \nu^{\alpha'_2}\delta(\rho, m'_2) \otimes \cdots \otimes \nu^{\alpha'_k}\delta(\rho, m'_k)$$

is the Langlands data for π.

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Proof First, we focus on showing

$$\nu^{\alpha_1'}\delta(\rho,m_1')\times\nu^{\alpha_2'}\delta(\rho,m_2')\times\cdots\times\nu^{\alpha_k'}\delta(\rho,m_k')$$
$$\cong\nu^{\alpha_1}\delta(\rho,m_1)\times\nu^{\alpha_2}\delta(\rho,m_2)\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k).$$

Suppose that $\nu^{\alpha_1'}\delta(\rho,m_1')=\nu^{\alpha_i}\delta(\rho,m_i)$. For $j<i$, we have $a_j\leq a_i$ (definition of $\delta_0(\pi)$). Further, since $\alpha_i\geq\alpha_j$, we must have $b_j\geq b_i$. Thus, by Lemma 2.1.1,

$$\nu^{\alpha_j}\delta(\rho,m_j)\times\nu^{\alpha_i}\delta(\rho,m_i)\cong\nu^{\alpha_i}\delta(\rho,m_i)\times\nu^{\alpha_j}\delta(\rho,m_j)$$

is irreducible. Thus, we can commute $\nu^{\alpha_i}\delta(\rho,m_i)$ to the front as follows:

$$\begin{aligned}
&\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_{i-2}}\delta(\rho,m_{i-2})\times\nu^{\alpha_{i-1}}\delta(\rho,m_{i-1})\times\nu^{\alpha_i}\delta(\rho,m_i)\\
&\qquad\times\nu^{\alpha_{i+1}}\delta(\rho,m_{i+1})\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\\
&\cong\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_{i-2}}\delta(\rho,m_{i-2})\times\nu^{\alpha_i}\delta(\rho,m_i)\times\nu^{\alpha_{i-1}}\delta(\rho,m_{i-1})\\
&\qquad\times\nu^{\alpha_{i+1}}\delta(\rho,m_{i+1})\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\\
&\cong\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_i}\delta(\rho,m_i)\times\nu^{\alpha_{i-2}}\delta(\rho,m_{i-2})\times\nu^{\alpha_{i-1}}\delta(\rho,m_{i-1})\\
&\qquad\times\nu^{\alpha_{i+1}}\delta(\rho,m_{i+1})\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\\
&\ \ \vdots\\
&\cong\nu^{\alpha_i}\delta(\rho,m_i)\times\nu^{\alpha_1}\delta(\rho,m_1)\times\nu^{\alpha_{i-2}}\delta(\rho,m_{i-2})\times\nu^{\alpha_{i-1}}\delta(\rho,m_{i-1})\\
&\qquad\times\nu^{\alpha_{i+1}}\delta(\rho,m_{i+1})\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\\
&=\nu^{\alpha_1'}\delta(\rho,m_1')\times\nu^{\alpha_1}\delta(\rho,m_1)\times\nu^{\alpha_{i-2}}\delta(\rho,m_{i-2})\times\nu^{\alpha_{i-1}}\delta(\rho,m_{i-1})\\
&\qquad\times\nu^{\alpha_{i+1}}\delta(\rho,m_{i+1})\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k).
\end{aligned}$$

Next, we identify $\nu^{\alpha_2'}\delta(\rho,m_2')$ among the remaining terms. We can then use the same argument to commute it into the second position, giving

$$\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\cong\nu^{\alpha_1'}\delta(\rho,m_1')\times\nu^{\alpha_2'}\delta(\rho,m_2')\times\cdots.$$

Iterating this procedure, after $k-1$ steps, we obtain

$$\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\cong\nu^{\alpha_1'}\delta(\rho,m_1')\times\cdots\times\nu^{\alpha_k'}\delta(\rho,m_k'),$$

as claimed.

The claim regarding the Langlands data is now straightforward. By Corollary 2.2.4, we have

$$\pi\hookrightarrow\nu^{\alpha_1}\delta(\rho,m_1)\times\cdots\times\nu^{\alpha_k}\delta(\rho,m_k)\cong\nu^{\alpha_1'}\delta(\rho,m_1')\times\cdots\times\nu^{\alpha_k'}\delta(\rho,m_k'),$$

which has a unique irreducible subrepresentation whose Langlands data is $\delta_0'(\pi)$. Thus, $\delta_0'(\pi)$ must be the Langlands data for π. ■

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The lemma below will be needed in Section 4.2. Suppose

$$\chi_1 = (\nu^{\alpha_1}\rho \otimes \nu^{\alpha_2}\rho \otimes \cdots \otimes \nu^{\alpha_j}\rho)$$
$$\chi_2 = (\nu^{\beta_1}\rho \otimes \nu^{\beta_2}\rho \otimes \cdots \otimes \nu^{\beta_k}\rho).$$

We let

$$\text{m.l.s. } (\chi_1, \chi_2) = \nu^{\gamma_1}\rho \otimes \nu^{\gamma_2}\rho \otimes \cdots \otimes \nu^{\gamma_{j+k}}\rho$$

with $\nu^{\gamma_1}\rho \otimes \nu^{\gamma_2}\rho \otimes \cdots \otimes \nu^{\gamma_{j+k}}\rho$ the shuffle of χ_1 and χ_2 which is minimal with respect to the lexicographic order (m.l.s. for "minimal lexicographic shuffle").

Lemma 2.4.2 *Let π_1, π_2 be finite-length representations supported on $\{\nu^\alpha\rho\}_{\alpha\in\alpha_0+\mathbb{Z}}$. Then,*

$$\chi_0(\pi_1 \times \pi_2) = m.l.s.\ \big(\chi_0(\pi_1), \chi_0(\pi_2)\big).$$

Proof First, by the characterization of the minimal Jacquet module of an induced representation via shuffles, we have

$$\text{m.l.s. } \big(\chi_0(\pi_1), \chi_0(\pi_2)\big) \le r_{\min}(\pi_1 \times \pi_2).$$

Thus, $\chi_0(\pi_1 \times \pi_2) \le$ m.l.s. $\big(\chi_0(\pi_1), \chi_0(\pi_2)\big)$ lexicographically.

On the other hand, by the characterization of the minimal Jacquet module of an induced representation via shuffles, we have

$$\chi_0(\pi_1 \times \pi_2) = \text{sh}_0(\chi_1, \chi_2)$$

for some $\chi_1 \le r_{\min}(\pi_1)$, $\chi_2 \le r_{\min}(\pi_2)$, and a shuffle sh_0. By definition, $\chi_0(\pi_1) \le \chi_1$ and $\chi_0(\pi_2) \le \chi_2$ lexicographically. Therefore, $\text{sh}_0\big(\chi_0(\pi_1), \chi_0(\pi_2)\big) \le \text{sh}_0(\chi_1, \chi_2)$ lexicographically. Thus,

$$\text{m.l.s. } \big(\chi_0(\pi_1), \chi_0(\pi_2)\big) \le \text{sh}_0\big(\chi_0(\pi_1), \chi_0(\pi_2)\big) \le \text{sh}_0(\chi_1, \chi_2) = \chi_0(\pi_1 \times \pi_2)$$

lexicographically. Combining the inequalities gives the lemma. ∎

3 Supercuspidal Supports

3.1 Background material

In this section, we give some additional background material for $S_n(F)$. In particular, we review the Casselman criterion, the Langlands classification, as well as some additional structures on R, $R[S]$ which we need later.

First, we review the Casselman criterion for the temperedness (resp., square-integrability) of representations of $S_n(F)$. Let π be an irreducible representation of $S_n(F)$ and

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$\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \otimes \sigma \leq s_{\min}(\pi)$, with ρ_i an irreducible unitary supercuspidal representation of $\mathrm{GL}_{m_i}(F)$, σ an irreducible supercuspidal representation of $S_m(F)$ and $\alpha_i \in \mathbb{R}$. Then, if π is tempered,

$$\begin{cases} m_1\alpha_1 \geq 0, \\ m_1\alpha_1 + m_2\alpha_2 \geq 0, \\ \ \vdots \\ m_1\alpha_1 + m_2\alpha_2 + \cdots + m_k\alpha_k \geq 0. \end{cases}$$

Conversely, if the corresponding inequalities hold for every element of $s_{\min}(\pi)$, then π is tempered. The criterion for square-integrability is the same except that the weak inequalities are replaced by strict inequalities.

Next, we review the Langlands classification for $S_n(F)$. As in Section 2.4, if δ is an irreducible, essentially square-integrable representation of $\mathrm{GL}_n(F)$, we have $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. Let $\delta_1, \ldots, \delta_k$ be irreducible essentially square-integrable representations of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_k}(F)$ satisfying $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k) < 0$ and τ a tempered representation of $S_{n-m}(F)$. Then, $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote by $L(\delta_1, \ldots, \delta_k; \tau)$. (Equivalently, we could formulate the Langlands classification with $\delta_1, \ldots, \delta_k$ essentially tempered and $\varepsilon(\delta_1) < \cdots < \varepsilon(\delta_k) < 0$.) Further, any irreducible representation may be realized this way. As with general linear groups, we favor the subrepresentation version of the Langlands classification for the following reason: In the subrepresentation version, $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau \leq s_{(n_1,\ldots,n_k)}\big(L(\delta_1, \ldots, \delta_k; \tau)\big)$.

The Langlands classification is done in its general form in [Sil1] and [B-W]; the Casselman criterion in [Cas]. The discussion above is largely based on [Tad1]. The reader is referred there for more details. We now turn to some structures we will need later.

Definition 3.1.1

1. If τ is a representation of $\mathrm{GL}_n(F)$, set

$$m^*(\tau) = \sum_{i=0}^{n} r_{(i)}(\tau).$$

2. If π is a representation of $S_n(F)$, set

$$\mu^*(\pi) = \sum_{i=0}^{n} s_{(i)}(\pi).$$

Observe that we may lift m^* to a map $m^*\colon R \longrightarrow R \otimes R$. With multiplication given by $\times$ and comultiplication given by m^*, R has the structure of a Hopf algebra (*cf.* [Zel1, Section 1.7]). In particular, if we define $\times\colon (R \otimes R) \otimes (R \otimes R) \longrightarrow (R \otimes R)$ by taking $(\tau_1 \otimes \tau_2) \times (\tau_1' \otimes \tau_2') = (\tau_1 \times \tau_1') \otimes (\tau_2 \times \tau_2')$ and extending bilinearly, we have $m^*(\tau_1 \times \tau_2) = m^*(\tau_1) \times m^*(\tau_2)$.

Now, define $s\colon R \otimes R \longrightarrow R \otimes R$ by taking the map $s\colon \tau_1 \otimes \tau_2 \longmapsto \tau_2 \otimes \tau_1$ and extending it bilinearly. For notational convenience, write $m\colon R \otimes R \longrightarrow R$ for multiplication. Set

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$M^* = (m \otimes 1) \circ (\tilde{} \otimes m^*) \circ s \circ m^*$. If we define $\rtimes : (R \otimes R) \otimes (R \otimes R[S]) \longrightarrow R \otimes R[S]$ by taking $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \theta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \theta)$ and extending bilinearly, we have the following:

***Theorem 3.1.2* (Tadić)** *For $\tau \in R$, $\theta \in R[S]$, we have*

$$\mu^*(\tau \rtimes \theta) = M^*(\tau) \rtimes \mu^*(\theta).$$

In other words, with $\rtimes$ and μ^*, $R[S]$ acquires the structure of an M^*-Hopf module over R (*cf.* [Tad2]). We note that there is a corresponding result for the groups $O_{2n}(F)$ [Ban]. Therefore, we expect that once certain issues related to disconnectedness are addressed, it will be possible to bring $O_{2n}(F)$ into this discussion as well.

Finally, we shall make use of the following:

Definition 3.1.3 Suppose ρ is an irreducible, unitary, supercuspidal representation of $GL_n(F)$ and σ and irreducible, supercuspidal representation of $S_r(F)$. For $\alpha \geq 0$, we say that (ρ, σ) satisfies (Cα) if $\nu^\alpha \rho \rtimes \sigma$ is reducible and $\nu^\beta \rtimes \sigma$ is irreducible for all $\beta \in \mathbb{R} \setminus \{\pm\alpha\}$.

It is well-known that if $\rho \not\cong \tilde{\rho}$, then $\nu^\beta \rho \rtimes \sigma$ is irreducible for all $\beta \in \mathbb{R}$. If $\rho \cong \tilde{\rho}$, then there is an $\alpha \geq 0$ such that $\nu^\alpha \rho \rtimes \sigma$ is reducible (*cf.* [Tad5]). Further, it is then the case that $\nu^\beta \rho \rtimes \sigma$ is irreducible for all $\beta \in \mathbb{R} \setminus \{\pm\alpha\}$ (*cf.* [Sil2]).

If σ is generic and (ρ, σ) satisfies (Cα), then $\alpha \in \{0, \frac{1}{2}, 1\}$ (*cf.* [Sha1], [Sha2]). If σ is nongeneric, one can have (ρ, σ) satisfying (Cα) for $\alpha > 1$ (*cf.* [Re]). In general, it is expected—and we shall assume—that $\alpha \in \frac{1}{2}\mathbb{Z}$. (Assuming certain conjectures, this has recently been verified in [Mœ2] and [Zh].) The problem of determining α for a given pair (ρ, σ) is difficult. However, we note that in the case $\sigma = 1$, much is known. In particular, when $n \geq 2$ and $\rho \cong \tilde{\rho}$ is tamely ramified (*cf.* [Adl]), the value of α has been explicitly calculated in [M-R] for a large collection of such ρ using a criterion from [Sha2].

