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TRANSCENDENCE MEASURES FOR EXPONENTIALS AND LOGARITHMS

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Dedicated to K. Mahler on his 75th birthday

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Abstract

In the present paper, we derive transcendence measures for the numbers $\log\alpha$, e^{β}, α^{β}, $(\log\alpha_1)/(\log\alpha_2)$ from a previous lower bound of ours on linear forms in the logarithms of algebraic numbers.

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1. Introduction

A great deal of work has already been done on finding transcendence measures for the numbers listed above; see, for example, the contributions of Mahler [Ma 1, 2, 3, 5], Gelfond [G] and Feldman [F 1, 2, 3, 4]. A systematic study of this subject is given in Cijsouw's thesis [Ci 1], and the sharpest results to date are due mainly to Cijsouw [Ci 1, 2, 3, 4]. However, until now, special arguments were needed for each given class of numbers. The novelty of the present paper is to give a single uniform proof of all these results, by using a lower bound for linear forms in the logarithms of algebraic numbers [W].

Our main results are summarized by the following diagrams. The notation is as follows. Let $\varphi(X, Y)$ be a real-valued function which is defined for $X\geqslant 1$ and $Y\geqslant\log 16$ (this is for convenience only). If ω is a transcendental number, we say that φ is a transcendence measure for ω if $\log|P(\omega)|\geqslant-\varphi(N,\log H)$ for all non-trivial polynomials P in $\mathbf{Z}[X]$ with degree at most N and height (in the usual sense) at most H. In addition, we say that a real number $\tau\geqslant 2$ is a transcendence type

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for ω if there exists a constant $c(\omega,\tau)>0$ such that $c(\omega,\tau)(\log H+N)^{\tau}$ is a transcendence measure for ω. Finally, for real x, we write $\operatorname{Log}_+ x = \operatorname{Log}(\max(1,x))$.

In Fig. 1 we give the consequences of the lower bound of [W] for the transcendence measures of the numbers π, $\log\alpha$, e^{β} and $(\log\alpha_1)/(\log\alpha_2)$, when α, β, α_1, α_2 are non-zero algebraic numbers with $\log\alpha\neq 0$ and $(\log\alpha_1)/(\log\alpha_2)$ irrational. Moreover, we give a result concerning α^{β} (with α and β algebraic, $\alpha\neq 0$, $\log\alpha\neq 0$ and β irrational) which is obtained by combining our lower bound with a result of Choodnovsky [Ch].

Number	Transcendence measure	Type $\leqslant$
π	$2^{40} N(\operatorname{Log} H+N\operatorname{Log} N)(1+\operatorname{Log} N)$	$2+\varepsilon$
$\log\alpha$	$C_1 N^2(\operatorname{Log} H+N\operatorname{Log} N)(1+\operatorname{Log} N)^{-1}$	3
e^{β}	$C_2 N^2(\operatorname{Log} H+\operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H+\operatorname{Log} N)^2$ $\times(\operatorname{Log}\operatorname{Log} H+\operatorname{Log}_+\operatorname{Log} N)^{-2}$	3
$(\log\alpha_1)/(\log\alpha_2)$	$C_3 N^3(\operatorname{Log} H+N\operatorname{Log} N)(1+\operatorname{Log} N)^{-2}$	4
α^{β}	$C_4 N^3(\operatorname{Log} H+\operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H+\operatorname{Log} N)(1+\operatorname{Log} N)^{-2}$	4

FIG. 1. Transcendence measures for classical numbers.

We are interested with transcendence measure which are explicit in N and H (some older papers are better when H is large with respect to N). The estimate concerning π is due to Fel'dman and Cijsouw; the result concerning $\log\alpha$ is due to Cijsouw. All the other results improve earlier known transcendence measures with respect to the degree N.

The number e^{π} is worthy of special consideration (Fig. 2). The transcendence measure which is provided by our linear form improves earlier results in the case of large height. In [Ch], Choodnovsky announces a result which is much more precise when the degree is very large. In the middle case, a claim of the "Stellingen" of Cijsouw's thesis [Ci 1] leads to a still better result. From Cijsouw's result it follows that e^{π} has a transcendence type at most 3.

$\operatorname{Log} H\geqslant N(\operatorname{Log} N)^3$	$2^{62} N^2(\operatorname{Log} H)(\operatorname{Log}\operatorname{Log} H)(1+\operatorname{Log} N)$	
$N(\operatorname{Log} N)^3\geqslant\operatorname{Log} H\geqslant N(\operatorname{Log} N)^{-1}$	$C_5 N(\operatorname{Log} H+N)^2(\operatorname{Log}\operatorname{Log} H)^{-1}$	(Cijsouw)
$N(\operatorname{Log} N)^{-1}\geqslant\operatorname{Log} H$	$C_6 N(\operatorname{Log} H+\operatorname{Log} N)^2(\operatorname{Log} N)$	(Choodnovsky)

FIG. 2. Transcendence measures for e^{π}.

Finally in Figure 3 (see p. 447) we give four general results which are consequences of our estimate [W], and which in fact contain several of the above-mentioned results (see Theorems 5.1 and 5.3 below).

The idea of deriving transcendence measures from a lower bound for linear forms in logarithms was used already in [M–W], and even earlier by Baker to give an irrationality measure for the number e^{π} (cf. [B] Chap. 3).

In a subsequent paper we will consider simultaneous approximations and improve several results of [C–W] and [M–W].

Number	Transcendence measure	Type $\leqslant$
$e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_m^{\beta_m}$	$C_7\, N^{m+2}(\text{Log}\, H+\text{Log}\, N)$ $\times(\text{Log Log}\, H+\text{Log}\, N)^{m+3}$ $\times(\text{Log Log}\, H)^{-m-2}\,(1+\text{Log}\, N)^{-m-1}$	$m+3$
$e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_m^{\beta_m}e^{\gamma i\pi}$	$C_8\, N^{m+2}(\text{Log}\, H+\text{Log}\, N)$ $\times(\text{Log Log}\, H+\text{Log}\, N)\,(1+\text{Log}\, N)$	$m+3+\varepsilon$
$\dfrac{\beta_0+\beta_1\log\alpha_1+\dots+\beta_s\log\alpha_s}{\beta_0'+\beta_1'\log\alpha_1'+\dots+\beta_t'\log\alpha_t'}$	$C_9\, N^{s+t+1}(\text{Log}\, H+N\,\text{Log}\, N)$ $\times(1+\text{Log}\, N)^{-s-t}$	$s+t+2$
$\dfrac{\beta_0+\beta_1\log\alpha_1+\dots+\beta_s\log\alpha_s+\gamma i\pi}{\beta_0'+\beta_1'\log\alpha_1'+\dots+\beta_t'\log\alpha_t'+\gamma' i\pi}$	$C_{10}\, N^{s+t+1}(\text{Log}\, H+N\,\text{Log}\, N)\,(1+\text{Log}\, N)$	$s+t+2+\varepsilon$

FIG. 3. Four general results.

