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ON THE FREQUENT UNIVERSALITY OF UNIVERSAL TAYLOR SERIES IN THE COMPLEX PLANE

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Abstract. We prove that the classical universal Taylor series in the complex plane are never frequently universal. On the other hand, we prove the 1-upper frequent universality of all these universal Taylor series.

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1. Introduction and definitions. The theory of universal series is a very active branch of mathematical analysis, which has received an increasing interest since the work of Nestoridis in 1996 [**12**]. The first notion of universal Taylor series in the complex plane was obtained in 1951 by Seleznev [**15**], who showed that there exists a formal power series $\sum_{n\geq 1} a_n z^n$ with coefficients in $\mathbb{C}$ such that for every compact set $K \subset \mathbb{C} \setminus \{0\}$, with connected complement, and for every function h continuous on K and holomorphic on the interior K° (if nonempty) of K, there is a sequence of positive integers (λ_n) such that

$$\sup_{z\in K}\left|\sum_{j=0}^{\lambda_n} a_j z^j - h(z)\right| \to 0 \text{ as } n \to +\infty. \tag{1}$$

Clearly, Seleznev universal series must have radius of convergence equal to 0. In the sequel, we denote by $A(K)$ the family of functions continuous on K and holomorphic on the interior K° of K. Later, Luh [**14**] and Chui and Parnes [**7**] independently constructed in the early 1970's universal power series with a nonzero radius of convergence. More precisely, they proved that there exists a series $\sum_{n\geq 0} a_n z^n$, which is convergent on the open unit disc $\mathbb{D} = \{z \in \mathbb{C}\ ;\ |z| < 1\}$, such that, for every compact set $K \subset \{z \in \mathbb{C}\ ;\ |z| > 1\}$ with connected complement, and for every function $h \in A(K)$, there exists an increasing sequence (λ_n) of non-negative integers such that the approximation property (1) holds. Clearly, the radius of convergence of such series is exactly 1. Then,

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in 1996 Nestoridis refined this result by observing that the compact set is allowed to contain pieces of the boundary $\partial\mathbb{D}$; thus, the approximations also holds on the points of the boundary [**12**]. Moreover, the idea of the use of Baire's theorem (as in [**8**, **9**]) yields a simplification of the proofs of the existence of universal series together with a G_δ-dense set of such elements. This leads to the following general statement.

THEOREM 1.1. *There exists a power series* $\sum_{n\geq 1} a_n z^n$ *with radius of convergence* 1 *(resp.* 0*) such that for every compact set* $K \subset \{z \in \mathbb{C};\ |z| \geq 1\}$ *(resp.* $K \subset \mathbb{C} \setminus \{0\}$*), with connected complement, and for every function* $h \in A(K)$*, there is an increasing sequence* (λ_n) *of non-negative integers such that*

$$\sup_{z\in K}\left|\sum_{j=0}^{\lambda_n} a_j z^j - h(z)\right| \to 0 \text{ as } n \to +\infty.$$

Moreover, the set $\mathcal{U}(\mathbb{D})$ *(resp.* $\mathcal{U}$*) of such universal Taylor series is a* G_δ*-dense subset of* $H(\mathbb{D})$ *(resp.* $\mathbb{C}^{\mathbb{N}}$*) endowed with the topology of uniform convergence on compact sets (resp. with the Cartesian topology).*

Since then, many results on universal series have appeared. We refer the reader to [**3**] and the references therein. Kyrezi, Nestoridis and Papachristodoulos have recently been interested in the densities of the sub-sequences (λ_n) (see Definition 2.1) which realize the approximations (1). In fact, they studied this property in the context of the abstract theory of universal series. Here, let us define the classes of frequently and 1-upper frequently universal Taylor series in the complex plane.

DEFINITION 1.2. A power series $\sum_{j\geq 0} a_j z^j$ of radius of convergence 1 (resp. 0) is said to be frequently universal if for every $\varepsilon > 0$, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ (resp. $\mathbb{C} \setminus \{0\}$) with connected complement, and any function $h \in A(K)$, we have

$$\underline{\text{dens}}\left\{n \in \mathbb{N};\ \sup_{z\in K}\left|\sum_{k=0}^{n} a_k z^k - h(z)\right| < \varepsilon\right\} > 0.$$

We denote by $\mathcal{FU}(\mathbb{D})$ (resp. $\mathcal{FU}$) the set of such power series.