3.2 Reducing the Problem Based on Supercuspidal Supports

Let ρ be an irreducible unitary supercuspidal representation of $GL_n(F)$, $\beta \in \mathbb{R}$. Set

$$\mathcal{S}(\rho, \beta) = \{\nu^\alpha \rho, \nu^{-\alpha} \tilde{\rho}\}_{\alpha \in \beta + \mathbb{Z}}.$$

If $\rho = \tilde{\rho}$, we may take $0 \leq \beta \leq \frac{1}{2}$; otherwise $0 \leq \beta < 1$. Suppose $\rho_1, \rho_2, \ldots, \rho_m$ are irreducible, unitary, supercuspidal representations of $GL_{n_1}(F), \ldots, GL_{n_m}(F)$, and $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, with $0 \leq \beta_i \leq \frac{1}{2}$ if $\rho_i \cong \tilde{\rho}_i$, $0 \leq \beta_i < 1$ if not. Further, assume that $\mathcal{S}(\rho_1, \beta_1), \ldots, \mathcal{S}(\rho_m, \beta_m)$ are disjoint. Set

$$\mathcal{S}\big((\rho_1, \beta_1), (\rho_2, \beta_2), \ldots, (\rho_m, \beta_m)\big) = \mathcal{S}(\rho_1, \beta_1) \cup \mathcal{S}(\rho_2, \beta_2) \cup \cdots \cup \mathcal{S}(\rho_m, \beta_m).$$

If σ is an irreducible supercuspidal representation of $S_r(F)$, set

$$\mathcal{S}\big((\rho_1, \beta_1), (\rho_2, \beta_2), \ldots, (\rho_m, \beta_m); \sigma\big) = \mathcal{S}(\rho_1, \beta_1) \cup \mathcal{S}(\rho_2, \beta_2) \cup \cdots \cup \mathcal{S}(\rho_m, \beta_m) \cup \{\sigma\}.$$

We note that every irreducible representation of $S_n(F)$ has supercuspidal support on a set of this form.

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We now recall some results from [Jan3]. Suppose π is an irreducible representation supported on $\mathcal{S}\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m);\sigma\big)$. Then, there exist irreducible representations $\tau_1,\tau_2,\dots,\tau_{m-1}$ of $\mathrm{GL}_{k_1}(F),\mathrm{GL}_{k_2}(F),\dots,\mathrm{GL}_{k_{m-1}}(F)$ and an irreducible representation θ_m of $S_{k_m+r}(F)$ such that

$\pi \hookrightarrow \tau_1\times\tau_2\times\cdots\times\tau_{m-1}\rtimes\theta_m$

2. τ_i is supported on $\mathcal{S}(\rho_i,\beta_i)$ and θ_m is supported on $\mathcal{S}\big((\rho_m,\beta_m);\sigma\big)$.

Further, θ_m is unique. Similarly, one could single out $(\rho_1,\beta_1),\dots,(\rho_{m-1},\beta_{m-1})$, resp., to produce $\theta_1,\dots,\theta_{m-1}$, resp., supported on $\mathcal{S}\big((\rho_1,\beta_1);\sigma\big),\dots,\mathcal{S}\big((\rho_{m-1},\beta_{m-1});\sigma\big)$, resp. Write $(\rho_i,\beta_i)(\pi)=\theta_i$.

Theorem 3.2.1 *Let* $\mathrm{Irr}\big((\rho_1,\beta_1),\dots,(\rho_m,\beta_m);\sigma\big)$ *denote the set of all irreducible representations of all* $S_n(F)$, $n\geq 0$, *supported on* $\mathcal{S}\big((\rho_1,\beta_1),\dots,(\rho_m,\beta_m);\sigma\big)$. *Then, the map*

$$\pi \longmapsto (\rho_1,\beta_1)(\pi)\otimes\cdots\otimes(\rho_m,\beta_m)(\pi)$$

implements a bijective correspondence

$$\mathrm{Irr}\big((\rho_1,\beta_1),\dots,(\rho_m,\beta_m);\sigma\big)\longleftrightarrow \mathrm{Irr}\big((\rho_1,\beta_1);\sigma\big)\otimes\cdots\otimes\mathrm{Irr}\big((\rho_m,\beta_m);\sigma\big).$$

Further, π *is square-integrable (resp., tempered) if and only if* $(\rho_1,\beta_1)(\pi),\dots,(\rho_m,\beta_m)(\pi)$ *are all square-integrable (resp., tempered).*

Remark 3.2.2 The correspondence described above also respects contragredience, duality, Langlands data, induction, and Jacquet modules in a sense made precise in [Jan3]. For our purposes, the key feature is that it respects square-integrability.

3.3 A Conjecture of Tadić

Suppose that $(\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m),\sigma$ are as in Section 3.2. We let $R\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m)\big)$ denote the subalgebra of R generated by the representations supported in $\mathcal{S}\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m)\big)$. Then,

$$R\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m)\big)\cong R\big((\rho_1,\beta_1)\big)\otimes R\big((\rho_2,\beta_2)\big)\otimes\cdots\otimes R\big((\rho_m,\beta_m)\big)$$

as Hopf subalgebras of R (*cf.* Remark 8.7 of [Zel1]). On irreducible representations, this tensor product decomposition is determined by the (appropriate) Jacquet module (*cf.* Lemma 2.1.3 2.). If we let $R\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m);\sigma\big)$ denote the additive subgroup of $R[S]$ generated by representations supported in $\mathcal{S}\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m);\sigma\big)$, then

$$R\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m);\sigma\big)\cong R\big((\rho_1,\beta_1);\sigma\big)\otimes R\big((\rho_2,\beta_2);\sigma\big)\otimes\cdots\otimes R\big((\rho_m;\beta_m);\sigma\big)$$

as $R\big((\rho_1,\beta_1),(\rho_2,\beta_2),\dots,(\rho_m,\beta_m)\big)\cong R\big((\rho_1,\beta_1)\big)\otimes R\big((\rho_2,\beta_2)\big)\otimes\cdots\otimes R\big((\rho_m,\beta_m)\big)$ M^*-Hopf modules (*cf.* [Jan3, Proposition 10.10]). On irreducible representations, this tensor product decomposition corresponds to that described in Theorem 3.2.1 above.

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Conjecture 3.3.1 Suppose $(\rho_1, \sigma_1), (\rho_2, \sigma_2)$ both satisfy $(C\alpha)$ (same value of α), $0 \leq \beta \leq \frac{1}{2}$. Then,

$$R\big((\rho_1, \beta); \sigma_1\big) \cong R\big((\rho_2, \beta); \sigma_2\big)$$

as $R\big((\rho_1, \beta)\big) \cong R\big((\rho_2, \beta)\big)$ M^*-Hopf modules. A similar result should hold if $\rho_i \ncong \tilde{\rho}_i$ for $i = 1, 2$.

Of course, if two irreducible representations correspond under the isomorphism, they should have supercuspidal support of the same parabolic rank. We also note that this isomorphism should send tempered (resp., square-integrable) representations to tempered (resp., square-integrable) ones and commute with $\tilde{}$ and $\hat{}$ =duality operator (*cf.* [Aub], [S-S]).

That $R\big((\rho_1, \beta)\big) \cong R\big((\rho_2, \beta)\big)$ is conjectured in [Zel2] (or more precisely, is an immediate consequence of a conjecture in [Zel2]). That it holds follows from the results in chapter 7 of [B-K]. The conjecture above was suggested by Marko Tadić.

It is not difficult to describe the conjectured isomorphism. For concreteness, suppose that $(\rho_1, \sigma_1), (\rho_2, \sigma_2)$ both satisfy $(C1/2)$. For $R\big((\rho_1, \beta)\big) \cong R\big((\rho_2, \beta)\big)$, we want

$$\delta([\nu^{\beta+x}\rho_1, \nu^{\beta+y}\rho_1]) \longleftrightarrow \delta([\nu^{\beta+x}\rho_2, \nu^{\beta+y}\rho_2])$$

and

$$\delta([\nu^{-\beta+x}\rho_1, \nu^{-\beta+y}\rho_1]) \longleftrightarrow \delta([\nu^{-\beta+x}\rho_2, \nu^{-\beta+y}\rho_2])$$

for all $x, y \in \mathbb{Z}$ with $x \leq y$. This gives a bijective correspondence between irreducible essentially square-integrable representations in $R\big((\rho_1, \beta)\big)$ and those in $R\big((\rho_2, \beta)\big)$. This immediately extends to a bijective correspondence between irreducible essentially tempered representations in $R\big((\rho_1, \beta)\big)$ and those in $R\big((\rho_2, \beta)\big)$ (since any irreducible essentially tempered representation in $R\big((\rho_i, \beta)\big)$ may be written as an irreducible product of irreducible essentially square-integrable representations in $R\big((\rho_i, \beta)\big)$). Finally, in general, two irreducible representations correspond if their Langlands data correspond. Equivalently, two irreducible representations $\pi_1 \in R\big((\rho_1, \beta)\big)$, $\pi_2 \in R\big((\rho_2, \beta)\big)$ correspond if $\delta_0(\pi_1)$ and $\delta_0(\pi_2)$ correspond (*cf.* Chapter 2).

We cannot be quite as explicit about the isomorphism $R\big((\rho_1, \beta); \sigma_1\big) \cong R\big((\rho_2, \beta); \sigma_2\big)$, but we can describe an inductive procedure. Suppose we know the map for irreducible representations whose supercuspidal support has parabolic rank $\leq n - 1$. Let $\pi_1 \in R\big((\rho_1, \beta); \sigma_1\big)$, $\pi_2 \in R\big((\rho_2, \beta); \sigma_2\big)$ be irreducible representations with supercuspidal support of parabolic rank n. If π_1, π_2 are nontempered, $\pi_1 \longleftrightarrow \pi_2$ if their Langlands data correspond. This is a question of whether a collection of irreducible essentially tempered representations from $R\big((\rho_1, \beta)\big)$ and $R\big((\rho_2, \beta)\big)$ correspond and whether a tempered representation from $R\big((\rho_1, \beta); \sigma_1\big)$ and one from $R\big((\rho_2, \beta); \sigma_2\big)$—both having supercuspidal support of parabolic rank $< n$—correspond. Thus, if π_1, π_2 are nontempered, we can check if $\pi_1 \longleftrightarrow \pi_2$. If π_1, π_2 are tempered but $\hat{\pi}_1$, $\hat{\pi}_2$ are nontempered, we can simply check whether $\hat{\pi}_1 \longleftrightarrow \hat{\pi}_2$. Thus, the only problem is when π_1, π_2 and $\hat{\pi}_1$, $\hat{\pi}_2$ are all tempered. By [Jan3, Corollary 4.2], this forces $\beta = 0$ and

$$\pi_1 = \underbrace{\rho_1 \times \rho_1 \times \cdots \times \rho_1}_{n} \rtimes \sigma \quad \pi_2 = \underbrace{\rho_2 \times \rho_2 \times \cdots \times \rho_2}_{n} \rtimes \sigma_2,$$

noting that the hypothesis $(C1/2)$ forces both of these to be irreducible (by [Gol]). These should correspond under the isomorphism, finishing the inductive procedure. The same argument works for $(C\alpha)$ with $\alpha \neq 0$ or $(C0)$ when $\beta \neq 0$ (or in the case where $\rho_i \ncong \tilde{\rho}_i$). In the case where $\alpha = 0$ and $\beta = 0$, we have $\rho_i \rtimes \sigma_i = T_1(\rho_i; \sigma_i) \oplus T_2(\rho_i; \sigma_i)$. For isomorphism purposes, these cannot be distinguished, giving rise to two such isomorphisms.

While it is easy enough to describe the expected isomorphism, it is likely to be very difficult to show that it respects $\rtimes$ and μ^*. It is included here mainly for motivation; which we take up momentarily. As for evidence for this conjecture, we point to the fact that the conditions $(C\alpha)$, $\alpha = 0, \frac{1}{2}, \ldots$ generally seem to be enough to determine how induced representations supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ decompose, especially when Jacquet module methods are employed (*cf.* [Tad3], [Tad4], [Tad5], [Jan1], [Jan2]).

To see the significance of this conjecture to the problem at hand, suppose, *e.g.*, (ρ, σ) satisfies $(C1/2)$. Then, the conjecture gives a bijective correspondence between irreducible square-integrable representations supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ and those supported on $\mathcal{S}\big((\rho',\beta);\sigma'\big)$ for any other (ρ', σ') satisfying $(C1/2)$. Now, if we let ρ' be the trivial representation of $F^\times$ and σ' the trivial representation of $\mathrm{SO}_1(F)$, we have (ρ', σ') satisfying $(C1/2)$. In this case, Mœglin [Mœ1] has parameterized the irreducible square-integrable representations based on the results of Kazhdan-Lusztig [K-L]. Thus, we can expect an analogous parameterization for any pair (ρ, σ) satisfying $(C1/2)$.

4 Basic Results

4.1 A Basic Lemma for $S_n(F)$

Let ρ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_n(F)$, σ an irreducible supercuspidal representation of $S_r(F)$.

Definition 4.1.1 Let π be an irreducible representation supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$. Set

$$X(\pi) = \{\chi \leq s_{\min}(\pi) \mid \chi = \nu^{\alpha_1}\rho \otimes \cdots \otimes \nu^{\alpha_m}\rho \otimes \sigma \text{ has } \alpha_1 + \cdots + \alpha_m \text{ minimal for } s_{\min}(\pi)\}.$$

Then, let $\chi_0(\pi) \in X(\pi)$ which is minimal in the lexicographic ordering.

Lemma 4.1.2 *Assume $\beta \in \frac{1}{2}\mathbb{Z}$. Then, $\chi_0(\pi)$ has the form*

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k-1}\rho \otimes \cdots \otimes \nu^{b_k}\rho) \otimes \sigma,$$

with $a_1 \leq a_2 \leq \cdots \leq a_k$.