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2. Notations and preliminary lemmas

Let $P=a_0X^N+\dots+a_N\in\mathbf{C}[X]$ be any non-zero polynomial with complex coefficients, with $a_0\neq 0$. Denote by $N(P)=N$ its degree, by

$$H(P)=\max_{0\leqslant j\leqslant N}|a_j|$$

its "usual height", by

$$L(P)=\sum_{j=0}^{N}|a_j|$$

its "length", and by

$$M(P)=\exp\int_0^1 \text{Log}|P(e^{2i\pi t})|\,dt$$

its "measure". If $\alpha_1,\dots,\alpha_N$ are the roots of P (according to multiplicity), then by Jensen's formula we have [Ma 4]:

$$M(P)=|a_0|\prod_{j=1}^{N}\max\{1,|\alpha_j|\}.$$

This measure has been introduced by Mahler who used the trivial formula

$$M(P_1 P_2) = M(P_1)\, M(P_2) \quad \text{for } P_1, P_2 \in \mathbf{C}[X]$$

to deduce non-trivial relations between $H(P_1 . P_2)$ and $H(P_1) . H(P_2)$. Here we need only the inequality

$$\text{(2.1)} \qquad \operatorname{Log} M(P) \leqslant \operatorname{Log} H(P) + \operatorname{Log} N(P),$$

(see [Ma 4]), which can be deduced from

$$M(P) \leqslant \|P\| \leqslant (N+1)^{\frac{1}{2}} H(P)$$

where

$$\|P\| = \left(\sum_{j=0}^{N} |a_j|^2 \right)^{\frac{1}{2}}$$

(see [M–W] Lemma 1).

The most important properties of Mahler's measure M concern algebraic numbers. Let α be an algebraic number, and $P \in \mathbf{Z}[X]$ be its minimal polynomial. We define the degree, usual height, length and measure of α by

$$N(\alpha) = N(P), \quad H(\alpha) = H(P), \quad L(\alpha) = L(P), \quad M(\alpha) = M(P).$$

From (2.1) we get

$$\text{(2.2)} \qquad \operatorname{Log} M(\alpha) \leqslant \operatorname{Log} H(\alpha) + \operatorname{Log} N(\alpha).$$

The multiplicative property of the measure of polynomials is no longer true for the measure of algebraic numbers (because the minimal polynomial of the product of algebraic numbers is usually different from the product of the minimal polynomials). However, if we define the "absolute logarithmic height" of α by

$$h(\alpha) = \frac{1}{N(\alpha)} \operatorname{Log} M(\alpha),$$

then $h(\alpha_1 \alpha_2) \leqslant h(\alpha_1) + h(\alpha_2)$ and $h(\alpha^m) = m h(\alpha)$ for all algebraic numbers $\alpha_1, \alpha_2, \alpha$ and any non-zero rational integer m. These properties follow from the very useful connection between h and the height on projective spaces of Neron and Lang (cf. [W]). It follows that the absolute logarithmic height is far easier to handle for transcendence proofs than other "size" functions.

We have followed the tradition in Section 1 by defining transcendence measure in terms of the usual height. But it will be much more convenient to define "approximation measures" in a slightly unusual way (compare with [Ci 1]) which will be essential in certain cases.

Let M, N be positive real numbers; for convenience we assume $M \geqslant e^e$ (to avoid some trivialities connected with the factors $\operatorname{Log}\operatorname{Log} M$), and since $15 < e^e < 16$ we assume in fact $M \geqslant 16$. Since the set $B(N, M)$ of all algebraic numbers of degree at most N and measure at most M is finite, in view of the trivial inequalities

$$H(P) \leqslant \|P\| \leqslant 2^{N(P)} M(P),$$

if ω is a transcendental number we have

$$\min_{\xi \in B(N,M)} |\omega - \xi| > 0.$$

Our aim is to give lower bounds for such positive numbers.

DEFINITION. Let $\omega \in \mathbf{C}$ be a transcendental number. A real-valued function $\psi(N, \operatorname{Log} M)$, which is defined for N, M positive real numbers, with $N \geqslant 1$, $M \geqslant 16$, is an *approximation measure for* ω if

$$|\omega - \xi| \geqslant \exp\{-\psi(N, \operatorname{Log} M)\}$$

for all algebraic numbers ξ of degree $\leqslant N$ and measure $\leqslant M$.

In the present paper we first obtain approximation measures, and then deduce transcendence measures. To carry out this deduction we use the following lemma.

LEMMA 2.3. *Let $\omega \in \mathbf{C}$ be a transcendental number, and $\psi(N, \operatorname{Log} M)$ an approximation measure for ω. Assume*

$$\psi(N_2, \operatorname{Log} M_2) \geqslant k\psi(N_1, \operatorname{Log} M_1)$$

for all positive integer k and all real numbers N_1, M_1, N_2, M_2 satisfying

$$N_1 \geqslant 1, \quad M_1 \geqslant 16, \quad N_2 \geqslant kN_1, \quad \operatorname{Log} M_2 \geqslant k \operatorname{Log} M_1.$$

Then the function

$$\psi(N, \operatorname{Log} H + \operatorname{Log} N) + 2N(\operatorname{Log} H + N)$$

is a transcendence measure for ω.

This kind of result is well known (see, for example, [Ci 1] Lemmas 2.15 and 4.3), but what is important here is that we have $\psi(N, \operatorname{Log} H + \operatorname{Log} N)$ instead of $\psi(N, \operatorname{Log} H + N)$. This improvement will be important, for example, for the transcendence measure of e^β when $\operatorname{Log} H < N$.

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PROOF OF LEMMA 2.3. Let $P \in \mathbf{Z}[X]$ be a non-trivial polynomial, and let ξ be the root of P which is at minimal distance from ω. From a well-known result of Güting (see [M–W] Lemma 9) we deduce

$$|\omega - \xi|^k \leqslant 4^{N_2^2}(2N_2 H(P))^{N_2} |P(\omega)|,$$

where $N_2 = N(P)$ and k is the multiplicity of the root ξ of P.

Let Q denote the minimal polynomial of ξ over $\mathbf{Z}$ and let N_1 be its degree. Since Q^k divides P, we have

$$N_1 \leqslant N_2/k$$

and

$$\operatorname{Log} M(Q) \leqslant \frac{1}{k} \operatorname{Log} M(P) \leqslant \frac{1}{k}(\operatorname{Log} H(P) + \operatorname{Log} N_2).$$

From the inequality (for $H = \max(16, H(P))$)

$$\operatorname{Log} H + N_2 \operatorname{Log} 4 + \operatorname{Log}(2N_2) \leqslant 2(\operatorname{Log} H + N_2)$$

we conclude

$$\operatorname{Log}|P(\omega)| \geqslant -k\psi(N_1, \operatorname{Log} M_1) - 2N_2(\operatorname{Log} H + N_2)$$

with $M_1 = \max(M(Q), 16)$. Using our assumption

$$k\psi(N_1, \operatorname{Log} M_1) \leqslant \psi(N_2, \operatorname{Log} H + \operatorname{Log} N_2)$$

we obtain the desired result.