Clearly, we have $\mathcal{FU}(\mathbb{D}) \subset \mathcal{U}(\mathbb{D})$ or $\mathcal{FU} \subset \mathcal{U}$.

DEFINITION 1.3. A power series $\sum_{j\geq 0} a_j z^j$ of radius of convergence 1 (resp. 0) is said to be 1-upper frequently universal if for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ (resp. $\mathbb{C} \setminus \{0\}$) with connected complement, and any function $h \in A(K)$, there exists an increasing sequence $\lambda = (\lambda_n) \subset \mathbb{N}$, with $\overline{\text{dens}}(\lambda) = 1$, such that

$$\sup_{z\in K}\left|\sum_{k=0}^{\lambda_n} a_k z^k - h(z)\right| \to 0, \text{ as } n \to +\infty.$$

We denote by $\tilde{\mathcal{U}}(\mathbb{D})$ (resp. $\tilde{\mathcal{U}}$) the set of such power series.

We have $\tilde{\mathcal{U}}(\mathbb{D}) \subset \mathcal{U}(\mathbb{D})$ and $\tilde{\mathcal{U}} \subset \mathcal{U}$ again. Notice that if we replace $\overline{\text{dens}}(\lambda) = 1$ by $\underline{\text{dens}}(\lambda) > 0$ in Definition 1.3, the corresponding set of universal series is empty [**13**].

These classes of frequently universal series are related to the notion of frequently hypercyclic (or universal) operators introduced by Bayart and Grivaux ([**1**, **2**], see also [**6**]).

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DEFINITION 1.4. Let Y (resp. Y, Z) be topological vector space(s). An operator $T : Y \to Y$ (resp. a sequence of operators $T_n : Y \to Z$) is called *frequently hypercyclic* (resp. *frequently universal*) if there exists a vector $y \in Y$ such that for every non empty open set $U \subset Y$ (resp. $U \subset Z$),

$$\underline{\text{dens}}(\{k \in \mathbb{N};\ T^k y \in U\ (resp.\ T_k y \in U)\} > 0.$$

Roughly speaking this notion quantifies how often the orbit of a hypercyclic (or universal) vector visits a non-empty open set. For example the translation operator [**5**] or the differentiation operator [**11**] in the space $H(\mathbb{C})$ of entire functions, endowed with the topology of uniform convergence on compact sets, are hypercyclic operators [**11**] which are frequently hypercyclic [**2**].

Concerning the theory of universal series, we know that the sets of frequently universal series are of first category [**10**, **13**]. But in [**13**, Question 3.6] the author wonders about the existence of frequently universal Taylor series. In this short paper, we show that frequently universal Taylor series in the sense of Seleznev do not exist. The proof comes in a natural way. Using an additional argument, we will obtain the same result concerning universal Taylor series in the sense of Nestoridis, i.e. $\mathcal{FU}(\mathbb{D}) = \emptyset$. Moreover, in [**13**] the author proves that the set of Taylor series in $H(\mathbb{D})$ which are 1-upper frequently universal with respect to a single compact set $K \subset \mathbb{C} \setminus \mathbb{D}$, with connected complement, is a G_δ-dense subset of $H(\mathbb{D})$ endowed with the topology of uniform convergence on compact sets. Combining this result with a diagonal argument (as in [**3**, Theorem 3]), it is easy to check that $\tilde{\mathcal{U}}(\mathbb{D})$ is a G_δ-dense subset of $H(\mathbb{D})$. In our paper, we show that in fact *all* the elements of $\mathcal{U}(\mathbb{D})$ are 1-upper frequently universal. Hence $\tilde{\mathcal{U}}(\mathbb{D}) = \mathcal{U}(\mathbb{D})$. The same property holds for the set $\mathcal{U}$.

The paper is organized as follows: in Section 2 we recall some definitions and some useful lemmas. Next, Section 3 is devoted to the statements and proofs of the main results.