Proof Take $\tau \otimes \sigma \leq s_{\mathrm{GL}}(\pi)$ irreducible such that $\chi_0(\pi) \leq s_{\min}(\tau \otimes \sigma)$. Since $\beta \in \frac{1}{2}\mathbb{Z}$, we may apply Lemma 2.2.2 for τ to finish the proof. ■

Definition 4.1.3 With notation as above, if $\beta \in \frac{1}{2}\mathbb{Z}$ and

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k-1}\rho \otimes \cdots \otimes \nu^{b_k}\rho),$$

set

$$\delta_0(\pi) = \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \otimes \cdots \otimes \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \otimes \sigma.$$

Lemma 4.1.4 *Suppose $\chi_0(\pi)$, $\delta_0(\pi)$ as above, $\beta \in \frac{1}{2}\mathbb{Z}$. Let $M = \mathrm{GL}_{(a_1-b_1+1)n}(F) \times \cdots \times \mathrm{GL}_{(a_k-b_k+1)n}(F) \times S_r(F)$. Then,*

$$\pi \hookrightarrow i_{GM}\big(\delta_0(\pi)\big).$$

Proof The proof parallels that of Corollary 2.2.4. ■

4.2 A criterion for square-integrability

Theorem 4.2.1 *Suppose π is an irreducible representation with $\chi_0(\pi) = (\nu^{a_1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_m}\rho \otimes \cdots \otimes \nu^{b_m}\rho) \otimes \sigma$.*

1. *π is nontempered if and only if $a_i + b_i < 0$ for some i.*
2. *π is tempered but not square-integrable if and only if $a_i + b_i \geq 0$ for all i and equality occurs for at least one i.*
3. *π is square-integrable if and only if $a_i + b_i > 0$ for all i.*

Proof (3)($\Rightarrow$) Suppose not—say π is square-integrable but $a_i + b_i \leq 0$ for some i. Fix i to be the smallest value of i for which $a_i + b_i \leq 0$. Now, for $j < i$ we know that $a_j \leq a_i$ and $a_j + b_j > 0 \geq a_i + b_i$. Therefore, $b_j > b_i$. Thus,

$$\delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \cong \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_j}\rho, \nu^{a_j}\rho])$$

by irreducibility. Using these equivalences, we may commute $\delta([\nu^{b_i}\rho, \nu^{a_i}\rho])$ forward:

$$\begin{aligned}
\pi &\hookrightarrow \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \\
&\qquad \times \delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho]) \times \cdots \times \delta([\nu^{b_m}\rho, \nu^{a_m}\rho]) \rtimes \sigma \\
&\cong \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \\
&\qquad \times \delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho]) \times \cdots \times \delta([\nu^{b_m}\rho, \nu^{a_m}\rho]) \rtimes \sigma \\
&\ \vdots \\
&\cong \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{i-1}}\rho, \nu^{a_{i-1}}\rho]) \\
&\qquad \times \delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho]) \times \cdots \times \delta([\nu^{b_m}\rho, \nu^{a_m}\rho]) \rtimes \sigma.
\end{aligned}$$

Therefore, by Frobenius reciprocity,

$$s_{\min}\pi \geq (\nu^{a_i}\rho \otimes \cdots \otimes \nu^{b_i}\rho) \otimes (\nu^{a_1}\rho \otimes \cdots \otimes \nu^{b_i}\rho) \otimes \cdots .$$

However, since $a_i + b_i \leq 0$, this violates the Casselman criterion for square-integrability. Thus, π square-integrable implies $a_i + b_i > 0$ for all i.

(2)($\Rightarrow$) The proof that π tempered implies $a_i + b_i \geq 0$ for all i is essentially the same as that used in (3)($\Rightarrow$) above.

We now argue that if π is not square-integrable, then $a_i + b_i = 0$ for some i. If π is tempered but not square-integrable, we have

$$\pi \hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^{\alpha_1}\rho]) \times \cdots \times \delta([\nu^{-\alpha_k}\rho, \nu^{\alpha_k}\rho]) \rtimes \delta$$

for some square-integrable δ. We have $k \geq 1$. For convenience, we will use $\delta([\nu^{\beta_i}\rho, \nu^{\alpha_i}\rho])$ and $\delta([\nu^{-\alpha_i}\rho, \nu^{\alpha_i}\rho])$ interchangeably when $i \leq k$. Write $\delta_0(\delta) = \delta([\nu^{\beta_{k+1}}\rho, \nu^{\alpha_{k+1}}\rho]) \otimes \cdots \otimes \delta([\nu^{\beta_\ell}\rho, \nu^{\alpha_\ell}\rho])$. Note that

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^{\alpha_1}\rho]) \times \cdots \times \delta([\nu^{-\alpha_k}\rho, \nu^{\alpha_k}\rho]) \rtimes \delta \\ &\hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^{\alpha_1}\rho]) \times \cdots \times \delta([\nu^{-\alpha_k}\rho, \nu^{\alpha_k}\rho]) \\ &\qquad \times \delta([\nu^{\beta_{k+1}}\rho, \nu^{\alpha_{k+1}}\rho]) \times \cdots \times \delta([\nu^{\beta_\ell}\rho, \nu^{\alpha_\ell}\rho]) \rtimes \sigma.\end{aligned}$$

Let $\delta([\nu^{\beta_1'}\rho, \nu^{\alpha_1'}\rho]), \ldots, \delta([\nu^{\beta_\ell'}\rho, \nu^{\alpha_\ell'}\rho])$ be the permutation of $\delta([\nu^{\beta_1}\rho, \nu^{\alpha_1}\rho]), \ldots, \delta([\nu^{\beta_\ell}\rho, \nu^{\alpha_\ell}\rho])$ satisfying

1. $\alpha_1' \leq \alpha_2' \leq \cdots \leq \alpha_\ell'$
2. if $\alpha_i' = \alpha_{i+1}'$, then $\beta_{i+1}' \geq \beta_i'$.

We claim that

$$\delta([\nu^{\beta_1}\rho, \nu^{\alpha_1}\rho]) \times \cdots \times \delta([\nu^{\beta_\ell}\rho, \nu^{\alpha_\ell}\rho]) \cong \delta([\nu^{\beta_1'}\rho, \nu^{\alpha_1'}\rho]) \times \cdots \times \delta([\nu^{\beta_\ell'}\rho, \nu^{\alpha_\ell'}\rho]).$$

Since the proof of this claim is very similar to the proof of Proposition 2.4.1, we omit the details. (To apply that argument here, one also needs the following observation: By (1)($\Rightarrow$) above, $\beta_j + \alpha_j > 0$ for $j > k$.) As a consequence,

$$\pi \hookrightarrow \delta([\nu^{\beta_1'}\rho, \nu^{\alpha_1'}\rho]) \times \cdots \times \delta([\nu^{\beta_\ell'}\rho, \nu^{\alpha_\ell'}\rho]) \rtimes \sigma.$$

Therefore, by Frobenius reciprocity we see that

$$s_{\min}(\pi) \geq (\nu^{\alpha_1'}\rho \otimes \cdots \otimes \nu^{\beta_1'}\rho) \otimes \cdots \otimes (\nu^{\alpha_\ell'}\rho \otimes \cdots \otimes \nu^{\beta_\ell'}\rho) \otimes \sigma.$$

We shall show that this is, in fact, $\chi_0(\pi)$.

We now check that

$$\chi_0(\pi) = (\nu^{\alpha_1'}\rho \otimes \cdots \otimes \nu^{\beta_1'}\rho) \otimes \cdots \otimes (\nu^{\alpha_\ell'}\rho \otimes \cdots \otimes \nu^{\beta_\ell'}\rho) \otimes \sigma.$$

In fact, we show more—we check that

$$\begin{aligned}&\chi_0\big(\delta([\nu^{-\alpha_1}\rho, \nu^{\alpha_1}\rho]) \times \cdots \times \delta([\nu^{-\alpha_k}\rho, \nu^{\alpha_k}\rho]) \rtimes \delta\big) \\ &\qquad = (\nu^{\alpha_1'}\rho \otimes \cdots \otimes \nu^{\beta_1'}\rho) \otimes \cdots \otimes (\nu^{\alpha_\ell'}\rho \otimes \cdots \otimes \nu^{\beta_\ell'}\rho) \otimes \sigma.\end{aligned}$$

To do this, consider $\tau\otimes\sigma \leq s_{\mathrm{GL}}\big(\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\rtimes\delta\big)$ with τ irreducible and $\chi_0\big(\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\rtimes\delta\big)\leq s_{\min}(\tau\otimes\sigma)$. To calculate this, we write

$$M^*_{\mathrm{GL}}\big(\delta([\nu^{\beta}\rho,\nu^{\alpha}\rho])\big)=\sum_{i=\beta}^{\alpha+1}\delta([\nu^{-i+1}\rho,\nu^{-\beta}\rho])\times\delta([\nu^{i}\rho,\nu^{\alpha_1}\rho])$$

as in [Tad5]. It follows from Theorem 3.2 that

$$\begin{aligned}&s_{\mathrm{GL}}\big(\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\rtimes\delta\big)\\&\quad=M^*_{\mathrm{GL}}\big(\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\big)\times\cdots\times M^*_{\mathrm{GL}}\big(\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\big)\times s^0_{\mathrm{GL}}(\delta)\otimes\sigma,\end{aligned}$$

where we use $s^0_{\mathrm{GL}}(\delta)$ to denote that part of $s_{\mathrm{GL}}(\delta)$ attached to the general linear group (*i.e.*, $s_{\mathrm{GL}}(\delta) = s^0_{\mathrm{GL}}(\delta)\otimes\sigma$). For $\delta([\nu^{-\alpha_j}\rho,\nu^{\alpha_j}\rho])$ with $j\leq k$, the $i=-\alpha_j,\alpha_j+1$ terms in $M^*_{\mathrm{GL}}\big(\delta([\nu^{\beta_j}\rho,\nu^{\alpha_j}\rho])\big)$ (which both give rise to a copy of $\delta([\nu^{-\alpha_j}\rho,\nu^{\alpha_j}\rho])$) minimize $[(-i+1)+(-i+2)+\cdots+\alpha_j]+[i+(i+1)+\cdots+\alpha_j]$. Thus, $\tau\leq\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\times s^0_{\mathrm{GL}}(\delta)$. If we let

$$s^{X(\delta)}_{\mathrm{GL}}(\delta)=\{\tau'\leq s^0_{\mathrm{GL}}(\delta)\mid \tau' \text{ irreducible and } s_{\min}(\tau'\otimes\sigma)\leq X(\delta)\},$$

we must clearly have $\tau\leq\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\times s^{X(\delta)}_{\mathrm{GL}}(\delta)$. Now, by Lemma 2.4.2 (writing $\chi_0(\delta)=\chi^0_0(\delta)\otimes\sigma$ as above)

$$\begin{aligned}&\chi_0\big(\delta([\nu^{-\alpha_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{-\alpha_k}\rho,\nu^{\alpha_k}\rho])\times s^{X(\delta)}_{\mathrm{GL}}(\delta)\otimes\sigma\big)\\&\quad=\mathrm{m.l.s.}\big((\nu^{\alpha_1}\rho\otimes\cdots\otimes\nu^{-\alpha_1}\rho),\ldots,(\nu^{\alpha_k}\rho\otimes\cdots\otimes\nu^{-\alpha_k}\rho),\chi^0_0(\delta)\big)\otimes\sigma\\&\quad=(\nu^{\alpha'_1}\rho\otimes\cdots\otimes\nu^{\beta'_1}\rho)\otimes\cdots\otimes(\nu^{\alpha'_\ell}\rho\otimes\cdots\otimes\nu^{\beta'_\ell}\rho)\otimes\sigma,\end{aligned}$$

by the construction of $(\nu^{\alpha'_1}\rho\otimes\cdots\otimes\nu^{\beta'_1}\rho)\otimes\cdots\otimes(\nu^{\alpha'_\ell}\rho\otimes\cdots\otimes\nu^{\beta'_\ell}\rho)\otimes\sigma$, as needed.

Since π is assumed not to be square-integrable, we have $k\geq1$ and $-\alpha_1+\alpha_1=0$. Therefore, $\beta'_i+\alpha'_i=0$ for some i. This finishes the case (2)($\Rightarrow$).

(1)($\Rightarrow$) Write $\pi=L\big(\delta([\nu^{\beta_1}\rho,\nu^{\alpha_1}\rho]),\ldots,\delta([\nu^{\beta_k}\rho,\nu^{\alpha_k}\rho]);T\big)$. Write

$$\delta_0(T)=\delta([\nu^{\beta_{k+1}}\rho,\nu^{\alpha_{k+1}}\rho])\otimes\cdots\otimes\delta([\nu^{\beta_\ell}\rho,\nu^{\alpha_\ell}\rho])\otimes\sigma.$$

Then,

$$\begin{aligned}\pi\hookrightarrow\ &\delta([\nu^{\beta_1}\rho,\nu^{\alpha_1}\rho])\times\cdots\times\delta([\nu^{\beta_k}\rho,\nu^{\alpha_k}\rho])\\&\times\delta([\nu^{\beta_{k+1}}\rho,\nu^{\alpha_{k+1}}\rho])\times\cdots\times\delta([\nu^{\beta_\ell}\rho,\nu^{\alpha_\ell}\rho])\rtimes\sigma.\end{aligned}$$

We can now argue as we did to show $a_i+b_i=0$ for some i in (2)($\Rightarrow$).

The converse directions now follow immediately. ■

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4.3 Supports for Square-Integrable Representations

Proposition 4.3.1 *Suppose ρ is an irreducible unitary supercuspidal representation of* $\mathrm{GL}_n(F)$ *and σ an irreducible supercuspidal representation of $S_r(F)$. Let $\mathcal{S}\big((\rho,\beta);\sigma\big)$ be as in Section 3.*

1. *If $\rho \not\cong \tilde{\rho}$, there are no irreducible square-integrable representation supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ for any β.*
2. *Suppose $\rho \cong \tilde{\rho}$ and (ρ,σ) satisfies $(C\alpha)$. Then, there are irreducible square-integrable representations supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ if and only if $\beta \equiv \alpha \mod 1$.*

Proof Claim 1 follows from [Tad4, Theorem 6.2]. In the case where $\beta \notin \frac{1}{2}\mathbb{Z}$, the second claim also follows from [Tad4, Theorem 6.2].

We consider 2. when $\beta \in \frac{1}{2}\mathbb{Z}$. First, assume that (ρ,σ) satisfies $(C\alpha)$ and $\beta \not\equiv \alpha \mod 1$. Now, suppose there were an irreducible square-integrable representation π supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$. Write

$$\begin{aligned}\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes (\nu^{a_2}\rho \otimes \nu^{a_2-1}\rho \otimes \cdots \otimes \nu^{b_2}\rho)\\ \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k-1}\rho \otimes \cdots \otimes \nu^{b_k}\rho) \otimes \sigma,\end{aligned}$$

noting that $a_i, b_i \equiv \beta \equiv -\beta \mod 1$. By Lemma 4.1.4, we have

$$\pi \hookrightarrow \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{k-1}}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma.$$

Observe that, by [Tad3, Theorem 13.2], $\delta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \rtimes \sigma \cong \delta([\nu^{-a_k}\rho, \nu^{-b_k}\rho]) \rtimes \sigma$ is irreducible. Thus,

$$\pi \hookrightarrow \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{k-1}}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{-a_k}\rho, \nu^{-b_k}\rho]) \rtimes \sigma.$$

Therefore, by Frobenius reciprocity,

$$\begin{aligned}s_{\min}(\pi) \geq (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_{k-1}}\rho \otimes \nu^{a_{k-1}-1}\rho \otimes \nu^{b_{k-1}}\rho)\\ \otimes (\nu^{-b_k}\rho \otimes \nu^{-b_k-1}\rho \otimes \cdots \otimes \nu^{-a_k}\rho) \otimes \sigma.\end{aligned}$$

However, it follows from Theorem 4.2.1 that we have $-b_k + (-b_k - 1) + \cdots + (-a_k) < 0 < a_k + (a_k - 1) + \cdots + b_k$, contradicting the construction of $\chi_0(\pi)$. Thus, there can be no irreducible square-integrable representation supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$, as claimed.