Finally, we give a very simple lemma which will be needed in the study of exponentials and powers.

LEMMA 2.4. *Let v and w be two complex numbers satisfying*

$$|w - e^v| \leqslant \tfrac{1}{3}|e^v|.$$

then there exists a determination of the logarithm of w such that

$$|w - e^v| \geqslant \tfrac{2}{3}|e^v|\,|\log w - v|.$$

PROOF OF LEMMA 2.4. Since the principal value of the logarithm (say Log) satisfies

$$\sup_{|z| \leqslant \frac{1}{3}} |\operatorname{Log}(1+z)| < \tfrac{1}{2},$$

by the maximum modulus principle applied to the function $(1/z)\operatorname{Log}(1+z)$ we have, for $|z| \leqslant \frac{1}{3}$,

$$|\operatorname{Log}(1+z)| \leqslant \tfrac{3}{2}|z|.$$

Therefore

$$|\operatorname{Log}(we^{-v})| \leqslant \tfrac{3}{2}|we^{-v}-1|.$$

We define

$$\log w = \operatorname{Log}(we^{-v}) + v.$$

This completes the proof of Lemma 2.4.

3. The difference between an algebraic number and the logarithm of an algebraic number

In this part we derive several consequence of the following lower bound for $|\beta - \log\alpha|$.

THEOREM A. *Let* α, β *be two non-zero algebraic numbers, and let* $\log\alpha$ *be any determination of the logarithm of* α. *Let* D *be a positive integer, and* V, E *be positive real numbers, satisfying*

$$D \geqslant [Q(\alpha,\beta) : Q],$$

$$V \geqslant \max\{h(\alpha);\ |\log\alpha|/D;\ 1/D\}$$

and

$$1 < E \leqslant \min\{e^{DV};\ 4DV/|\log\alpha|\}.$$

Finally set $V^+ = \max\{V, 1\}$. *Then*

$$|\beta - \log\alpha| > \exp\{-2^{35} D^3 V(h(\beta) + \operatorname{Log}(EDV^+))(\operatorname{Log}(ED))(\operatorname{Log} E)^{-2}\}.$$

This estimate is a special case of Theorem C, Section 5, and is proved in [W]. We show how it can be used to study the algebraic approximations of the numbers π, $\log\alpha$ and e^β.

1. *Algebraic approximations to the number* π

According to a remarkable result of Mahler [Ma 3, 5] and an improved version of Mignotte [Mi], for every rational number p/q with $q \geqslant 2$ we have

$$\left|\pi - \frac{p}{q}\right| \geqslant q^{-21}.$$

Here we consider algebraic approximations of π. Let ξ be a real algebraic number of degree N, with $N \geqslant 2$. Define $\alpha_1 = -1$, $\log\alpha_1 = i\pi$, $\beta = i\xi$, $D = 2N$, $V = \pi/D$ and $E = 4$. From Theorem A we deduce

$$\begin{aligned}|\xi - \pi| &= |\beta - i\pi| \\ &\geqslant \exp\{-2^{35}\pi D^2(h(\beta) + \operatorname{Log} D + \operatorname{Log} 4)(\operatorname{Log} D + \operatorname{Log} 4).(\operatorname{Log} 4)^{-2}\}.\end{aligned}$$

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Let M satisfy $M \geqslant \max\{e^{Nh(\xi)}, 16\}$. Since $h(\xi) = h(\beta)$, the inequalities

$$\operatorname{Log} M + N \operatorname{Log} N + 3N \operatorname{Log} 2 \leqslant \tfrac{21}{10}(\operatorname{Log} M + N \operatorname{Log} N)$$

$$\operatorname{Log} N + 3N \operatorname{Log} 2 \leqslant \tfrac{17}{10}(1 + \operatorname{Log} N)$$

and

$$\pi \tfrac{21}{10} \cdot \tfrac{17}{10} (\operatorname{Log} 2)^{-2} < 24$$

lead to the following result.

THEOREM 3.1. *An approximation measure for* π *is*

$$3 . 2^{38} . N(\operatorname{Log} M + N \operatorname{Log} N)(1 + \operatorname{Log} N).$$

From Lemma 2.3 and the inequality

$$\operatorname{Log} H + N \operatorname{Log} N + \operatorname{Log} N \leqslant \tfrac{6}{5}(\operatorname{Log} H + N \operatorname{Log} N)$$

(valid for $H \geqslant 16$ and $N \geqslant 2$) we deduce

COROLLARY 3.2. *A transcendence measure for* π *is*

$$2^{40} N(\operatorname{Log} H + N \operatorname{Log} N)(1 + \operatorname{Log} N).$$

The existence of an absolute constant $C_{11} > 0$ such that

$$C_{11} N(\operatorname{Log} \mathrm{H} + N \operatorname{Log} N)(1 + \operatorname{Log} N)$$

is a transcendence measure for π was first announced (and proved) by Cijouw in [Ci 5]. Actually it can be deduced from two earlier results of Fel'dman, [F 2], namely

$$(3.3) \qquad |P(\pi)| > \exp\{-C_{12} N(\operatorname{Log} H + N \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)\}$$

and, provided that $\operatorname{Log} H > N^2(\operatorname{Log} N)^4$,

$$(3.4) \qquad |P(\pi)| > \exp\{-C_{13} N(\operatorname{Log} H)(1 + \operatorname{Log} N)\}.$$

For if $\operatorname{Log} H \geqslant N^3$, then (3.4) gives what we want, while if $\operatorname{Log} H \leqslant N^3$, then (3.3) gives

$$|P(\pi)| > \exp\{-3C_{12} N(\operatorname{Log} H + N \operatorname{Log} N)(1 + \operatorname{Log} N)\}$$

which is also what we want.

Earlier transcendence measures for π were due to Popken (1929), Siegel (1930) and Mahler [Ma 1, 3]. (See [F–S], [Ci 1, 5] and [F 4]).

In some special cases, as already remarked by Fel'dman in 1959 (cf. F 4]), Theorem 3.1 can be improved.