2. Notations and preliminary lemmas. First, we recall the notions of the densities of subsets of $\mathbb{N}$.

DEFINITION 2.1. The *lower density* of a strictly increasing sequence (n_k) of positive integers is defined as the lower density of the corresponding subset of $\mathbb{N}$, that is,

$$\underline{\text{dens}}(n_k) = \liminf_{N \to +\infty} \frac{\#\{k \in \mathbb{N} : n_k \leq N\}}{N},$$

where as usual # denotes the cardinality of the corresponding set.

Similarly the *upper density* of a strictly increasing sequence (n_k) of positive integers is defined as

$$\overline{\text{dens}}(n_k) = \limsup_{N \to +\infty} \frac{\#\{k \in \mathbb{N} : n_k \leq N\}}{N}.$$

We have also the following useful simple facts [**6**].

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LEMMA 2.2. *Let (n_k) be a strictly increasing sequence of positive integers.*

(*i*) *We have*

$$\underline{dens}(n_k) = \liminf_{k\to+\infty}\left(\frac{k}{n_k}\right),$$

(*ii*) *the sequence (n_k) is of positive lower density if and only if* $\sup_{k\geq 1}\left(\frac{n_k}{k}\right) < +\infty.$

Now, we state the nice Turán inequality [**16**] (see also [**4**]), which forms the starting point of our proofs.

LEMMA 2.3. *Let Q be a polynomial of arbitrary degree which possesses only n non zero coefficients. Then for any $r > 0$ and any δ $(0 < \delta < 2\pi)$*

$$\sup_{|z|=r}|Q(z)| \leq \left(\frac{4\pi e}{\delta}\right)^n \sup_{|t|\leq\delta/2}|Q(re^{it})|.$$

Throughout the paper, for $r > 0$ and $0 < \delta < 2\pi$, $\Gamma_{r,\delta}$ will be the set

$$\Gamma_{r,\delta} = \left\{z \in \mathbb{C};\ |z| = r \text{ and } -\frac{\delta}{2} \leq \arg(z) \leq \frac{\delta}{2}\right\},$$

and $C_\delta = \frac{4\pi e}{\delta}$ the constant of the above Turán inequality.

Finally for a power series $f = \sum_{j\geq 0} a_j z^j$ we will denote its n-th partial sum as $S_n(f)$.

3. Main results. First, we address the question [**13**, Question 3.6], by showing that the class of frequently universal Taylor series in the sense of Seleznev is empty.

THEOREM 3.1. *No Taylor series is frequently universal in the sense of Seleznev, i.e.* $\mathcal{FU} = \emptyset$.

Proof. The proof is based on the use of Turán's inequality. Let $f = \sum_{n\geq 1} a_n z^n$ be a Seleznev universal series and assume that f is frequently universal. Let $r > 0$ and $0 < \delta < 2\pi$. Denote by A the following set of positive integers

$$A = \{n \in \mathbb{N};\ \sup_{z\in\Gamma_{r,\delta}} |S_n(f)(z)| < 1\},$$

and let us write A as a strictly increasing sequence (n_l). Since f is frequently universal, the sequence (n_l) is of positive lower density. According to Lemma 2.2, we have $\sup_{l\geq 1}\left(\frac{n_l}{l}\right) < +\infty$. Let us consider

$$M := \sup_{l\geq 1}\left(\frac{n_l}{l}\right) < +\infty. \tag{2}$$

Now, we have $S_{n_{l+1}}(f)(z) - S_{n_l}(f)(z) = \sum_{k=1+n_l}^{n_{l+1}} a_k z^k$. By construction, we easily deduce

$$\sup_{z\in\Gamma_{r,\delta}}\left|\sum_{k=1+n_l}^{n_{l+1}} a_k z^k\right| < 2. \tag{3}$$