The converse direction of 2. follows immediately from the fact that $\nu^{\alpha} \rtimes \sigma$ has a square-integrable subrepresentation. ■

We note that in the case where σ is generic, the above proposition follows from [Mu].

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4.4 Some Constraints on $\chi_0(\pi)$

Lemma 4.4.1 *Suppose (ρ,σ) satisfies $C(\alpha)$ and π is an irreducible representation supported on $\mathcal{S}\big((\rho,\beta);\sigma\big)$ with $\beta \equiv \alpha \mod 1$. If*

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \cdots \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \cdots \otimes \nu^{b_k}\rho) \otimes \sigma,$$

then for each $1 \leq i \leq k$, we have $b_i \leq \alpha$. Further, if $\alpha > 0$, there is at most one i for which $b_i = \alpha$.

Proof First, we define a representation we will need in the proof. If $\alpha \geq 1$, let $\theta_\alpha = \nu^\alpha\rho \rtimes \delta(\nu^\alpha\rho;\sigma)$, which is irreducible (*cf.* [Tad3]). If $\alpha = \frac{1}{2}$, let θ_α denote the irreducible subquotient common to $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho;\sigma)$ and $\delta([\nu^{-\frac{1}{2}}\rho,\nu^{\frac{1}{2}}\rho]) \rtimes \sigma$ (θ_α is needed only for $\alpha > 0$). We observe that θ_α is the only irreducible representation containing $\nu^\alpha\rho \otimes \nu^\alpha\rho \otimes \sigma$ in its minimal Jacquet module. Further, $s_{\min}(\theta_\alpha) \geq \nu^\alpha\rho \otimes \nu^{-\alpha}\rho \otimes \sigma$.

Let i be the largest value for which $b_i > 0$. Then, $[\nu^{b_i}\rho,\nu^{a_i}\rho] \subset [\nu^{b_j}\rho,\nu^{a_j}\rho]$ for all $j > i$. Therefore, commuting arguments give

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{b_1}\rho,\nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{i-1}}\rho,\nu^{a_{i-1}}\rho]) \times \delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\\ &\quad \times \delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho,\nu^{a_k}\rho]) \rtimes \sigma\\ &\cong \delta([\nu^{b_1}\rho,\nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_{i-1}}\rho,\nu^{a_{i-1}}\rho]) \times \delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\\ &\quad \times \cdots \times \delta([\nu^{b_k}\rho,\nu^{a_k}\rho]) \times \delta([\nu^{b_i}\rho,\nu^{a_i}\rho]) \rtimes \sigma.\end{aligned}$$

If $b_i \neq \alpha$, we get

$$\begin{aligned}\pi &\hookrightarrow (\nu^{a_1}\rho \times \cdots \times \nu^{b_1}\rho) \times \cdots \times (\nu^{a_{i-1}}\rho \times \cdots \times \nu^{b_{i-1}}\rho) \times (\nu^{a_{i+1}}\rho \times \cdots \times \nu^{b_{i+1}}\rho)\\ &\quad \times \cdots \times (\nu^{a_k}\rho \times \cdots \times \nu^{b_k}\rho) \times (\nu^{a_i}\rho \times \cdots \times \nu^{b_i+1}\rho \times \nu^{b_i}\rho) \rtimes \sigma\\ &\cong (\nu^{a_1}\rho \times \cdots \times \nu^{b_1}\rho) \times \cdots \times (\nu^{a_{i-1}}\rho \times \cdots \times \nu^{b_{i-1}}\rho) \times (\nu^{a_{i+1}}\rho \times \cdots \times \nu^{b_{i+1}}\rho)\\ &\quad \times \cdots \times (\nu^{a_k}\rho \times \cdots \times \nu^{b_k}\rho) \times (\nu^{a_i}\rho \times \cdots \times \nu^{b_i+1}\rho \times \nu^{-b_i}\rho) \rtimes \sigma\end{aligned}$$

since $\nu^{b_i}\rho \rtimes \sigma \cong \nu^{-b_i}\rho \rtimes \sigma$ is irreducible. However, by Frobenius reciprocity, this contradicts the minimality of $\chi_0(\pi)$ above. Thus, $b_i = \alpha$.

Now, suppose $b_i = \alpha$ and $j < i$ is the largest value for which $b_j > 0$. Then, a commuting argument gives

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{b_1}\rho,\nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho,\nu^{a_k}\rho]) \rtimes \sigma\\ &\cong \delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{k-2}}\rho,\nu^{a'_{k-2}}\rho])\\ &\quad \times \delta([\nu^{b_j}\rho,\nu^{a_j}\rho]) \times \delta([\nu^{b_i}\rho,\nu^{a_i}\rho]) \rtimes \sigma,\end{aligned}$$

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where $a'_m = a_m$ if $m < j$, a_{m+1} if $j \leq m < i$, a_{m+2} if $i \leq m$, and similarly for b'_m. If $b_j > \alpha = b_i$, we can use a commuting argument to get

$$\begin{aligned}\pi \hookrightarrow \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{k-2}}\rho, \nu^{a'_{k-2}}\rho])\\ \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \rtimes \sigma\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\pi &\hookrightarrow (\nu^{a'_1}\rho \times \cdots \times \nu^{b'_1}\rho) \times \cdots \times (\nu^{a'_{k-2}}\rho \times \cdots \times \nu^{b'_{k-2}}\rho)\\ &\qquad \times (\nu^{a_i}\rho \times \cdots \times \nu^{b_i}\rho) \times (\nu^{a_j}\rho \times \cdots \times \nu^{b_j+1}\rho \times \nu^{b_j}\rho) \rtimes \sigma\\ &\hookrightarrow (\nu^{a'_1}\rho \times \cdots \times \nu^{b'_1}\rho) \times \cdots \times (\nu^{a'_{k-2}}\rho \times \cdots \times \nu^{b'_{k-2}}\rho)\\ &\qquad \times (\nu^{a_i}\rho \times \cdots \times \nu^{b_i}\rho) \times (\nu^{a_j}\rho \times \cdots \times \nu^{b_j+1}\rho \times \nu^{-b_j}\rho) \rtimes \sigma\end{aligned}$$

since $\nu^{b_j}\rho \rtimes \sigma \cong \nu^{-b_j}\rho \rtimes \sigma$ is irreducible. This contradicts the minimality of $\chi_0(\pi)$ above. If $b_j = \alpha = b_i$, observe that $\delta([\nu^\alpha\rho, \nu^{a_i}\rho]) \times \delta([\nu^\alpha\rho, \nu^{a_j}\rho]) \hookrightarrow \nu^{a_i}\rho \times \cdots \times \nu^{a_j+1}\rho \times (\nu^{a_j}\rho \times \nu^{a_j}\rho) \times \cdots \times (\nu^\alpha\rho \times \nu^\alpha\rho)$. Thus, arguing as above, we get

$$\begin{aligned}\pi \hookrightarrow (\nu^{a'_1}\rho \times \cdots \times \nu^{b'_1}\rho) \times \cdots \times (\nu^{a'_{k-2}}\rho \times \cdots \times \nu^{b'_{k-2}}\rho)\\ \times (\nu^{a_i}\rho \times \cdots \times \nu^{a_j+1}\rho) \times \big((\nu^{a_j}\rho \times \nu^{a_j}\rho) \times \cdots \times (\nu^\alpha\rho \times \nu^\alpha\rho)\big) \rtimes \sigma\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}s_{(p,\dots,p)}(\pi) \geq (\nu^{a'_1}\rho \otimes \cdots \otimes \nu^{b'_1}\rho) \otimes \cdots \otimes (\nu^{a'_{k-2}}\rho \otimes \cdots \otimes \nu^{b'_{k-2}}\rho)\\ \otimes (\nu^{a_i}\rho \otimes \cdots \otimes \nu^{a_j+1}\rho) \otimes \big((\nu^{a_j}\rho \otimes \nu^{a_j}\rho) \otimes \cdots \otimes (\nu^{\alpha+1}\rho \otimes \nu^{\alpha+1}\rho)\big) \otimes \theta_\alpha\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}s_{\min}(\pi) \geq (\nu^{a'_1}\rho \otimes \cdots \otimes \nu^{b'_1}\rho) \otimes \cdots \otimes (\nu^{a'_{k-2}}\rho \otimes \cdots \otimes \nu^{b'_{k-2}}\rho) \otimes (\nu^{a_i}\rho \otimes \cdots \otimes \nu^{a_j+1}\rho)\\ \otimes \big((\nu^{a_j}\rho \otimes \nu^{a_j}\rho) \otimes \cdots \otimes (\nu^{\alpha+1}\rho \otimes \nu^{\alpha+1}\rho)\big) \otimes (\nu^\alpha\rho \otimes \nu^{-\alpha}\rho) \otimes \sigma,\end{aligned}$$

again contradicting the minimality of $\chi_0(\pi)$ above. This finishes the proof. ■

The following refinement is of interest when $\alpha \geq 1$.

Lemma 4.4.2 *Suppose (ρ, σ) satisfies $C(\alpha)$ and π is an irreducible representation supported on $\mathcal{S}\big((\rho, \alpha); \sigma\big)$. If (now using $-b_i$ for lower ends)*

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \cdots \otimes \nu^{-b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \cdots \otimes \nu^{-b_k}\rho) \otimes \sigma,$$

then there is a β with $\alpha + 1 \geq \beta > 0$ such that each of $\{-\beta, -\beta - 1, \dots, -\alpha\}$ appears exactly once among $b_1, b_2, \dots, b_k$ and there are no other negative b_i's.

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Proof Let $[-d_1, c_1], \dots, [-d_k, c_k]$ be the permutation of $[-b_1, a_1], \dots, [-b_k, a_k]$ having $d_1 \geq \cdots \geq d_k$ (and if $d_i = d_{i+1}$, then $c_i \leq c_{i+1}$). Then, we claim that

$$\pi \hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \rtimes \sigma.$$

To see this, observe that if $a_i \geq a_j$ and $b_i \leq b_j$, then $\delta([\nu^{-b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{-b_j}\rho, \nu^{a_j}\rho]) \cong \delta([\nu^{-b_j}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{-b_i}\rho, \nu^{a_i}\rho])$ (by irreducibility). One can get from $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{-b_k}\rho, \nu^{a_k}\rho])$ to $\delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho])$ through a sequence of such transpositions, hence equivalences are preserved. Therefore,

$$\pi \hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \rtimes \sigma,$$

as claimed.

If $d_k \geq 0$, we are done: $\beta = \alpha + 1$. So, suppose $d_k < 0$. By Lemma 4.4.1, $d_k \geq -\alpha$. Then, we need to check that $d_k = -\alpha$. If $d_k > -\alpha$, we would have

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{k-1}}\rho, \nu^{c_{k-1}}\rho]) \\ &\quad \times \delta([\nu^{-d_k+1}\rho, \nu^{c_k}\rho]) \times \nu^{-d_k}\rho \rtimes \sigma \\ &\cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{k-1}}\rho, \nu^{c_{k-1}}\rho]) \\ &\quad \times \delta([\nu^{-d_k+1}\rho, \nu^{c_k}\rho]) \times \nu^{d_k}\rho \rtimes \sigma,\end{aligned}$$

which, by Frobenius reciprocity, contradicts the minimality of $\delta_0(\pi)$ (switching signs on $-d_k$ lowers the exponent total). Thus, if $d_k < 0$, we have $d_k = -\alpha$.

Now, consider d_{k-1}. If $d_{k-1} \geq 0$, we are done: $\beta = \alpha$. So, suppose $d_{k-1} < 0$. We have $d_{k-1} \geq d_k = -\alpha$. Further, by Lemma 4.4.1, we cannot have $d_{k-1} = d_k = -\alpha$, so $d_{k-1} > -\alpha$. We need to check that $d_{k-1} = -\alpha + 1$. Suppose this were not the case. Then,

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{k-2}}\rho, \nu^{c_{k-2}}\rho]) \\ &\quad \times \delta([\nu^{-d_{k-1}+1}\rho, \nu^{c_{k-1}}\rho]) \times \nu^{-d_{k-1}}\rho \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{k-2}}\rho, \nu^{c_{k-2}}\rho]) \\ &\quad \times \delta([\nu^{-d_{k-1}+1}\rho, \nu^{c_{k-1}}\rho]) \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \times \nu^{-d_{k-1}}\rho \rtimes \sigma \\ &\cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{k-2}}\rho, \nu^{c_{k-2}}\rho]) \\ &\quad \times \delta([\nu^{-d_{k-1}+1}\rho, \nu^{c_{k-1}}\rho]) \times \delta([\nu^{-d_k}\rho, \nu^{c_k}\rho]) \times \nu^{d_{k-1}}\rho \rtimes \sigma\end{aligned}$$

by the irreducibility of $\nu^{-d_{k-1}}\rho \rtimes \sigma$. Again, by Frobenius reciprocity, this contradicts the minimality of $\delta_0(\pi)$.