Let $m \geqslant 1$ be an integer and ξ a real algebraic number; let D_m denote the degree over Q of the field $Q(\xi, e^{i\pi/m}, i)$. From Theorem A applied to $(i\xi/m)-(i\pi/m)$, with $\alpha = e^{i\pi/m}$, $\log\alpha = i\pi/m$, $V = (\pi + \operatorname{Log} m)/D_m$, $E = em$, and since

$$h(\xi/m) \leqslant h(\xi) + \operatorname{Log} m,$$

we conclude

$$|\pi - \xi| \geqslant \left|\frac{i\xi}{m} - \frac{i\pi}{m}\right|$$

$$> \exp\left\{-2^{35}\pi D_m^2(h(\xi) + 1 + 2\operatorname{Log} m + \operatorname{Log} D_m)\left(1 + \frac{\operatorname{Log} D_m}{1 + \operatorname{Log} m}\right)\right\}$$

(We choose V rather large to enable E to be large.)

THEOREM 3.5. *Let ξ be an algebraic number and m a positive integer. Define*

$$D_m = [Q(i, \xi, e^{i\pi/m}) : Q].$$

Then

$$|\pi - \xi| > \exp\left\{-2^{38} D_m^2(h(\xi) + \operatorname{Log} m + \operatorname{Log} D_m)\left(1 + \frac{\operatorname{Log} D_m}{1 + \operatorname{Log} m}\right)\right\}.$$

In the particular case where $\beta \in Q(e^{i\pi/m})$, we obtain

$$|\pi - \xi| > \exp\{-2^{40} D_m^2(h(\xi) + \operatorname{Log} D_m)\},$$

while Fel'dman obtains

$$|\pi - \xi| > \exp\{-C_{14} D_m^2(N(\xi)^{-1} \operatorname{Log} L(\xi) + 1)\},$$

where C_{14} is an effectively computable absolute constant. In spite of the inequality $N(\xi)\,h(\xi) \leqslant \operatorname{Log} L(\xi)$, Fel'dman's result is sometimes better in some cases where m is large. On the other hand, our Theorem 3.5 does not assume $\beta \in Q(e^{i\pi/m})$.

2. *Approximation of the logarithms of algebraic numbers*

Let α be a non-zero algebraic number and let $\log\alpha$ be a non-zero determination of the logarithm of α. Transcendence measures for $\log\alpha$ have been proved successively by Mahler [Ma 1, 2], Fel'dman [F 1, 3] and Cijsouw [Ci 1, 2, 4]. The best known result is due to Cijsouw [Ci 4], and we give a new proof of it.

We define

$$V = \max\{h(\alpha);\ |\log\alpha|;\ 1\}, \quad d = [Q(\alpha) : Q].$$

Let ξ be an algebraic number of degree N and measure at most M, with $M \geqslant 16$. From Theorem A with $D = Nd$, $E = eD$, we deduce

$$|\log\alpha - \xi| > \exp\left\{-2^{35} d^3 VN^3\left(\frac{1}{N}\operatorname{Log} M + 2\operatorname{Log} N + 2\operatorname{Log} d + 1 + \operatorname{Log} V\right)\right.$$
$$\left.\times (1 + 2\operatorname{Log} N + 2\operatorname{Log} d)(1 + \operatorname{Log} d + \operatorname{Log} N)^{-2}\right\}.$$

Let us define

$$C_{15}(\alpha) = 2^{38} d^3 V(1 + \operatorname{Log} d + \operatorname{Log} V).$$

We get:

THEOREM 3.6. *An approximation measure for* $\log\alpha$ *is*

$$C_{15}(\alpha) N^2(\operatorname{Log} M + N\operatorname{Log} N)(1 + \operatorname{Log} N)^{-1}.$$

COROLLARY 3.7. *A transcendence measure for* $\log\alpha$ *is*

$$2C_{15}(\alpha) N^2(\operatorname{Log} H + N\operatorname{Log} N)(1 + \operatorname{Log} N)^{-1}.$$

3. *Approximation of the exponentials of algebraic numbers*

The problem of finding a transcendence measure for the number e is very old (at least for the standard of the theory) since it was initiated as soon as 1899 by Borel. Later, Popken [P], Siegel (1930), Mahler (1932), [Ma 1] and Fel'dman (1963) improved the bound, and the sharpest result until now was due to Cijsouw [Ci 1, 2] and was valid also for the numbers e^β with β algebraic, $\beta \neq 0$. Our result will be slightly better.

Let β be a non-zero algebraic number and d be its degree. Let ξ be an algebraic number of degree N and measure at most M, with $M \geqslant 16$, satisfying $|e^\beta - \xi| \leqslant \frac{1}{3}|e^\beta|$. From Lemma 2.4 we can find a determination $\log\xi$ of the logarithm of ξ such that

$$|e^\beta - \xi| \geqslant \tfrac{2}{3}|e^\beta|\,|\beta - \log\xi|.$$

We now use Theorem A with

$$V = \frac{1}{N}(\operatorname{Log} M)(1 + |\beta|), \quad D = dN, \quad E = \operatorname{Log} M.$$

Thus

$$|\beta - \log\xi| \geqslant \exp\{-2^{35} d^3 N^2(\operatorname{Log} M)(1 + |\beta|)$$
$$\times (h(\beta) + \operatorname{Log} d + \operatorname{Log} N + 2\operatorname{Log}\operatorname{Log} M + \operatorname{Log}(1 + |\beta|))$$
$$\times (\operatorname{Log} d + \operatorname{Log}\operatorname{Log} M + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} M)^{-2}\}.$$

Let us define

$$C_{16}(\beta) = 2^{37} d^3(1 + |\beta|)(1 + \operatorname{Log} d)(h(\beta) + \operatorname{Log} d + \operatorname{Log}(1 + |\beta|) + 1).$$

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THEOREM 3.8. *An approximation measure for e^β is*

$$C_{16}(\beta)\,N^2(\operatorname{Log} M)\,(\operatorname{Log}\operatorname{Log} M+\operatorname{Log} N)^2\operatorname{Log}\operatorname{Log} M)^{-2}.$$

If we bound $\operatorname{Log} H+\operatorname{Log} N$ by $\operatorname{Log} H+N$, we deduce from Theorem 3.8 and Lemma 2.3 that

$$4C_{16}(\beta)\,N^2(\operatorname{Log} H+N)$$

is a transcendence measure for e^β. This is Cijsouw's result [Ci 1, 2]. But, for $H\leqslant e^N$, we actually get a better result.

COROLLARY 3.9. *A transcendence measure for e^β is*

$$8C_{16}(\beta)\,N^2(\operatorname{Log}\mathrm{H}+\operatorname{Log} N)\,(\operatorname{Log}\operatorname{Log} H+\operatorname{Log} N)^2\,(\operatorname{Log}\operatorname{Log} H+\operatorname{Log}_+\operatorname{Log} N)^{-2}.$$

It is interesting to notice that the old result of Mahler [Ma 1]:

$$|P(e^\beta)|>\exp\{-C_{17}\,N\operatorname{Log} H\}\quad\text{for } H\geqslant H_0(N)$$

is still better for large H.

4. The linear form $\beta\log\alpha_1-\log\alpha_2$

We study now several consequences of the following estimate.