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Hence, Lemma 2.3 ensures that

$$\sup_{|z|\leq r}\left|\sum_{k=1+n_l}^{n_{l+1}} a_k z^k\right| \leq 2C_\delta^{n_{l+1}-n_l}, \tag{4}$$

where C_δ does not depend on r. By Cauchy estimates, we get for every $j = 1 + n_l, \ldots, n_{l+1}$,

$$|a_j| \leq \frac{2}{r^j} C_\delta^{n_{l+1}-n_l}. \tag{5}$$

Thus, combining (2) and (5) with the inequality $n_l \geq l$, we obtain, for every $l \geq 1$ and for $j = 1 + n_l, \ldots n_{l+1}$,

$$|a_j|^{1/j} \leq \frac{2^{1/j}}{r} C_\delta^{M-1}. \tag{6}$$

We conclude that the power series $f = \sum_{n\geq 0} a_n z^n = \sum_{n=0}^{n_1} a_n z^n + \sum_{l=1}^{+\infty}\sum_{j=1+n_l}^{n_{l+1}} a_j z^j$ has a strictly positive radius of convergence. This contradicts the fact that $f \in \mathcal{U}$. □

Notice that this previous proof does not adapt to the case of Nestoridis universal series along the same lines. Indeed we obtain that the radius R of convergence of the universal series satisfies the inequality $R \geq r/C_\delta^{M-1}$, for some $r \geq 1$, and the quantity M depends on r. So we do not know if we can have $r/C_\delta^{M-1} > 1$ to obtain a contradiction. Nevertheless, a careful examination of the proof leads to the case of 1-upper frequently universal Taylor series (see Definition 1.3). First, we prove the following key result.

PROPOSITION 3.2. *Let $f \in \mathcal{U}(\mathbb{D})$ (resp. $\mathcal{U}$). Let also $K \subset \{z \in \mathbb{C};\ |z| \geq 1\}$ (resp. $K \subset \mathbb{C}\setminus\{0\}$) be a compact set with connected complement, $h \in A(K)$ and $\varepsilon > 0$. For every $\alpha > 1$, there exists a sub-sequence of positive integers $\mu = (\mu_k)$ such that*

$$\forall k \in \mathbb{N},\ \sup_{z\in K}|S_{\mu_k}(f)(z) - h(z)| < \varepsilon \text{ and } \overline{\text{dens}}(\mu) \geq 1 - \frac{1}{\alpha}.$$

Proof. Set $0 < \delta < 2\pi$. Let us choose $R > 0$ so that

$$\frac{R}{C_\delta^\alpha} > \sup_{z\in K}|z|. \tag{7}$$

Clearly the set $K_{R,\delta} := \Gamma_{R,\delta} \cup K$ is a compact set with connected complement. We then consider the function $\tilde{h}$ defined by $\tilde{h}(z) = h(z)$ for $z \in K$ and $\tilde{h}(z) = 0$ for $z \in \Gamma_{R,\delta}$. Since $f \in \mathcal{U}$ we get

$$\#\{n \in \mathbb{N};\ \sup_{z\in K_{R,\delta}}|S_n(f)(z) - \tilde{h}(z)| < \varepsilon/2\} = +\infty.$$

Let us write $(\lambda_k) = \{n \in \mathbb{N};\ \sup_{z\in K_{R,\delta}}|S_n(f)(z) - \tilde{h}(z)| < \varepsilon/2\}$ where (λ_k) is a strictly increasing sub sequence of positive integers. We set $f(z) = \sum_{k\geq 0} a_k z^k$. Now, by the

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triangle inequality we have

$$\sup_{z\in\Gamma_{R,\delta}}\left|\sum_{j=1+\lambda_k}^{\lambda_{k+1}} a_j z^j\right| < \varepsilon.$$

Using Lemma 2.3 and Cauchy estimates, we get for every $k \geq 1$ and $j = 1 + \lambda_k, \dots, \lambda_{k+1}$,

$$|a_j|^{1/j} \leq \frac{\varepsilon^{1/j}}{R} C_\delta^{(\lambda_{k+1}-\lambda_k)/j}. \tag{8}$$

Let us also consider the following subsets of $\mathbb{N}$

$$A = \{k \in \mathbb{N};\ \lambda_{k+1} - \lambda_k \leq \alpha(1+\lambda_k)\} \text{ and } B = \{k \in \mathbb{N};\ \lambda_{k+1} - \lambda_k > \alpha(1+\lambda_k)\}. \tag{9}$$