Finally, suppose we have $d_i = d_{i+1} + 1$ for all $i < j$ with $j \leq k-2$. If $d_j \geq 0$, we are done: $\beta = -d_{j+1}$. So, suppose $d_j < 0$. An argument similar to that in the preceding paragraph shows that we cannot have $d_j > d_{j+1} + 1$. Thus, it remains to show that we cannot have

$d_j = d_{j+1}$. Suppose that were the case—say $d_j = d_{j+1} = \gamma$. Then,

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ &\quad \times \nu^{-\gamma}\rho \times \delta([\nu^{-\gamma}\rho, \nu^{c_{j+1}}\rho]) \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ &\quad \times \delta([\nu^{-\gamma}\rho, \nu^{c_{j+1}}\rho]) \times \nu^{-\gamma}\rho \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma \\ &\hookrightarrow \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ &\quad \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \times \nu^{-\gamma}\rho \times \nu^{-\gamma}\rho \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \times \nu^{-\gamma+1}\rho \\ &\quad \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ &\quad \times \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \times \nu^{-\gamma}\rho \times \nu^{-\gamma}\rho \times \nu^{-\gamma+1}\rho \\ &\quad \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \rtimes \sigma.\end{aligned}$$

Now, the only irreducible representation of $GL_{3n}(F)$ containing $\nu^{-\gamma}\rho \otimes \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho$ in its $r_{\min}$ is $\nu^{-\gamma}\rho \times \zeta([\nu^{-\gamma}\rho, \nu^{-\gamma+1}\rho])$. Since

$$r_{\min}\big(\nu^{-\gamma}\rho \times \zeta([\nu^{-\gamma}\rho, \nu^{-\gamma+1}\rho])\big) \geq \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho \otimes \nu^{-\gamma}\rho,$$

we see that

$$\begin{aligned}s_{\mathrm{app}}(\pi) \geq\ & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ & \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \otimes \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho \\ & \otimes \nu^{-\gamma}\rho \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \otimes \sigma,\end{aligned}$$

where s_{app} denotes the Jacquet module taken with respect to the appropriate parabolic subgroup. Now, the only irreducible representation of $GL_{n(c_{j+3}+\gamma)}(F)$ containing $\nu^{-\gamma}\rho \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho])$ in its m^* is $\nu^{-\gamma}\rho \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho])$. However,

$$m^*\big(\nu^{-\gamma}\rho \times \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho])\big) \geq \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \otimes \nu^{-\gamma}\rho.$$

Thus,

$$\begin{aligned}s_{\mathrm{app}}(\pi) \geq\ & \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ & \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \otimes \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho \\ & \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \otimes \nu^{-\gamma}\rho \otimes \delta([\nu^{-\gamma+3}\rho, \nu^{c_{j+4}}\rho]) \\ & \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \otimes \sigma.\end{aligned}$$

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Iterating this argument, we can eventually commute $\nu^{-\gamma}\rho$ to the right end to get

$$s_{\mathrm{app}}(\pi) \geq \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \otimes \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho \\ \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \otimes \nu^{-\gamma}\rho \otimes \sigma.$$

Finally, we observe that the only irreducible representation of $S_{n+r}(F)$ containing $\nu^{-\gamma}\rho \otimes \sigma$ in its $s_{\min}$ is $\nu^{-\gamma}\rho \rtimes \sigma$. However,

$$s_{\min}(\nu^{-\gamma}\rho \rtimes \sigma) \geq \nu^{\gamma}\rho \otimes \sigma.$$

Therefore,

$$s_{\mathrm{app}}(\pi) \geq \delta([\nu^{-d_1}\rho, \nu^{c_1}\rho]) \otimes \cdots \otimes \delta([\nu^{-d_{j-1}}\rho, \nu^{c_{j-1}}\rho]) \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_j}\rho]) \\ \otimes \delta([\nu^{-\gamma+1}\rho, \nu^{c_{j+1}}\rho]) \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+2}}\rho]) \otimes \nu^{-\gamma}\rho \otimes \nu^{-\gamma+1}\rho \\ \otimes \delta([\nu^{-\gamma+2}\rho, \nu^{c_{j+3}}\rho]) \otimes \cdots \otimes \delta([\nu^{\alpha}\rho, \nu^{c_k}\rho]) \otimes \nu^{\gamma}\rho \otimes \sigma,$$

contradicting the minimality of $\delta_0(\pi)$ (by total exponent considerations). This finishes the proof. ■

Remark 4.4.3 It is an easy consequence of the arguments above that if $i < j$ with both $b_i, b_j < 0$, then $b_i > b_j$.

5 Example: the Two-Segment, ($C\frac{1}{2}$) Case

In this chapter, we give an example to show how our results can be applied. Throughout this chapter, we assume (ρ, σ) satisfies ($C\frac{1}{2}$). We then use the results of the preceding sections to help classify the square-integrable representations supported on $\mathcal{S}\big((\rho, \frac{1}{2}); \sigma\big)$ in the case where δ_0 consists of two generalized Steinbergs; $k = 2$ in the notation of Definition 4.1.3. In light of Lemma 4.4.1, we may write δ_0 for such a representation in the form $\delta([\nu^{-b_1}\rho, \nu^{a_1}\rho]) \otimes \delta([\nu^{-b_2}\rho, \nu^{a_2}\rho]) \otimes \sigma$ with $b_1, b_2 \geq -\frac{1}{2}$; for convenience, we do so.

5.1 The One-Segment Case

In this section, we discuss the square-integrable representations with δ_0 consisting of one generalized Steinberg representation (*i.e.*, $k = 1$ in the notation of Definition 4.1.3). This discussion is based on results from [Tad5].

Lemma 5.1.1 *Suppose $b \in \frac{1}{2} + \mathbb{Z}$ with $b \geq \frac{1}{2}$. Then, $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma$ decomposes as a direct sum of two irreducible representations (both tempered, neither square-integrable). We write*

$$\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma \cong \delta([\nu^{-b}\rho, \nu^{b}\rho]; \sigma)_1 \oplus \delta([\nu^{-b}\rho, \nu^{b}\rho]; \sigma)_2.$$

Proof See [Tad5, Theorem 3.2]. ■

By convention, we let $\delta([\nu^{-b}\rho, \nu^b\rho]; \sigma)_1$ denote the component with the larger Jacquet module.

By Theorem 4.2.1 and Lemma 4.4.1, a square-integrable representation supported on a single segment appears as a subrepresentation of $\delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ for some $a, b \in \frac{1}{2} + \mathbb{Z}$ with $a > b \geq -\frac{1}{2}$. (Alternatively, we could use a more direct argument for the one-segment case to reduce to just considering the above induced representations; *cf.* [Tad5, Proposition 4.4]).

Theorem 5.1.2 *Suppose $a, b \in \frac{1}{2} + \mathbb{Z}$ with $a > b \geq -\frac{1}{2}$.*

1. *If $b = -\frac{1}{2}$, we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma) + L\big(\delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma\big).$$

The unique irreducible quotient (Langlands quotient) is $L\big(\delta([\nu^{-a}\rho, \nu^{\frac{1}{2}}\rho]); \sigma\big)$; the unique irreducible subrepresentation is $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma)$. Further, $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma)$ is square-integrable.

2. *If $b > -\frac{1}{2}$, we have*

$$\delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma = \delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_1 + \delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_2 + L\big(\delta([\nu^{-a}\rho, \nu^b\rho]); \sigma\big).$$

The unique irreducible quotient (Langlands quotient) is $L\big(\delta([\nu^{-a}\rho, \nu^b\rho]); \sigma\big)$; $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_1$ and $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_2$ are subrepresentations. Further, $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_1$ and $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_2$ are square-integrable. We note that $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_1$ (resp., $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_2$) may be characterized as the unique irreducible subquotient common to $\delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ and $\delta([\nu^{b+1}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^b\rho]; \sigma)_1$ (resp., $\delta([\nu^{b+1}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^b\rho]; \sigma)_2$).

We note that $\delta_0\big(\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t\big) = \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ (by convention, if $b = -\frac{1}{2}$, we take $t = 1$ and write $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma)_1 = \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma)$).

Proof For 1., the properties of $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]; \sigma)$ are given in [Tad5, Theorem 2.1]. That $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ has two irreducible subquotients follows from [Jan1, Proposition 3.6] and [Aub] or [S-S]. The identity of the other irreducible subquotient follows immediately from $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma$ (*cf.* [BDK, Lemma 5.4]).

For 2., the properties of $\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t$ are given in Theorems 3.3, 4.2, and 4.3 of [Tad5]. That $\delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ has three irreducible subquotients follows from [Jan1, Proposition 3.6] and [Aub] or [S-S]. The identity of the other irreducible subquotient follows immediately from $\delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma = \delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$.

Finally, the fact that $\delta_0\big(\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t\big) = \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ follows from Theorems 2.1, 3.3(ii), and 4.2(iv) of [Tad5]. ■

We will need the following lemma in the next section.

Lemma 5.1.3 *Suppose $a, b, f, g \in \frac{1}{2} + \mathbb{Z}$.*

1. *Suppose $a > f \geq b \geq -\frac{1}{2}$. Then,*

$$\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t \hookrightarrow \delta([\nu^{f+1}\rho, \nu^{a}\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^{f}\rho]; \sigma)_t$$

and is the unique irreducible subrepresentation.

2. *Suppose $a > b > g > -\frac{1}{2}$. Then,*

$$\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t \hookrightarrow \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$$

and is the unique irreducible subrepresentation.

Proof We start with 2. First, we check that $\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$ appears with multiplicity one in $s_{((b-g)n)}\big(\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t\big)$. Write $\mu^*\big(\delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t\big) = \sum_k \tau_k \otimes \theta_k$ with τ_k, θ_k irreducible (repetition possible). We note that since

$$\begin{aligned} s_{(n)}\big(\delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t\big) &\leq s_{(n)}\big(\delta([\nu^{-g}\rho, \nu^{a}\rho]) \rtimes \sigma\big) \\ &= \nu^{g}\rho \otimes \delta([\nu^{-g+1}\rho, \nu^{a}\rho]) \rtimes \sigma + \nu^{a}\rho \otimes \delta([\nu^{-g}\rho, \nu^{a-1}\rho]) \rtimes \sigma, \end{aligned}$$

we have that an element of $r_{\min}(\tau_k)$ begins with either $\nu^{g}\rho$ or $\nu^{a}\rho$ unless $\tau_k = 1$. Now, by Theorem 3.1.2,

$$\begin{aligned} &\mu^*\big(\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t\big) \\ &\quad = \sum_k \sum_{i=g+1}^{b+1} \sum_{j=i}^{b+1} \delta([\nu^{-i+1}\rho, \nu^{-g-1}\rho]) \times \delta([\nu^{j}\rho, \nu^{b}\rho]) \times \tau_k \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \theta_k. \end{aligned}$$

To obtain a copy of $\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$, we must have $\tau_k = 1$ (since $r_{\min}\big(\delta([\nu^{g+1}\rho, \nu^{b}\rho])\big)$ contains neither $\nu^{g}\rho$ nor $\nu^{a}\rho$), hence $\theta_k = \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$. Further, since $r_{\min}\big(\delta([\nu^{g+1}\rho, \nu^{b}\rho])\big)$ does not contain $\nu^{-g-1}\rho$, we must have $i = g + 1$. Then, $j = g + 1$ gives the only copy of $\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$. Note that by Frobenius reciprocity, we immediately see that $\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_t$ has a unique irreducible subrepresentation.

Calculate:

$$\begin{aligned} \delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t &\hookrightarrow \delta([\nu^{-b}\rho, \nu^{a}\rho]) \rtimes \sigma \\ &\hookrightarrow \delta([\nu^{-g}\rho, \nu^{a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-g-1}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-g}\rho, \nu^{a}\rho]) \times \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-g}\rho, \nu^{a}\rho]) \rtimes \sigma, \end{aligned}$$

where $\delta([\nu^{-b}\rho, \nu^{-g-1}\rho]) \rtimes \sigma \cong \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \sigma$ is irreducible by [Tad3, Theorem 9.1]. By [Jan3, Lemma 5.5], we have $\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t \hookrightarrow \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \theta$ for some irreducible $\theta \leq \delta([\nu^{-g}\rho, \nu^{a}\rho]) \rtimes \sigma$.

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Next, we claim that $\theta \neq L\big(\delta([\nu^{-a}\rho, \nu^{g}\rho]); \sigma\big)$. If we had $\theta = L\big(\delta([\nu^{-a}\rho, \nu^{g}\rho]); \sigma\big)$, then by Frobenius reciprocity,

$$s_{\min}\big(\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t\big) \geq (\nu^{b}\rho \otimes \nu^{b-1}\rho \otimes \cdots \otimes \nu^{g+1}\rho) \otimes (\nu^{g}\rho \otimes \nu^{g-1}\rho \otimes \cdots \otimes \nu^{-a}\rho) \otimes \sigma.$$

By the Casselman criterion, this would contradict the square-integrability of $\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t$ (or alternatively, the fact that $\chi_0\big(\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_t\big) = (\nu^{a}\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{-b}\rho) \otimes \sigma$, which follows from the preceding theorem).

Finally, that $\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_1 \hookrightarrow \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_1$ follows from the fact that $\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_1$ may be characterized as the irreducible subquotient of $\delta([\nu^{-b}\rho, \nu^{a}\rho]) \rtimes \sigma$ containing $\nu^{a}\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{b+1}\rho \otimes (\nu^{b}\rho \otimes \nu^{b}\rho) \otimes \cdots \otimes (\nu^{\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho) \otimes \sigma$ in its $s_{\min}$ (an easy consequence of Theorems 3.3(i) and 4.2(iv) of [Tad5]). Then, the fact that $\delta([\nu^{-b}\rho, \nu^{a}\rho]; \sigma)_2 \hookrightarrow \delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_2$ follows from the fact that $\delta([\nu^{g+1}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-g}\rho, \nu^{a}\rho]; \sigma)_1$ admits a unique irreducible subrepresentation.

A similar argument may be used to verify 1. (it is slightly easier). ■

5.2 Constraints on $\chi_0(\pi)$

Definition 5.2.1 Suppose τ is an irreducible representation of $GL_m(F)$ and π a representation of $S_n(F)$. Write

$$\mu^*(\pi) = \sum_i m_i \xi_i \otimes \theta_i,$$

where $\xi_i \otimes \theta_i$ is irreducible and m_i is its multiplicity. Let $I_\tau = \{i \mid \xi_i = \tau\}$. We set

$$\mu^*_\tau(\pi) = \sum_{i \in I_\tau} m_i \xi_i \otimes \theta_i = \sum_{i \in I_\tau} m_i \tau \otimes \theta_i.$$

Similarly, if ξ is a representation of $GL_r(F)$ and

$$M^*(\xi) = \sum_j n_j \xi_j^{(1)} \otimes \xi_j^{(2)},$$

let $J_\tau = \{j \mid \xi_j^{(1)} = \tau\}$. We set

$$M^*_\tau(\xi) = \sum_{j \in J_\tau} n_j \xi_j^{(1)} \otimes \xi_j^{(2)} = \sum_{j \in J_\tau} n_j \tau \otimes \xi_j^{(2)}.$$

Lemma 5.2.2 *Let τ, ξ, π be as in Definition 5.2.1.*

1. *Suppose* $\operatorname{supp}(\tau) \cap [\operatorname{supp}(\xi) \cup \operatorname{supp}(\tilde{\xi})] = \varnothing$. *Then,*

$$\mu^*_\tau(\xi \rtimes \pi) = (1 \otimes \xi) \rtimes \mu^*_\tau(\pi).$$

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2. *Suppose* $\mathrm{supp}(\tau) \cap \mathrm{supp}(\pi) = \varnothing$. *Then,*

$$\mu^*_\tau(\xi \rtimes \pi) = M^*_\tau(\xi) \rtimes (1 \otimes \pi).$$

Proof First, with notation as above,

$$\mu^*(\xi \rtimes \pi) = \sum_i \sum_j m_i n_j (\xi_j^{(1)} \times \xi_i) \otimes (\xi_j^{(2)} \rtimes \theta_i).$$

For 1., observe that if $\xi_j^{(1)} \neq 1$, then $\xi_j^{(1)} \times \xi_i \not\geq \tau$ (since $\mathrm{supp}(\xi_j^{(1)}) \subset [\mathrm{supp}(\xi) \cup \mathrm{supp}(\tilde{\xi})]$). Therefore, $\xi_j^{(1)} = 1$. From the formula $M^* = (m \otimes 1) \circ (\tilde{\ } \otimes m^*) \circ s \circ m^*$, this forces $\xi_j^{(2)} = \xi$ and $n_j = 1$. Therefore, we must have $\xi_i = \tau$. The claim 1. follows.