THEOREM B. *Let α_1, α_2, β be non-zero algebraic numbers, and* $\log\alpha_1$, $\log\alpha_2$ be *determinations of the logarithms of α_1, α_2 respectively. We assume*

$$\beta\log\alpha_1\neq\log\alpha_2.$$

Let D be a positive integer, and V_1, V_2, E be positive real numbers, satisfying

$$D\geqslant[Q(\alpha_1,\alpha_2,\beta):Q],$$

$$V_j\geqslant\max\{h(\alpha_j),\,|\log\alpha_j|/D,\,1/D\}\quad(j=1,2),$$

and

$$1<E\leqslant\min\{e^{DV_1},e^{DV_2},4DV_1/|\log\alpha_1|,\,4DV_2/|\log\alpha_2|\}.$$

Further define

$$V_j^+=\max\{V_j,1\}\quad(j=1,2),\quad V^*=\max\{V_1^+,V_2^+\},\quad\text{and}\quad V_*=\min\{V_1^+,V_2^+\}.$$

Then

$$|\beta\log\alpha_1-\log\alpha_2|>\exp\{-2^{53}\,D^4\,V_1\,V_2(h(\beta)+\operatorname{Log}(EDV^*))\,(\operatorname{Log}(EDV_*))\,(\operatorname{Log} E)^{-3}\}.$$

This result, which is proved in [W] (see Theorem C, Section V below), will be our main tool in the investigation of diophantine approximations of the numbers $(\log\alpha_1)/(\operatorname{Log}\alpha_2)$ and α^β. We begin with the number e^π.

1. *Approximation to e^π*

We first consider rational approximations to e^π. The best known result is due to Baker [B]: there exists an absolute constant C_{18} such that

$$\left|e^\pi - \frac{p}{q}\right| > q^{-C_{18}\operatorname{Log}\operatorname{Log} q}$$

for all rational numbers p/q with $q \geqslant 3$ (it is not yet known whether the extra $\operatorname{Log}\operatorname{Log} q$ is superfluous). We deduce from Theorem B an upper bound for C_{18} as follows.

Assume first $q > (e^\pi + 1)^2$ and $|e^\pi - (p/q)| \leqslant 1$, so that

$$\left|\pi - \operatorname{Log}\frac{p}{q}\right| \leqslant \left|e^\pi - \frac{p}{q}\right| \quad \text{and} \quad p < (e^\pi + 1)q < q^{\frac{3}{2}}.$$

We choose

$$\beta = -i,\quad \alpha_1 = -1,\quad \log\alpha_1 = i\pi,\quad \alpha_2 = p/q,\quad D = 2,\quad V_1 = \tfrac{1}{2}\pi,\quad V_2 = \operatorname{Log} p,\quad E = 4.$$

Thus

$$\left|e^\pi - \frac{p}{q}\right| > \exp\left\{-2^{53}2^4\frac{\pi}{2}(\operatorname{Log} p)(\operatorname{Log} 8 + \operatorname{Log}\operatorname{Log} p)(\operatorname{Log} 4\pi)(\operatorname{Log} 4)^{-3}\right\}$$
$$> \exp\{-2^{59}(\operatorname{Log} p)(\operatorname{Log}\operatorname{Log} p)\}$$
$$> \exp\{-2^{60}(\operatorname{Log} q)(\operatorname{Log}\operatorname{Log} q)\}.$$

From the continued fraction expansion

$$e^\pi = [23, 7, 9, 3, 1, 1, 591, \ldots]$$

we see that these results hold also for $3 \leqslant q < 583$. Therefore

$$C_{18} \leqslant 2^{60}.$$

We now consider algebraic approximations. Let ξ be a real algebraic number with $|\xi - e^\pi| \leqslant 1$. The (usual) logarithm of ξ satisfies, by Lemma 2.4,

$$|\pi - \operatorname{Log}\xi| \leqslant |e^\pi - \xi|.$$

We use Theorem *B* with

$$D = 2N,\quad N \geqslant 2,\quad V_1 = \pi/D,\quad V_2 = (1/N)\operatorname{Log} M,\quad E = 4.$$

We obtain

$$|e^\pi - \xi| > \exp\left\{-2^{53}\, 2^4 \frac{\pi}{2} N^2 (\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M + \operatorname{Log} N + \operatorname{Log} 8)\right.$$
$$\left.(\operatorname{Log} N + \operatorname{Log} 8)(\operatorname{Log} 4)^{-3}\right\}$$

THEOREM 4.1. *An approximation measure for e^π is*

$$2^{59} N^2 (\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M + \operatorname{Log} N)(1 + \operatorname{Log} N).$$

COROLLARY 4.2. *A transcendence measure for e^π is*

$$2^{60} N^2 (\operatorname{Log} H + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)(1 + \operatorname{Log} N).$$

Several other estimates for e^π are known: Koksma and Popken [K–P], Gel'fond [G], Cijsouw [Ci 1] and Choodnovsky [Ch]. Two of them are not consequences of Corollary 4.2, namely

$$C_{19} N (\operatorname{Log} H + N)^2 (\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)^{-1}$$

(Cijsouw, "Stellingen" of [Ci 1]), and

$$C_{20} N (\operatorname{Log} H + \operatorname{Log} N)^2 (\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)$$

(Choodnovsky [Ch]), where C_{19}, C_{20} are effectively computable absolute constants.

The three above-mentioned transcendence measures for e^π hold uniformly ($N \geqslant 1$, $H \geqslant 16$), but our Corollary 4.2 is better when $\operatorname{Log} H \geqslant N(\operatorname{Log} \mathrm{N})^3$ (in which case we can bound $(\operatorname{Log} H + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)$ by $3(\operatorname{Log} H)(\operatorname{Log}\operatorname{Log} H)$), Cijsouw's result is better when $N(\operatorname{Log} N)^3 \geqslant \operatorname{Log} \mathrm{H} \geqslant N(\operatorname{Log} N)^{-1}$ (and then $\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N \geqslant \operatorname{Log}\operatorname{Log} H$ does not weaken the result), while Choodnovsky's result is better when $N(\operatorname{Log} N)^{-1} \geqslant \operatorname{Log} H$ (which implies $\operatorname{Log}\operatorname{Log} H \leqslant \operatorname{Log} N$). Consequently we obtain the results which we announced in Fig. 2.