We have $f(z) = \sum_{j=0}^{\lambda_1} a_j z^j + f_A(z) + f_B(z)$, with

$$f_A(z) := \sum_{k\in A}\sum_{j=1+\lambda_k}^{\lambda_{k+1}} a_j z^j \text{ and } f_B(z) := \sum_{k\in B}\sum_{j=1+\lambda_k}^{\lambda_{k+1}} a_j z^j.$$

By (8) and (7), the power series f_A converges on the compact set K. On the other hand, notice that, for $k \in B$, if we have $\lfloor \frac{1}{\alpha}(\lambda_{k+1} - \lambda_k)\rfloor + 1 \leq j \leq \lambda_{k+1}$, then $|a_j|^{1/j} \leq \varepsilon^{1/j} C_\delta^\alpha / R$ and the power series

$$f_{B_1}(z) := \sum_{k\in B}\sum_{j=\lfloor \frac{1}{\alpha}(\lambda_{k+1}-\lambda_k)\rfloor+1}^{\lambda_{k+1}} a_j z^j,$$

converges on the compact set K again. Let us also consider the subset of $\mathbb{N}$

$$C = \{k \in \mathbb{N};\ \sup_{z\in K} |S_k(f)(z) - h(z)| < \varepsilon\}.$$

Clearly, we have $(\lambda_k) \subset C$. Further, since the power series f_A and f_{B_1} converge on the compact set K, one can find $k_0 \in \mathbb{N}$, such that, for every $k \geq k_0$, we have

$$\sup_{z\in K} |S_{\lambda_{k+1}}(f)(z) - S_j(f)(z)| < \frac{\varepsilon}{2},$$

for $1 + \lambda_k \leq j \leq \lambda_{k+1}$ if $k \in A$ or for $\lfloor \frac{1}{\alpha}(\lambda_{k+1} - \lambda_k)\rfloor + 1 \leq j \leq \lambda_{k+1}$ if $k \in B$. Thus by the triangle inequality, we have

$$\left(\bigcup_{\substack{k\in A;\\ k\geq k_0}} \{j \in \mathbb{N}; 1 + \lambda_k \leq j \leq 8\lambda_{k+1}\}\right)$$
$$\times \cup \left(\bigcup_{\substack{k\in B;\\ k\geq k_0}} \left\{j \in \mathbb{N}; \lfloor \frac{1}{\alpha}(\lambda_{k+1} - \lambda_k)\rfloor + 1 \leq j \leq \lambda_{k+1}\right\}\right) \subset C. \tag{10}$$

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Let us write

$$(\mu_i) = \left(\bigcup_{\substack{k\in A;\\ k\geq k_0}} \{j \in \mathbb{N}; 1+\lambda_k \leq j \leq \lambda_{k+1}\} \right)$$
$$\times \cup \left(\bigcup_{\substack{k\in B;\\ k\geq k_0}} \left\{ j \in \mathbb{N}; \lfloor \frac{1}{\alpha}(\lambda_{k+1}-\lambda_k) \rfloor + 1 \leq j \leq \lambda_{k+1} \right\} \right). \tag{11}$$

From (11), we deduce, for N large enough,

$$\begin{aligned}
\#\{k \in \mathbb{N};\ \mu_k \leq \lambda_{N+1}\} &= \sum_{k\in A; k=k_0}^{N} (\lambda_{k+1}-\lambda_k) + \sum_{k\in B; k=k_0}^{N} \left(\lambda_{k+1} - \left\lfloor \frac{1}{\alpha}(\lambda_{k+1}-\lambda_k) \right\rfloor \right) \\
&\geq \sum_{k\in A; k=k_0}^{N} (\lambda_{k+1}-\lambda_k) + \sum_{k\in B; k=k_0}^{N} \left(\lambda_{k+1} - \frac{1}{\alpha}(\lambda_{k+1}-\lambda_k) \right) \\
&\geq \sum_{k\in A; k=k_0}^{N} (\lambda_{k+1}-\lambda_k) + \left(1-\frac{1}{\alpha}\right) \sum_{k\in B; k=k_0}^{N} (\lambda_{k+1}-\lambda_k) \\
&\geq \left(1-\frac{1}{\alpha}\right)(\lambda_{N+1}-\lambda_{k_0}).
\end{aligned}$$

Thus, we have $\limsup_{N\to+\infty} \dfrac{\#\{k \in \mathbb{N};\ \mu_k \leq N\}}{N} \geq 1-\dfrac{1}{\alpha}$, which gives the desired conclusion. □

Now, we are ready to show that all the Seleznev or Nestoridis universal Taylor series are 1-upper frequently universal.