For 2., observe that if $\xi_i \neq 1$, then $\xi_j^{(1)} \times \xi_i \not\geq \tau$ (since $\mathrm{supp}(\xi_i) \subset \mathrm{supp}(\pi)$). Therefore, $\xi_i = 1$. Then, we must have $\theta_i = \pi$ and $m_i = 1$. Therefore, we must have $\xi_j^{(1)} = \tau$. The claim 2. follows. ■

Lemma 5.2.3 *Let* $a, b \in \frac{1}{2} + \mathbb{Z}$ *with* $a > b \geq -\frac{1}{2}$.

1. *Suppose* $b \leq f < a$. *Then,*

$$\mu^*_{\delta([\nu^{f+1}\rho,\nu^a\rho])}\big(\delta([\nu^{-b}\rho, \nu^a\rho];\sigma)_t\big) = \delta([\nu^{f+1}\rho, \nu^a\rho]) \otimes \delta([\nu^{-b}\rho, \nu^f\rho];\sigma)_t.$$

2. *Suppose* $-\frac{1}{2} < g < b$. *Then,*

$$\mu^*_{\delta([\nu^{g+1}\rho,\nu^b\rho])}\big(\delta([\nu^{-b}\rho, \nu^a\rho];\sigma)_t\big) = \delta([\nu^{g+1}\rho, \nu^b\rho]) \otimes \delta([\nu^{-g}\rho, \nu^a\rho];\sigma)_t.$$

Proof This follows immediately from the proof of Lemma 5.1.3. ■

Suppose $a \geq b \geq c \geq d \geq -\frac{1}{2}$ have $a, b, c, d \in \frac{1}{2} + \mathbb{Z}$. Set

$$\pi_t = \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^b\rho];\sigma)_t$$
$$\pi_t' = \delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho];\sigma)_t$$
$$\pi_t'' = \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^c\rho];\sigma)_t.$$

Consider the following multisets:

$$X_t = \{\theta \leq \pi_t \mid \theta \text{ is irreducible and } \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0\}$$
$$X_t' = \{\theta \leq \pi_t' \mid \theta \text{ is irreducible and } \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0\}$$
$$X_t'' = \{\theta \leq \pi_t'' \mid \theta \text{ is irreducible and } \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0\}.$$

We note the following:

Lemma 5.2.4 *Suppose* π *is an irreducible subrepresentation of* π_t. *Then,* $\pi \in X_t$.

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Proof This follows immediately from Frobenius reciprocity and the fact that $\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])\otimes\delta([\nu^{-c}\rho,\nu^{c}\rho])\times\delta([\nu^{-d}\rho,\nu^{c}\rho])\leq m^*\big(\delta([\nu^{-c}\rho,\nu^{a}\rho])\times\delta([\nu^{-d}\rho,\nu^{b}\rho])\big)$. ■

Lemma 5.2.5 $X_t = X_t' = X_t''$.

Proof 1. $X_t = X_t''$.

If $b = c$, there is nothing to prove. Suppose $b \neq c$.

Let $\pi_t^{**} = \delta([\nu^{-c}\rho,\nu^{a}\rho])\times\delta([\nu^{c+1}\rho,\nu^{b}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t$. From Lemma 5.1.3, we have $\pi_t \leq \pi_t^{**}$. Also,

$$\pi_t'' \leq \delta([\nu^{-c}\rho,\nu^{a}\rho])\times\delta([\nu^{-b}\rho,\nu^{-c-1}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t = \pi_t^{**}.$$

Now, Lemma 5.2.2 2. and an easy calculation give

$$\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])}(\pi_t^{**}) = \delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho]) \otimes\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t.$$

Since $M^*_{\delta([\nu^{c+1}\rho,\nu^{a}\rho])}\big(\delta([\nu^{-c}\rho,\nu^{a}\rho])\big) = \delta([\nu^{c+1}\rho,\nu^{a}\rho])\otimes\delta([\nu^{-c}\rho,\nu^{c}\rho])$ and (by Lemma 5.2.3) $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])}\big(\delta([\nu^{-d}\rho,\nu^{b}\rho];\sigma)_t\big) = \delta([\nu^{c+1}\rho,\nu^{b}\rho])\otimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t$, we get

$$\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])}(\pi_t) \geq \delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho]) \otimes\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t.$$

Since $\pi_t \leq \pi_t^{**}$, we must have equality. Similarly, since

$$M^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])}\big(\delta([\nu^{-b}\rho,\nu^{a}\rho])\big) = \delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])\otimes\delta([\nu^{-c}\rho,\nu^{c}\rho]),$$

Lemma 5.2.2 2. tells us

$$\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])}(\pi_t'') = \delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho]) \otimes\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t.$$

From this, it is immediate that

$$X_t = X_t'' = \{\theta \leq \pi_t^{**} \mid \theta \text{ is irreducible and } \mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])\times\delta([\nu^{c+1}\rho,\nu^{a}\rho])}(\theta)\neq 0\}.$$

2. $X_t = X_t'$.

If $a = b$, there is nothing to prove. Suppose $a \neq b$.

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The proof is similar to the proof that $X_t = X_t''$. Let $\pi_t^* = \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{b+1}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^b\rho]; \sigma)_t$. Then one can easily check that $\pi_t \leq \pi_t^*$, $\pi_t' \leq \pi_t^*$. Now, calculations like those above tell us

$$\mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\pi_t) = \mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\pi_t') = \mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\pi_t^*)$$
$$= \delta([\nu^{b+1}\rho, \nu^a\rho]) \otimes \delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^b\rho]; \sigma)_t.$$

Since $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0$ implies $\mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\theta) \neq 0$, we get

$$X_t = X_t' = \{\theta \leq \pi_t^* \mid \theta \text{ is irreducible and } \mu_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0\}.$$

as needed. ∎

Suppose π is an irreducible, square-integrable representation supported on $\mathcal{S}\big((\rho, \frac{1}{2}); \sigma\big)$. Further, suppose $\delta_0(\pi)$ consists of two generalized Steinberg representations (and σ). By the results in chapter 4, $\delta_0(\pi)$ has one of the following forms:

1. $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma$
2. $\delta_0(\pi) = \delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma$
3. $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$

for $a, b, c, d \in \frac{1}{2} + \mathbb{Z}$ with $a \geq b \geq c \geq d \geq -\frac{1}{2}$. The three possible forms are distinct if the inequalities among a, b, c, d are strict. In the next two propositions, we show that the inequalities are strict and that the first possible form does not actually occur. We note that if σ is generic, the fact that the inequalities are strict follows from [Mu] (and holds for δ_0 with an arbitrary number of segments).

Proposition 5.2.6 *For π, $\delta_0(\pi)$ as above, $a > b > c > d \geq -\frac{1}{2}$.*

Proof First, suppose that $\delta_0(\pi)$ is of the form 1. above. Then,

$$\pi \hookrightarrow \delta([\nu^{-d}\rho, \nu^b\rho]) \times \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \sigma \cong \delta([\nu^{-c}\rho, \nu^a\rho]) \times \delta([\nu^{-d}\rho, \nu^b\rho]) \rtimes \sigma$$
$$\Downarrow$$
$$\pi \hookrightarrow \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^b\rho]; \sigma)_t \text{ or } \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes L\big(\delta([\nu^{-b}\rho, \nu^d\rho]); \sigma\big)$$

(some t) by [Jan3, Lemma 5.5] and Theorem 5.1.2. However, if $\pi \hookrightarrow \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes L\big(\delta([\nu^{-b}\rho, \nu^d\rho]); \sigma\big)$, we would have

$$\chi_0' = (\nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{-c}\rho) \otimes (\nu^d\rho \otimes \nu^{d-1}\rho \otimes \cdots \otimes \nu^{-b}\rho) \otimes \sigma \leq s_{\min}(\pi),$$

giving the contradiction $\chi_0' < \chi_0(\pi)$. Thus, if $\delta_0(\pi)$ is of the form 1. above, we must have $\pi \hookrightarrow \pi_t$ (notation as above). Similarly, if $\delta_0(\pi)$ is of the form 2. (resp., 3.) above, then $\pi \hookrightarrow \pi_t'$ (resp., $\pi \hookrightarrow \pi_t''$).

Suppose $\delta_0(\pi)$ is of one of the forms above with $b = c$. By Theorem 4.2.1, it is not of the second form; for $b = c$, forms 1. and 3. are the same. Note that in this case, $\pi_t = \pi_t''$. By

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Lemmas 5.2.4 and 5.2.5, $\pi \in X_t = X'_t$, so $\pi \le \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$. Since $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$ is unitary, Frobenius reciprocity tells us

$$\chi'_0 = (\nu^b\rho \otimes \nu^{b-1}\rho \otimes \cdots \otimes \nu^{-b}\rho) \otimes (\nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{-d}\rho) \otimes \sigma \le s_{\min}(\pi),$$

giving the contradiction $\chi'_0 < \chi_0(\pi)$. Thus, $b > c$.

A similar argument gives $a > b$.

Suppose $c = d$. We write c for both. By Lemma 4.4.1, $c > -\frac{1}{2}$. By Theorem 4.2.1, $\delta_0(\pi)$ is not of form 3.; forms 1. and 2. are the same. First, it is an easy consequence of Lemma 5.2.2 that

$$\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}\big(\delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t\big) \\ = \delta([\nu^{c+1}\rho, \nu^b\rho]) \times \delta([\nu^{c+1}\rho, \nu^a\rho]) \otimes \delta([\nu^{-c}\rho, \nu^c\rho]; \sigma)_t.$$

Therefore, an argument like that in the proof that $X_t = X''_t$ (Lemma 5.2.5) tells us

$$\begin{aligned}\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\pi_t) &= \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\pi^\dagger_t) \\ &= \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\pi^\sharp_t) \\ &= \delta([\nu^{c+1}\rho, \nu^b\rho]) \times \delta([\nu^{c+1}\rho, \nu^a\rho]) \\ &\qquad \otimes \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-c}\rho, \nu^c\rho]; \sigma)_t.\end{aligned}$$

where $\pi^\dagger_t = \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t$ and $\pi^\sharp_t = \delta([\nu^{-c}\rho, \nu^c\rho]) \times \delta([\nu^{c+1}\rho, \nu^b\rho]) \times \delta([\nu^{c+1}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-c}\rho, \nu^c\rho]; \sigma)_t$. Now, $\pi_t \le \pi^\sharp_t$ and $\pi^\dagger_t \le \pi^\sharp_t$. Thus, $X_t = X^\dagger_t$, where

$$X^\dagger_t = \{\theta \le \pi^\dagger_t \mid \theta \text{ is irreducible and } \mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0\}.$$

Then, $\pi \in X_t = X^\dagger_t$ has

$$\pi \le \pi^\dagger_t = \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^a\rho]; \sigma)_t.$$

As above, this tells us

$$\chi'_0 = (\nu^c\rho \otimes \nu^{c-1}\rho \otimes \cdots \otimes \nu^{-c}\rho) \otimes (\nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{-b}\rho) \otimes \sigma \le s_{\min}(\pi),$$

giving the contradiction $\chi'_0 < \chi_0(\pi)$. ■

Proposition 5.2.7 *Suppose $a > b > c > d \ge -\frac{1}{2}$. Then, $\delta_0(\pi)$ cannot have the form $\delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma$.*

Proof Suppose this were possible. As noted in the proof of the previous proposition, if $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma$, then $\pi \hookrightarrow \pi_t$ (some t). Thus, by Lemma 5.1.3,

$$\pi \hookrightarrow \pi_t \hookrightarrow \pi^{**}_t \cong \delta([\nu^{c+1}\rho, \nu^b\rho]) \times \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t.$$

Therefore, by [Jan3, Lemma 5.5],

$$\pi \hookrightarrow \delta([\nu^{c+1}\rho, \nu^b\rho]) \rtimes \theta$$

for some (irreducible) $\theta \leq \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t$. Further, since $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\pi) \neq 0$, we claim that $\mu^*_{\delta([\nu^{c+1}\rho,\nu^a\rho])}(\theta) \neq 0$. (In the proof of Lemma 5.2.5, we observed that

$$\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])\times\delta([\nu^{c+1}\rho,\nu^a\rho])}(\pi_t^{**}) = \delta([\nu^{c+1}\rho, \nu^b\rho]) \times \delta([\nu^{c+1}\rho, \nu^a\rho]) \otimes \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t,$$

which may be rewritten as $M^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])}\big(\delta([\nu^{c+1}\rho, \nu^b\rho])\big) \rtimes \mu^*_{\delta([\nu^{c+1}\rho,\nu^a\rho])}\big(\delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t\big)$. The claim follows.) Therefore, by Lemma 5.2.5 (with $b = c$), we have $\theta \leq \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$. Since $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$ is unitary, the fact that $\pi \hookrightarrow \delta([\nu^{c+1}\rho, \nu^b\rho]) \rtimes \theta$ and Frobenius reciprocity combine to tell us

$$\chi_0' = (\nu^b\rho \otimes \nu^{b-1}\rho \otimes \cdots \otimes \nu^{c+1}\rho) \otimes (\nu^c\rho \otimes \nu^{b-1}\rho \otimes \cdots \otimes \nu^{-c}\rho) \otimes (\nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{-d}\rho) \otimes \sigma \leq s_{\min}(\pi).$$

However, this gives the contradiction $\chi_0' < \chi_0(\pi)$. ■

We summarize these results below:

Corollary 5.2.8 *Suppose (ρ, σ) satisfies $(C\frac{1}{2})$ and π is an irreducible, square-integrable representation supported on $\mathcal{S}\big((\rho, \frac{1}{2}); \sigma\big)$. If $\delta_0(\pi)$ consists of two generalized Steinberg representations (and σ)—i.e., $k = 2$ in the notation of Definition 4.1.3—then $\delta_0(\pi)$ has one of the following forms:*

(a) $\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma$; or
(b) $\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$

for $a, b, c, d \in \frac{1}{2} + \mathbb{Z}$ with $a > b > c > d \geq -\frac{1}{2}$.