We can use Theorem B in the same way as we did for the proof of Theorem 3.5: let $m \geqslant 1$ be an integer; we choose

$$\alpha_1 = \xi, \quad \beta = i/m, \quad \alpha_2 = e^{i\pi/m}, \quad \log \alpha_2 = i\pi/m,$$
$$V_1 = h(\xi) + m(\pi - 1) D_m^{-1}, \quad V_2 = (\pi + \operatorname{Log} m) D_m^{-1}, \quad E = em.$$

THEOREM 4.3. *Let ξ be an algebraic number and m a positive integer. Denote by D_m the degree of the field $Q(i, \xi, e^{i\pi/m})$. Then*

$$|e^\pi - \xi| > \exp\{-2^{60} D_m^2 (D_m h(\xi) + m)(\operatorname{Log} h(\xi) + \operatorname{Log} D_m + \operatorname{Log} m + 1)$$
$$\times (\operatorname{Log} D_m + \operatorname{Log} m + 1)(\operatorname{Log} m + 1)^{-2}\}.$$

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2. *Approximation of the quotient of the logarithms of algebraic numbers*

Let α_1, α_2 be two non-zero algebraic numbers, and $\log\alpha_1, \log\alpha_2$ be non-zero determinations of their logarithms. We assume that $(\log\alpha_1)/(\log\alpha_2)$ is irrational. Transcendence measures for this number are due to Gel'fond (1935, 1939, 1949) (see [F–S] and [G]) and to Cijsouw [Ci 1]. Our Theorem B leads to a sharpening of these measures. Let us define

$$d = [Q(\alpha_1, \alpha_2) : Q], \quad V_j = 1 + h(\alpha_j) + |\log\alpha_j|, \quad (j = 1, 2),$$

and

$$C_{21}(\alpha_1, \alpha_2) = 2^{55} d^4 V_1 V_2 (1 + \operatorname{Log} V_1)(1 + \operatorname{Log} V_2).$$

We choose $D = dN$, $E = eD$.

THEOREM 4.4. *An approximation measure for* $(\log\alpha_1)/(\log\alpha_2)$ *is*

$$C_{21}(\alpha_1, \alpha_2) N^3 (\operatorname{Log} M + N \operatorname{Log} N)(1 + \operatorname{Log} N)^{-2}.$$

COROLLARY 4.5. *A transcendence measure for* $(\log\alpha_1)/(\log\alpha_2)$ *is*

$$2C_{21}(\alpha_1, \alpha_2) N^3 (\operatorname{Log} H + N \operatorname{Log} N)(1 + \operatorname{Log} N)^{-2}.$$

3. *Approximation of algebraic powers of algebraic numbers*

Let α and β be non-zero algebraic numbers, with β irrational, and let $\log\alpha$ be a non-zero determination of the logarithm of α. Define as usual $\alpha^\beta = \exp(\beta\log\alpha)$. Further let

$$d = [Q(\alpha, \beta) : Q], \quad V = 1 + h(\alpha) + |\log\alpha|.$$

Let ξ be an algebraic number of degree N and measure at most M, with $M \geqslant 16$, such that

$$|\alpha^\beta - \xi| \leqslant \tfrac{1}{3}|\alpha^\beta|.$$

By Lemma 2.4 we can choose a determination $\log\xi$ of the logarithm of ξ such that

$$|\alpha^\beta - \xi| \geqslant \tfrac{2}{3}|\alpha^\beta| . |\beta\log\alpha - \log\xi|.$$

We now choose

$$D = Nd, \quad V_1 = V, \quad V_2 = \frac{1}{N}(\operatorname{Log} M)(1 + |\beta\log\alpha|),$$

and

$$E = \min\{eN, \operatorname{Log} M\}.$$

Finally define

$$C_{22}(\beta, \log\alpha) = 6 . 2^{53} d^4 V (1 + |\beta\log\alpha|)$$
$$(1 + h(\beta) + \operatorname{Log} V + \operatorname{Log} d + |\beta\log\alpha|)(1 + \operatorname{Log} V + \operatorname{Log} d).$$

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Then Theorem B gives

$$|\alpha^\beta - \xi| > \exp\{-\tfrac{1}{6}C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M + \operatorname{Log} N + \operatorname{Log} E) \times (\operatorname{Log} N + \operatorname{Log} E)(\operatorname{Log} E)^{-3}\}.$$

In the case $\operatorname{Log} M \geqslant eN$ we get

$$|\alpha^\beta - \xi| > \exp\{-C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M)(1 + \operatorname{Log} N)^{-2}\}.$$

While in the case $\operatorname{Log} M \leqslant eN$ we obtain

$$|\alpha^\beta - \xi| > \exp\{-C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} M)(1 + \operatorname{Log} N)^2(\operatorname{Log}\operatorname{Log} M)^{-3}\}.$$

We conclude

THEOREM 4.6. *An approximation measure for* α^β *is*

$$C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M + \operatorname{Log} N)^4(\operatorname{Log}\operatorname{Log} M)^{-3}(1 + \operatorname{Log} N)^{-2}.$$

From Lemma 2.3 we deduce that

$$2^4 C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} H + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)^4 \times (\operatorname{Log}\operatorname{Log} H + \operatorname{Log}_+ \operatorname{Log} N)^{-3}(1 + \operatorname{Log} N)^{-2}$$

is a transcendence measure for α^β. This result can be improved in the following way. If $\operatorname{Log} H > N$, this transcendence measure is less than

$$2^9 C_{22}(\beta, \log\alpha)\, N^3(\operatorname{Log} H)(\operatorname{Log}\operatorname{Log} H)(1 + \operatorname{Log} N)^{-2}.$$

On the other hand, we know by [Ch] Theorem 1.5 that there exists a constant $C_{23}(\beta, \log\alpha)$ such that

$$C_{23}(\beta, \log\alpha)\, N^3(\operatorname{Log} H + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)^2(1 + \operatorname{Log} N)^{-3}$$

is a transcendence measure for α^β. When $\operatorname{Log} H \leqslant N$, this measure is less than

$$4C_{23}(\beta, \log\alpha)\, N^3(\operatorname{Log} H + \operatorname{Log} N)(1 + \operatorname{Log} N)^{-1}.$$

These two estimates enable us to conclude

THEOREM 4.7. *There exists a positive real number* $C_{24}(\beta, \log\alpha)$ *such that*

$$C_{24}(\beta, \log\alpha)\, N^3(\operatorname{Log} H + \operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H + \operatorname{Log} N)(1 + \operatorname{Log} N)^{-2}$$

is a transcendence measure for α^β.

This measure improves earlier results of Gel'fond (see [G], [F–S], [(Ci 1]). In the special case where β is quadratic, Šidlovskii proved in 1951 a result which is sharper for large N (see [F–S]).

5. Two general transcendence measures

In this last part we give two results, one concerning the quotient of linear forms in logarithms and the other concerning $e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_m^{\beta_m}$. Our main tool is the following estimate which is proved in [W]. (For earlier estimates we refer to the paper of Baker in the same volume.)