In the next section, we shall see that both possibilities actually occur.

5.3 The Two-Segment Case

By Corollary 5.2.8, we need to look for square-integrable subrepresentations in $\delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ and $\delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma$ (where we continue to assume $a > b > c > d \geq -\frac{1}{2}$). The first of these is covered by the results in [Tad5, Section 8]; we summarize what we need in Theorem 5.3.1 below. The remainder of this section will focus on $\delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma$ (or more precisely, on $\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$).

Theorem 5.3.1 $\delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ *has exactly four irreducible subrepresentations. They are all square-integrable and are pairwise inequivalent. (In Remark 5.3.3 2. below, we will show that $\delta_0 = \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ for all four.)*

Proposition 5.3.2 *Suppose π is an irreducible subquotient of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\rtimes\sigma$. Assume that $a_1,\ldots,a_k\equiv\frac{1}{2}\mod 1$ with*

1. *$a_k\geq\cdots\geq a_1>0$.*
2. *$a_i>b_i$ for all i.*
3. *$b_i\geq-\frac{1}{2}$ and there is at most one value of i for which $b_i=-\frac{1}{2}$.*

Then, $\delta_0(\pi)$ has the form $\delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b'_k}\rho,\nu^{a'_k}\rho])\otimes\sigma$ with $a'_1,\ldots,a'_k$, $b'_1,\ldots,b'_k$ a permutation of $a_1,\ldots,a_k,b_1,\ldots,b_k$.

Proof First, for $c\equiv\frac{1}{2}\mod 1$ with $c\geq 0$, let $m(c)$ be the number of terms of the form $\nu^c\rho$ or $\nu^{-c}\rho$ which appear in a given element of $\chi_0(\pi)$. We note that $m(c)$ is well-defined and is the same for every subquotient of $\delta([\nu^{-b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{-b_k}\rho,\nu^{a_k}\rho])\rtimes\sigma$ (it is just a matter of the supercuspidal support).

Write $\delta_0(\pi)=\delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho])\otimes\cdots\otimes\delta([\nu^{-b'_m}\rho,\nu^{a'_m}\rho])\otimes\sigma$. We first claim $m=k$. Observe that $b_i=-\frac{1}{2}$ for some (exactly one) i if and only if $m(\frac{1}{2})$ is odd. Similarly, by Lemma 4.4.1, we have that $b'_j=-\frac{1}{2}$ for some (exactly one) j if and only if $m(\frac{1}{2})$ is odd. Now, suppose $m(\frac{1}{2})$ is even. Then, every $\delta([\nu^{-b_i}\rho,\nu^{a_i}\rho])$ contains $\nu^{\frac{1}{2}}\rho$ and $\nu^{-\frac{1}{2}}\rho$ exactly once each. Therefore $k=\frac{1}{2}m(\frac{1}{2})$. On the other hand, any term in $s_{\min}(\pi)$ must be of the form $\nu^x\rho\otimes\cdots$ with $x\in\{a_1,\ldots,a_k,b_1,\ldots,b_k\}$ (an easy consequence of Theorem 3.1.2, *e.g.*). In particular, we must have $a'_1>0$. Thus, $a'_m\geq\cdots\geq a'_1>0$. Since $m(\frac{1}{2})$ is even, $b'_j\geq\frac{1}{2}$ for all j. Therefore, each $\delta([\nu^{-b'_j}\rho,\nu^{a'_j}\rho])$ contains $\nu^{\frac{1}{2}}\rho$ and $\nu^{-\frac{1}{2}}\rho$ exactly once each. Thus, $m=\frac{1}{2}m(\frac{1}{2})=k$.

The argument when $m(\frac{1}{2})$ is odd is similar but a little more involved. The same sort of argument as above tells us $k=\frac{1}{2}[m(\frac{1}{2})+1]$. To relate m to $m(\frac{1}{2})$, we consider three cases: $a'_1\neq-\frac{1}{2}$, $a'_2>a'_1=-\frac{1}{2}$, and $a'_2=a'_1=-\frac{1}{2}$. If $a'_1\neq-\frac{1}{2}$, we can argue as above to get $m=\frac{1}{2}[m(\frac{1}{2})+1]$. Suppose $a'_2>a'_1=-\frac{1}{2}$. If $b'_j\neq-\frac{1}{2}$ for any j, we get one copy of $\nu^{-\frac{1}{2}}\rho$ from each of $\delta([\nu^{-b'_1}\rho,\nu^{a'_1}\rho]),\ldots,\delta([\nu^{-b'_m}\rho,\nu^{a'_m}\rho])$ and one copy of $\nu^{\frac{1}{2}}\rho$ from each of $\delta([\nu^{-b'_2}\rho,\nu^{a'_2}\rho]),\ldots,\delta([\nu^{-b'_m}\rho,\nu^{a'_m}\rho])$, giving a total of $2m-1$. Thus, $m=\frac{1}{2}[m(\frac{1}{2})+1]$. If we had $a'_2>a'_1=-\frac{1}{2}$ and $b'_j=-\frac{1}{2}$ for some j (exactly one), we would get a total of $2m-2$ copies of $\nu^{\pm\frac{1}{2}}\rho$. Since this has the wrong parity, we could not have had $b'_j=-\frac{1}{2}$. Finally, suppose $a'_2=a'_1=-\frac{1}{2}$. Since $\delta([\nu^{-b'_1}\rho,\nu^{-\frac{1}{2}}\rho])\times\delta([\nu^{-b'_2}\rho,\nu^{-\frac{1}{2}}\rho])$ is irreducible, we have that $\mu^*(\pi)$ contains a term of the form $\big(\delta([\nu^{-b'_1}\rho,\nu^{-\frac{1}{2}}\rho])\times\delta([\nu^{-b'_2}\rho,\nu^{-\frac{1}{2}}\rho])\big)\otimes\cdots$. Further, since $r_{\min}\big(\delta([\nu^{-b'_1}\rho,\nu^{-\frac{1}{2}}\rho])\times\delta([\nu^{-b'_1}\rho,\nu^{-\frac{1}{2}}\rho])\big)$ contains terms of the form $\nu^{-\frac{1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\rho\otimes\cdots$, we see that $s_{\min}(\pi)$ also contains terms of the form $\nu^{-\frac{1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\rho\otimes\cdots$. However, we claim this cannot occur. If b_i is the (unique) element of $\{a_1,\ldots,a_k,b_1,\ldots,b_k\}$ which has $b_i=-\frac{1}{2}$, then a term of the form $\nu^{-\frac{1}{2}}\rho\otimes\nu^x\rho\otimes\cdots$ must have $x\in\{a_1,\ldots,a_k,$ $b_1,\ldots,b_{i-1},b_i-1,b_{i+1},\ldots,b_k\}$. Since $-\frac{1}{2}$ is not in this set, we have a contradiction. Thus we cannot have $a'_2=-\frac{1}{2}$. We now have $m=\frac{1}{2}[m(\frac{1}{2})+1]=k$ for the case $m(\frac{1}{2})$ odd, as needed.

We now claim that $a'_1,\ldots,a'_k,b'_1,\ldots,b'_k$ are the same as $a_1,\ldots,a_k,b_1,\ldots b_k$ up to permutation. For $c\equiv\frac{1}{2}\bmod 1$, let $n(c)$ (resp., $n'(c)$) be the number of times c appears in

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$a_1, \dots, a_k, b_1, \dots, b_k$ (resp., $a'_1, \dots, a'_k, b'_1, \dots, b'_k$). Observe that

$$n(c) = n'(c) = \begin{cases} m(c) - m(c+1) & \text{for } c \geq \frac{1}{2}, \\ m(\frac{1}{2}) - 2\left\lfloor \frac{1}{2} m(\frac{1}{2}) \right\rfloor & \text{for } c = -\frac{1}{2}, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. The proposition follows. ∎

Remark 5.3.3

1. One can weaken the second hypothesis to $a_i \geq b_i$ for all i without changing the result.
2. It follows from Proposition 5.3.2 and Corollary 5.2.8 that if π is one of the irreducible, square-integrable subrepresentations of $\delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$ of Theorem 5.3.1, the only candidates for $\delta_0(\pi)$ are $\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ and $\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma$. Since $\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ corresponds to a lower χ_0 (and Frobenius reciprocity tells us $\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma \leq s_{((d+c+1)n,(b+a+1)n)}(\pi)$), we see that $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$.

Now, suppose π is an irreducible subquotient of $\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t \leq \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma$. By the preceding proposition, $\delta_0(\pi)$ must be one of the following:

$$\begin{array}{ll}
\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma, & \delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma, \\
\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma, & \delta([\nu^{-b}\rho, \nu^c\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma, \\
\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-a}\rho, \nu^b\rho]) \otimes \sigma, & \delta([\nu^{-a}\rho, \nu^c\rho]) \otimes \delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \sigma, \\
\delta([\nu^{-c}\rho, \nu^d\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma, & \delta([\nu^{-b}\rho, \nu^d\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma, \\
\delta([\nu^{-c}\rho, \nu^d\rho]) \otimes \delta([\nu^{-a}\rho, \nu^b\rho]) \otimes \sigma, & \delta([\nu^{-a}\rho, \nu^d\rho]) \otimes \delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \sigma, \\
\delta([\nu^{-b}\rho, \nu^d\rho]) \otimes \delta([\nu^{-a}\rho, \nu^c\rho]) \otimes \sigma, & \delta([\nu^{-a}\rho, \nu^d\rho]) \otimes \delta([\nu^{-b}\rho, \nu^c\rho]) \otimes \sigma.
\end{array}$$

We observe that by Corollary 5.2.8, we cannot actually have $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^b\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma$.

Next, write $\mu^*\big(\delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t\big) = \sum_k \tau_k \otimes \theta_k$. Then, by Theorem 3.1.2,

$$\mu^*\big(\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t\big)$$
$$= \sum_k \sum_{i=-c}^{b+1} \sum_{j=i}^{b+1} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^j\rho, \nu^b\rho]) \times \tau_k \otimes \delta([\nu^i\rho, \nu^{j-1}\rho]) \rtimes \theta_k.$$

Consider, *e.g.*, the question of whether we can have $\delta_0(\pi) = \delta([\nu^{-b}\rho, \nu^d\rho]) \otimes \delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma$. For this to happen, we must have $\delta([\nu^{-b}\rho, \nu^d\rho]) \otimes \theta \leq \mu^*\big(\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t\big)$ for some irreducible θ with $\delta([\nu^{-c}\rho, \nu^a\rho]) \otimes \sigma \leq$

https://doi.org/10.4153/CJM-2000-025-7 Published online by Cambridge University Press

$s_{\mathrm{GL}}(\theta)$. However, since $\delta([\nu^{-b}\rho, \nu^{d}\rho])$ contains neither $\nu^{c}\rho$ nor $\nu^{b}\rho$, we must have $i = -c$ and $j = b+1$ above. Thus, $\tau_k = \delta([\nu^{-b}\rho, \nu^{d}\rho])$. However, we then have

$$s_{\min}\big(\delta([\nu^{-d}\rho, \nu^{a}\rho]; \sigma)_t\big) \\ \geq s_{\min}\big(\delta([\nu^{-b}\rho, \nu^{d}\rho]) \otimes \theta_k\big) \geq (\nu^{d}\rho \otimes \nu^{d-1}\rho \otimes \cdots \otimes \nu^{-b}\rho) \otimes \cdots,$$

in violation of the Casselman criterion. Thus we cannot have $\delta_0(\pi) = \delta([\nu^{-b}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c}\rho, \nu^{a}\rho]) \otimes \sigma$. The same argument tells us $\delta_0(\pi)$ cannot be $\delta([\nu^{-a}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c}\rho, \nu^{b}\rho]) \otimes \sigma$, $\delta([\nu^{-b}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-a}\rho, \nu^{c}\rho]) \otimes \sigma$, $\delta([\nu^{-a}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-b}\rho, \nu^{c}\rho]) \otimes \sigma$, $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-b}\rho, \nu^{a}\rho]) \otimes \sigma$, or $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \sigma$.

To show that $\delta_0(\pi)$ cannot be $\delta([\nu^{-a}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-d}\rho, \nu^{b}\rho]) \otimes \sigma$, we use the same basic argument as above to conclude that $j = b+1$ and $\tau_k = \delta([\nu^{-a}\rho, \nu^{-i}\rho])$, also violating the Casselman criterion.

Finally, we consider the possibility that $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \sigma$. We apply the usual commuting argument:

$$\pi \hookrightarrow \delta([\nu^{-d}\rho, \nu^{c}\rho]) \times \delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \sigma$$

$$\Downarrow$$

$$\delta([\nu^{-d}\rho, \nu^{c}\rho]) \times \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \sigma \leq s_{\mathrm{GL}}(\pi)$$

$$\Downarrow$$

$$\delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-d}\rho, \nu^{c}\rho]) \otimes \sigma \leq s_{((a+b+1)n,(c+d+1)n)}(\pi)$$

since $\delta([\nu^{-d}\rho, \nu^{c}\rho]) \times \delta([\nu^{-a}\rho, \nu^{b}\rho])$ is irreducible. We now argue as above. We must have a term of the form $\delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \theta$ in $\mu^*\big(\delta([\nu^{-c}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^{a}\rho]; \sigma)_t\big)$. If $i = -c$, we get $\tau_k = \delta([\nu^{-a}\rho, \nu^{j-1}\rho])$, which leads to a violation of the Casselman criterion. If $i \neq -c$, then we must have $\tau_k = \delta([\nu^{-a}\rho, \nu^{-i}\rho]) \times \delta([\nu^{c+1}\rho, \nu^{j-1}\rho])$ (irreducible). Then,

$$s_{\min}\big(\delta([\nu^{-d}\rho, \nu^{a}\rho]; \sigma)_t\big) \geq s_{\min}\big(\delta([\nu^{-a}\rho, \nu^{-i}\rho]) \times \delta([\nu^{c+1}\rho, \nu^{j-1}\rho]) \otimes \theta_k\big) \\ \geq (\nu^{-i}\rho \otimes \nu^{-i-1}\rho \otimes \cdots \otimes \nu^{-a}\rho) \otimes \cdots,$$

again violating the Casselman criterion. Thus we have the following:

Corollary 5.3.4 *If π is an irreducible subquotient of $\delta([\nu^{-c}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^{a}\rho]; \sigma)_t$, then $\delta_0(\pi)$ must be one of the following: $\delta([\nu^{-c}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-d}\rho, \nu^{a}\rho]) \otimes \sigma$, $\delta([\nu^{-d}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-b}\rho, \nu^{a}\rho]) \otimes \sigma$, or $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-d}\rho, \nu^{a}\rho]) \otimes \sigma$.*

Corollary 5.3.5 *$\delta([\nu^{-c}\rho, \nu^{b}\rho]) \rtimes \delta([\nu^{-d}\rho, \nu^{a}\rho]; \sigma)_t$ has a unique irreducible quotient (Langlands quotient). All other irreducible subquotients are square-integrable.*

Proof The Langlands classification gives the first claim. By [Jan3, Lemma 3.4], there is no other subquotient with $\delta_0 = \delta([\nu^{-c}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-d}\rho, \nu^{a}\rho]) \otimes \sigma$. By the preceding corollary and Theorem 4.2.1, all other irreducible subquotients are square-integrable. ■

Lemma 5.3.6 *$\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ decomposes as the direct sum of (exactly) two inequivalent irreducible subrepresentations.*

Proof First, we claim that $\delta([\nu^{-c}\rho,\nu^{c}\rho])\otimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ appears with multiplicity 2 in $\mu^*\big(\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)$. To see this, write $\mu^*\big(\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)=\sum_k\tau_k\otimes\theta_k$. Then,

$$\mu^*\big(\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)$$
$$=\sum_k\sum_{i=-c}^{c+1}\sum_{j=i}^{c+1}\delta([\nu^{-i+1}\rho,\nu^{c}\rho])\times\delta([\nu^{j}\rho,\nu^{c}\rho])\times\tau_k\otimes\delta([\nu^{i}\rho,\nu^{j-1}\rho])\rtimes\theta_k.$$

The only terms which can contribute to $\mu^*_{\delta([\nu^{-c}\rho,\nu^{c}\rho])}$ have either $i=-c$ or $j=c+1$ (since $\nu^{c}\rho$ appears only once in $\delta([\nu^{-c}\rho,\nu^{c}\rho])$). Suppose $i=-c$. If $\tau_k=1$, we have $\theta_k=\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ and $j=-c$ contributes one copy of $\delta([\nu^{-c}\rho,\nu^{c}\rho])\otimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$. If $\tau_k\neq1$, then any term in $r_{\min}(\tau_k)$ must have the form $\nu^{d}\rho\otimes\cdots$ or $\nu^{a}\rho\otimes\cdots$. Therefore, $j=d+1$. So, $\tau_k=\delta([\nu^{-c}\rho,\nu^{d}\rho])$. However, this is not possible: having $\delta([\nu^{-c}\rho,\nu^{d}\rho])\otimes\theta_k$ in $\mu^*\big(\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)$ violates the Casselman criterion for the square-integrability of $\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$. The same argument works for the case $j=c+1$. The claim follows.

Next, a similar argument tells us that the multiplicity of $\delta([\nu^{-c}\rho,\nu^{c}\rho])\otimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ in $\delta([\nu^{-c}\rho,\nu^{a}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t)$ is one. (One needs the observation that $\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ appears with multiplicity one in $\delta([\nu^{c+1}\rho,\nu^{a}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t$, which follows from the proof of Lemma 5.1.3 or [Tad5]).

We can now verify the lemma. By Frobenius reciprocity (or [Go]), $\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$ has at most two components. By Lemma 5.2.5, one of those components is also a subquotient of $\delta([\nu^{-c}\rho,\nu^{a}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{c}\rho];\sigma)_t$. The preceding discussion of multiplicities tells us this component cannot be all of $\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t$, hence we have exactly two components. That they are inequivalent follows from [Go]. ■

Proposition 5.3.7 *π_t' admits exactly two irreducible subrepresentations, and they are inequivalent. Further, an irreducible subquotient π of π_t' appears as a subrepresentation if and only if $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])}(\pi)\neq0$.*

Proof First, we claim that $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])}(\pi_t')=\delta([\nu^{c+1}\rho,\nu^{b}\rho])\otimes\big(\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)$. To see this, write $\mu^*\big(\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t\big)=\sum_k\tau_k\otimes\theta_k$. Then,

$$\mu^*(\pi_t')=\sum_k\sum_{i=-c}^{b+1}\sum_{j=i}^{b+1}\delta([\nu^{-i+1}\rho,\nu^{c}\rho])\times\delta([\nu^{j}\rho,\nu^{b}\rho])\times\tau_k\otimes\delta([\nu^{i}\rho,\nu^{j-1}\rho])\rtimes\theta_k.$$

To contribute to $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])}$, we must certainly have $i=-c$. Further, if $\tau_k\neq1$, then $r_{\min}(\tau_k)$ consists of terms of the form $\nu^{d}\rho\otimes\cdots$ or $\nu^{a}\rho\otimes\cdots$. Since $\nu^{d}\rho$ and $\nu^{a}\rho$ do not appear in $\delta([\nu^{c+1}\rho,\nu^{b}\rho])$, we must also have $\tau_k=1$. Therefore, $j=c+1$ and the claim is immediate.

Next, write $\delta([\nu^{-c}\rho,\nu^{c}\rho])\rtimes\delta([\nu^{-d}\rho,\nu^{a}\rho];\sigma)_t=T_1\oplus T_2$ (*cf.* Lemma 5.3.6). The same basic argument used above tells us $\mu^*_{\delta([\nu^{c+1}\rho,\nu^{b}\rho])}\big(\delta([\nu^{c+1}\rho,\nu^{b}\rho])\rtimes T_i\big)=\delta([\nu^{c+1}\rho,\nu^{b}\rho])\otimes$

T_i. Therefore, by Frobenius reciprocity, $\delta([\nu^{c+1}\rho,\nu^b\rho]) \rtimes T_i$ has a unique irreducible subrepresentation—call it $\pi_t'^{(i)}$. Note that since $T_1 \not\cong T_2$, we have $\pi_t'^{(1)} \not\cong \pi_t'^{(2)}$. Since

$$\begin{aligned}\pi_t' &\hookrightarrow \delta([\nu^{c+1}\rho,\nu^b\rho]) \rtimes \big(\delta([\nu^{-c}\rho,\nu^c\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^a\rho];\sigma)_t\big)\\ &\cong \delta([\nu^{c+1}\rho,\nu^b\rho]) \rtimes (T_1 \oplus T_2)\end{aligned}$$

and $\pi_t'^{(1)}$, $\pi_t'^{(2)}$ appear with multiplicity one in $\delta([\nu^{c+1}\rho,\nu^b\rho]) \rtimes \big(\delta([\nu^{-c}\rho,\nu^c\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^a\rho];\sigma)_t\big)$ (by $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])}$ considerations), we see that $\pi_t'^{(1)},\pi_t'^{(2)} \hookrightarrow \pi_t'$ (just consider the subspace $V_{\pi_t'} + (V_{\pi_t'^{(1)}} \oplus V_{\pi_t'^{(2)}})$ inside the space for $\delta([\nu^{c+1}\rho,\nu^b\rho]) \rtimes \big(\delta([\nu^{-c}\rho,\nu^c\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^a\rho];\sigma)_t\big)$).

Now, it is an easy consequence of Frobenius reciprocity that $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])}$ must be nonzero for an irreducible subrepresentation of π_t'. Therefore, $\pi_t'^{(1)}$ and $\pi_t'^{(2)}$ are the only irreducible subrepresentations. Further, since $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])}(\pi_t') = \delta([\nu^{c+1}\rho,\nu^b\rho]) \otimes (T_1 + T_2)$, if π is an irreducible subquotient of π_t' with $\mu^*_{\delta([\nu^{c+1}\rho,\nu^b\rho])}(\pi) \neq 0$, then $\pi = \pi_t'^{(1)}$ or $\pi_t'^{(2)}$. The proposition follows. ∎

Now, Lemma 5.2.2 tells us $\mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\pi_t'') = \delta([\nu^{b+1}\rho,\nu^a\rho]) \otimes \delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^c\rho];\sigma)_t$. An argument like that used for Lemma 5.3.6 tells us $\delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^c\rho];\sigma)_t$ decomposes as the direct sum of two inequivalent irreducible subrepresentations. (Here, we need the observation that $\delta([\nu^{-b}\rho,\nu^b\rho]) \otimes \delta([\nu^{-d}\rho,\nu^c\rho];\sigma)_t$ appears with multiplicity one in $\delta([\nu^{-c}\rho,\nu^b\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^b\rho];\sigma)_t$. To see this, write $\mu^*\big(\delta([\nu^{-d}\rho,\nu^b\rho];\sigma)_t\big) = \sum_k \tau_k \otimes \theta_k$. Then,

$$\begin{aligned}&\mu^*\big(\delta([\nu^{-c}\rho,\nu^b\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^b\rho];\sigma)_t\big)\\ &\quad= \sum_{i=-c}^{b+1}\sum_{j=i}^{b+1}\sum_k \delta([\nu^{-i+1}\rho,\nu^c\rho]) \times \delta([\nu^j\rho,\nu^b\rho]) \times \tau_k \otimes \delta([\nu^i\rho,\nu^{j-1}\rho]) \rtimes \theta_k.\end{aligned}$$

If $i = b+1$, we get one copy $\delta([\nu^{-b}\rho,\nu^b\rho]) \otimes \delta([\nu^d\rho,\nu^c\rho];\sigma_t$. If $i \neq b+1$, we must have $\tau_k = \delta([\nu^{-b}\rho,\nu^{j-1}\rho])$ or $\tau_k = \delta([\nu^{-b}\rho,\nu^{-i}\rho]) \times \delta([\nu^{c+1}\rho,\nu^{j-1}\rho])$ (irreducible). In either case, we get a contradiction to the Casselman criterion for $\delta([\nu^{-d}\rho,\nu^b\rho];\sigma)_t$.) From these observations and Theorem 5.3.1, or by arguing as in the proof of the preceding proposition, we get the following:

Proposition 5.3.8 *π_t'' admits exactly two irreducible subrepresentations, and they are inequivalent. Further, an irreducible subquotient π of π_t'' appears as a subrepresentation if and only if $\mu^*_{\delta([\nu^{b+1}\rho,\nu^a\rho])}(\pi) \neq 0$.*

Theorem 5.3.9 *Suppose (ρ,σ) satisfies (C1/2). Let $a > b > c > d \geq -\frac{1}{2}$ with $a,b,c,d \in \frac{1}{2}+\mathbb{Z}$. Then, $\pi_t' = \delta([\nu^{-c}\rho,\nu^b\rho]) \rtimes \delta([\nu^{-d}\rho,\nu^a\rho];\sigma)_t$ has exactly 3 irreducible subquotients —denote them $\pi_t'^{(0)}$, $\pi_t'^{(1)}$, $\pi_t'^{(2)}$. We may characterize them as follows:*

1. *$\pi_t'^{(0)}$ is the unique irreducible quotient (Langlands quotient). It is nontempered and has $\delta_0(\pi_t'^{(0)}) = \delta([\nu^{-b}\rho,\nu^c\rho]) \otimes \delta([\nu^{-d}\rho,\nu^a\rho]) \otimes \sigma$.*

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2. *$\pi_t'^{(1)}$ is a subrepresentation. It is square-integrable and has $\delta_0(\pi_t'^{(1)}) = \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$ (and is therefore one of the square-integrable representations from [Tad5] discussed in Theorem 5.3.1 above).*
3. *$\pi_t'^{(2)}$ is a subrepresentation. It is square-integrable and has $\delta_0(\pi_t'^{(2)}) = \delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma$.*

Further, we note that $\pi_1'^{(2)} \not\cong \pi_2'^{(2)}$ (assuming $d > -\frac{1}{2}$ so that both are defined).

Proof We first address claims 1.–3. In light of Corollary 5.3.5 and Proposition 5.3.7, all we need to do is show that there are at most three irreducible subquotients and determine δ_0 for the subrepresentations.

First, we note that arguments similar those done in Propositions 5.3.7, 5.3.8, and Lemma 5.2.5 can be used to show that $\mu^*_{\delta([\nu^{-c}\rho,\nu^b\rho])}(\pi_t') = 2\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$ and $\mu^*_{\delta([\nu^{-b}\rho,\nu^a\rho])}(\pi_t') = \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t$. Therefore, we see that $\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]) \otimes \sigma$ appears in $s_{((c+b+1)n,(a+d+1)n)}(\pi_t')$ with multiplicity 2 and $\delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \sigma$ appears in $s_{((a+b+1)n,(c+d+1)n)}(\pi_t')$ with multiplicity 1. Observe that if π is an irreducible representation with $\delta_0(\pi) = \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma$, then the usual commuting argument tells us

$$\pi \hookrightarrow \delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma$$
$$\Downarrow$$
$$\delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \sigma \leq s_{\mathrm{GL}}(\pi)$$
$$\Downarrow$$
$$\delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \sigma \leq s_{((a+b+1)n,(c+d+1)n)}(\pi)$$

since $\delta([\nu^{-d}\rho, \nu^c\rho]) \times \delta([\nu^{-b}\rho, \nu^a\rho])$ is irreducible. Therefore, by Corollary 5.3.4 and (the proof of) Corollary 5.3.5, we see that π_t' has at most 4 irreducible subquotients.

By Lemma 5.2.5, choose $\pi_t'^{(1)} \in X_t = X_t' = X_t''$ (there will turn out to be only one possible choice). By Propositions 5.3.7 and 5.3.8, we have $\pi_t'^{(1)} \hookrightarrow \pi_t'$ and $\pi_t'^{(1)} \hookrightarrow \pi_t''$. Therefore, Frobenius reciprocity tells us $\mu^*(\pi_t'^{(1)})$ contains both a copy of $\delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$ and $\delta([\nu^{-b}\rho, \nu^a\rho]) \otimes \delta([\nu^{-d}\rho, \nu^c\rho]; \sigma)_t$. Thus, there are at most 3 irreducible subquotients. The claims about $\delta_0(\pi_t'^{(1)})$ and $\delta_0(\pi_t'^{(2)})$ are now immediate.

Finally, that $\pi_1'^{(2)} \not\cong \pi_2'^{(2)}$ follows immediately from the observation above that $\mu^*_{\delta([\nu^{-c}\rho,\nu^b\rho])}(\pi_t'^{(2)}) = \delta([\nu^{-c}\rho, \nu^b\rho]) \otimes \delta([\nu^{-d}\rho, \nu^a\rho]; \sigma)_t$. ■

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https://doi.org/10.4153/CJM-2000-025-7 Published online by Cambridge University Press