THEOREM C. *Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers, and $\beta_0,\beta_1,\dots,\beta_n$ be algebraic numbers. For $1\leqslant j\leqslant n$, let $\log\alpha_j$ be any determination of the logarithm of α_j. Let D be a positive integer, and $V_1,\dots,V_n$ W, E be positive real numbers, satisfying*

$$D\geqslant[Q(\alpha_1,\dots,\alpha_n,\beta_0,\dots,\beta_n):Q],$$

$$V_j\geqslant\max\{h(\alpha_j),|\log\alpha_j|/D,1/D\}\quad(1\leqslant j\leqslant n),$$

$$W\geqslant\max_{0\leqslant j\leqslant n}\{h(\beta_j)\},$$

$$V_1\leqslant\dots\leqslant V_n$$

and

$$1<E\leqslant\min\{e^{DV_1};\ \min_{1\leqslant j\leqslant n}4DV_j/|\log\alpha_j|.$$

Finally define $V_j^+=\max\{V_j,1\}$ for $j=n$ and $j=n-1$, with $V_0^+=1$ in the case $n=1$. If the number

$$\Lambda=\beta_0+\beta_1\log\alpha_1+\dots+\beta_n\log\alpha_n$$

does not vanish, then

$$|\Lambda|>\exp\{-C(n)\,D^{n+2}\,V_1\dots V_n(W+\mathrm{Log}(EDV_n^+))(\mathrm{Log}\,EDV_{n-1}^+))(\mathrm{Log}\,E)^{-n-1}\},$$

where

$$C(1)\leqslant 2^{35},\quad C(2)\leqslant 2^{53}\quad\text{and}\quad C(n)\leqslant 2^{8n+51}n^{2n}.$$

1. *Quotient of linear forms in logarithms*

Let $\alpha_1,\dots,\alpha_s$, $\alpha_1',\dots,\alpha_t'$ be non-zero algebraic numbers, $\beta_0,\dots,\beta_s$, $\beta_0',\dots,\beta_t'$ be algebraic numbers. For $1\leqslant j\leqslant s$ (resp. $1\leqslant k\leqslant t$) let $\log\alpha_j$ (resp. $\log\alpha_k'$) be a determination of the logarithm of α_j (resp. of α_k'). We assume that

$$\beta_0'+\beta_1'\log\alpha_1'+\dots+\beta_t'\log\alpha_t'$$

does not vanish, and that the number

$$\omega=\frac{\beta_0+\beta_1\log\alpha_1+\dots+\beta_s\log\alpha_s}{\beta_0'+\beta_1'\log\alpha_1'+\dots+\beta_t'\log\alpha_t'}$$

is transcendental. (Thanks to Baker [B] Chapter 3 we know when this assumption is fullfilled.) We give here an estimate for the approximation of ω which improves substantially previous results of Cijsouw [Ci 1, 3].

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THEOREM 5.1. *There exists an easily computable constant* C_{25}, *depending only on* $\log \alpha_1, \ldots, \log \alpha_s$, $\log \alpha'_1, \ldots, \log \alpha'_t$, $\beta_0, \ldots, \beta_s$, $\beta'_0, \ldots, \beta'_t$, *with the following property.*

1. *Let* r_1+1 *be the dimension of the Q-vector space generated by* $\log \alpha_1, \ldots, \log \alpha_s$, $\log \alpha'_1, \ldots, \log \alpha'_t$, $i\pi$. *Then*

$$C_{25} N^{r_1+1}(\operatorname{Log} M + N \operatorname{Log} N)(1+\operatorname{Log} N)$$

is an approximation measure for ω, *and consequently*

$$2C_{25} N^{r_1+1}(\operatorname{Log} H + N \operatorname{Log} N)(1+\operatorname{Log} N)$$

is a transcendence measure for ω.

2. *Let* r_2 *be the dimension of the Q-vector space generated by* $\log \alpha_1, \ldots, \log \alpha_s$, $\log \alpha'_1, \ldots, \log \alpha'_t$. *Then*

$$C_{25} N^{r_2+1}(\operatorname{Log} M + N \operatorname{Log} N)(1+\operatorname{Log} N)^{-r_2}$$

is an approximation measure for ω, *and consequently*

$$2C_{25} N^{r_2+1}(\operatorname{Log} H + N \operatorname{Log} N)(1+\operatorname{Log} N)^{-r_2}$$

is a transcendence measure for ω.

Plainly we have $r_1 = r_2$ or $r_1 = r_2 - 1$. If $r_1 = r_2$ the second result is slightly sharper, while if $r_1 = r_2 - 1$ (which means that $i\pi$ is a linear combination of $\log \alpha_1, \ldots, \log \alpha'_t$ with rational coefficients) the first estimate is quite sharp.

From Theorem 5.1 it follows that the transcendence type of ω is at most $r_1+2+\varepsilon$ for all $\varepsilon > 0$ and at most r_2+2.

It is readily seen that Theorem 5.1 contains the above-mentioned transcendence measures for π (with $r_1 = 0$; cf. (3.2)), for $\log \alpha$ (with $r_2 = 1$; cf. (3.7)) and for $(\log \alpha_1)/(\log \alpha_2)$ (with $r_2 = 2$; cf. (4.5)). Here is another consequence corresponding to $r_1 = 1$.

COROLLARY 5.2. *Let* $\log \alpha$ *be a logarithm of an algebraic number, and let* β_0, β_1, β'_0, β'_1, γ, γ' *be algebraic numbers such that the number*

$$\beta'_0 + \beta'_1 \log \alpha + \gamma' i\pi$$

does not vanish, and that the number

$$\frac{\beta_0 + \beta_1 \log \alpha + \gamma i\pi}{\beta'_0 + \beta'_1 \log \alpha + \gamma' i\pi}$$

is transcendental. Then this number has a transcendence measure

$$C_{26} N^2(\operatorname{Log} H + N \operatorname{Log} N)(1+\operatorname{Log} N),$$

where C_{26} *depends only on* $\log \alpha$, β_0, β_1, β'_0, β'_1, γ, γ'.

Therefore this number has a transcendence type at most $3+\varepsilon$ for all $\varepsilon > 0$.

PROOF OF THEOREM 5.1. Let $\log a_1, \ldots, \log a_{r_1+1}$ be a basis of the Q-vector space generated by $\log\alpha_1, \ldots, \log\alpha_s$, $\log\alpha'_1, \ldots, \log\alpha'_t$, $i\pi$, with $\log a_1 = i\pi$. Thus there exist rational numbers $b_{j,l}$, $b'_{k,l}$ $(1 \leqslant j \leqslant s,\ 1 \leqslant k \leqslant t,\ 1 \leqslant l \leqslant r_1+1)$ such that

$$\log\alpha_j = \sum_{l=1}^{r_1+1} b_{j,l} \log a_l \quad (1 \leqslant j \leqslant s),$$

and

$$\log\alpha'_k = \sum_{l=1}^{r_1+1} b'_{k,l} \log a_l \quad (1 \leqslant k \leqslant t).$$

Let ξ be an algebraic number of degree N and measure at most M, with $M \geqslant 16$. Then

$$0 < |\omega - \xi| = |\Lambda| . |\beta'_0 + \beta'_1 \log\alpha'_1 + \ldots + \beta'_t \log\alpha'_t|,$$

where

$$\Lambda = \beta_0 + \sum_{j=1}^{s} \beta_j \log\alpha_j - \xi\beta'_0 - \xi \sum_{k=1}^{t} \beta'_t \log\alpha'_k$$

$$= (\beta_0 - \xi\beta'_0) + \sum_{l=1}^{r_1+1} \left(\sum_{j=1}^{s} \beta_j b_{j,l} - \xi \sum_{k=1}^{t} \beta'_k b'_{k,l} \right) \log a_l.$$

Theorem C provides us a lower bound for $|\Lambda|$, with

$$n = r_1 + 1, \quad W = C_{27}(1/N)\operatorname{Log} M, \quad V_1 = 1/D, \quad D = C_{28} N, \quad E = 4$$

and C_{27}, C_{28}, $V_2, \ldots, V_n$ are easily computable constants (for instance

$$C_{28} = [Q(\beta_0, \ldots, \beta'_t, \ldots, \alpha'_t)]).$$

Now let $\log a'_1, \ldots, \log a'_{r_2}$ be a basis of the Q vector space generated by $\log\alpha_1, \ldots, \log\alpha'_t$. Using the previous arguments, we need a lower bound for

$$\Lambda = \beta_0 - \xi\beta'_0 + \sum_{l=1}^{r_2} \gamma_l \log a'_l,$$

where γ_l is a linear combination of 1 and ξ with fixed coefficients in $Q(\beta_0, \ldots, \beta'_t)$. We now choose $n = r_2$, $E = eN$, while $V_1, \ldots, V_n$ are constants. The desired result plainly follows from Theorem C. (The non-vanishing of Λ is a consequence of the transcendence of ω.)

2. *Product of algebraic powers of algebraic numbers*

Let $\alpha_1, \ldots, \alpha_m$ be non-zero algebraic numbers, $\beta_0, \ldots, \beta_m$ be algebraic numbers, and, for $1 \leqslant j \leqslant m$, let $\log\alpha_j$ be any determination of the logarithm of α_j. We assume that the number

$$\theta = \exp\left(\beta_0 + \sum_{j=1}^{m} \beta_j \log\alpha_j\right),$$

which we write for shortness

$$\theta = e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_m^{\beta_m},$$

is transcendental (once more we refer to [B] to see which are the only trivial circumstances where such a number could be algebraic).

THEOREM 5.3. *There exists an easily computable constant C_{29}, depending only on* $\log\alpha_1, \dots, \log\alpha_m, \beta_0, \dots, \beta_m$, *such that the following holds.*

1. *Denote by r_1+1 the dimension of the Q-vector space generated by* $\log\alpha_1, \dots, \log\alpha_m, i\pi$. *Then* $C_{29}N^{r_1+2}(\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M+\operatorname{Log} N)(1+\operatorname{Log} N)$ *is an approximation measure for θ. Therefore*

$$2C_{29}N^{r_1+2}(\operatorname{Log} H+\operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H+\operatorname{Log} N)(1+\operatorname{Log} N)$$

is a transcendence measure for θ.

2. *Denote by r_2 the dimension of the Q-vector space generated by* $\log\alpha_1, \dots, \log\alpha_m$. *Then*

$$C_{29}N^{r_2+2}(\operatorname{Log} M)(\operatorname{Log}\operatorname{Log} M+\operatorname{Log} N)^{r_2+3}(\operatorname{Log}\operatorname{Log} M)^{-r_2-2}(1+\operatorname{Log} N)^{-r_2-1}$$

is an approximation measure for θ, and therefore

$$2C_{29}N^{r_2+2}(\operatorname{Log} H+\operatorname{Log} N)(\operatorname{Log}\operatorname{Log} H+\operatorname{Log} N)^{r_2+3}$$
$$(\operatorname{Log}\operatorname{Log} H)^{-r_2-2}(1+\operatorname{Log} N)^{-r_2-1}$$

is a transcendence measure for θ.

Again the first estimate is much more interesting if $r_1 = r_2-1$ and the second if $r_1 = r_2$.

The transcendence type of θ is at most $r_1+3+\varepsilon$ for all $\varepsilon>0$, and at most r_2+3.

Theorem 5.3 generalizes our approximation measure for α^β (with $r_2 = 1$; cf. (4.6)) and our transcendence measure for e^π (with $r_1 = 0$; cf. (4.2)). Indeed, the case $r_1 = 0$ shows that the transcendence measure (4.2) of the number e^π holds also for the numbers $e^{\beta+\gamma\pi}$ with β and γ algebraic.

Earlier transcendence measures for θ were due to Cijsouw [Ci 1, 3]. In the case of bounded degree, a partial result of Smelev (1969) concerning the case $m = 1$ has been improved and extended to the general case by Baker [B] Chapter 3. All these results follow from Theorem 5.3.

PROOF OF THEOREM 5.3. Let ξ be an algebraic number with $|\xi-\theta|<|\theta|/3$. Using Lemma 2.4, we reduce the problem to a lower bound of

$$\beta_0+\gamma_1\log\alpha_1+\dots+\gamma_{n-1}\log\alpha_{n-1}-\log\xi,$$

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with $n=r_1+2$ and $\log\alpha_1 = i\pi$ in the first case, $n=r_2+1$ in the second case; $\gamma_1, \ldots, \gamma_n$ are fixed algebraic numbers.

We use Theorem C with $V_n = C_{30}(1/N)(\operatorname{Log} M)$, and

$$V_2, \ldots, V_{n-1}, \quad W \text{ fixed}, \quad V_1 = 1/N, \quad \text{and} \quad E=4$$

in the first case,

$$V_1, V_2, \ldots, V_{n-1}, \quad W \text{ fixed}, \quad \text{and} \quad E = \min\{eN, \operatorname{Log} M\}$$

in the second case.

This completes the proof of Theorem 5.3.

A final remark

It is rather surprising that so few results are not consequences of our lower bound [W]. However, it seems worthwhile to go on in this field by looking at special cases, like π (cf. [F 4]), or e^π (cf. [Ch] in the case of large degree) and to use the specific properties of these numbers to improve the known results. In fact, the sharpest known results are far from best possible (namely $C_{31}\, N \operatorname{Log} H$ for the transcendence measures) and there is still a lot of work to do.

From the continued fraction expansions of π and e^π up to 80 places, which were kindly provided to me by David Hunt, it follows that the only exceptions to

$$\left|\pi - \frac{p}{q}\right| > q^{-3}$$

in the range $2 \leqslant q \leqslant 10^{41}$ are for $q=7$ and $q=113$, and similarly the only exceptions to

$$\left|e^\pi - \frac{p}{q}\right| > q^{-3}$$

in the range $2 \leqslant q \leqslant 10^{44}$ are for $q=7$ and $q=462$.

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