THEOREM 3.3. *We have* $\tilde{\mathcal{U}}(\mathbb{D}) = \mathcal{U}(\mathbb{D})$ *(resp.* $\tilde{\mathcal{U}} = \mathcal{U}$*).*

Proof. The inclusion $\tilde{\mathcal{U}}(\mathbb{D}) \subset \mathcal{U}(\mathbb{D})$ (resp. $\tilde{\mathcal{U}} \subset \mathcal{U}$) is clear. Conversely, let $f \in \mathcal{U}(\mathbb{D})$ (resp. $f \in \mathcal{U}$). Let also $K \subset \mathbb{C} \setminus \mathbb{D}$ (resp. $\mathbb{C} \setminus \{0\}$) be a compact set, with connected complement and $h \in A(K)$. Set $N_0 = 0$ and $\mu_0^{(0)} = 0$. For $i = 1, 2, \ldots$, by Proposition 3.2 it follows that there exist positive integers $\mu_1^{(i)} < \mu_2^{(i)} < \cdots < \mu_{N_i}^{(i)}$, such that

$$\mu_{N_{i-1}}^{(i-1)} < \mu_1^{(i)}, \tag{12}$$

$$\sup_{z\in K} \left| S_{\mu_k^{(i)}}(f)(z) - h(z) \right| < \frac{1}{i+1}, \quad k = 1, \ldots, N_i, \tag{13}$$

and

$$\frac{N_i}{\mu_{N_i}^{(i)}} \geq 1 - \frac{1}{i+1}. \tag{14}$$

We set $E_i = \{\mu_1^{(i)}, \ldots, \mu_{N_i}^{(i)}\}$ and $\lambda = \bigcup_{i=1}^{+\infty} E_i = \{\lambda_1, \lambda_2, \ldots\}$. Then, by (14), we get that $\overline{\mathrm{dens}}(\lambda) = 1$ and by (13) that $\sup_{z\in K} \left| S_{\lambda_n}(f)(z) - h(z) \right| \to 0$, as $n \to +\infty$. Therefore, $f \in \tilde{\mathcal{U}}(\mathbb{D})$ (resp. $\tilde{\mathcal{U}}$). This finishes the proof. □

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Finally, we deduce the following result.

THEOREM 3.4. *No Taylor series is frequently universal in the sense of Nestoridis, i.e.* $\mathcal{FU}(\mathbb{D}) = \emptyset$.

Proof. Combining the inclusion $\mathcal{FU}(\mathbb{D}) \subset \mathcal{U}(\mathbb{D})$ with Theorem 3.3 it suffices to prove $\mathcal{FU}(\mathbb{D}) \cap \tilde{\mathcal{U}}(\mathbb{D}) = \emptyset$. To do this, we use the main ideas of the proof of [**13**, Proposition 3.5]. For the sake of completeness and clarity, we report them here. Let $f \in \mathcal{FU}(\mathbb{D}) \cap \tilde{\mathcal{U}}(\mathbb{D})$. Let also $K \subset \{z \in \mathbb{C};\ |z| \geq 1\}$ be a compact set, with connected complement, and $h \in A(K)$ a non-zero element. Since $f \in \tilde{\mathcal{U}}(\mathbb{D})$, one can find an increasing sub-sequence $\lambda = (\lambda_n)$ of positive integers so that

$$\overline{\text{dens}}(\lambda) = 1 \text{ and } \sup_{z\in K} |S_{\lambda_n}(f)(z) - h(z)| \to 0, \text{ as } n \to +\infty.$$

Let us consider the subset A of $\mathbb{N}$ defined by

$$A = \{n \in \mathbb{N};\ \sup_{z\in K} |S_n(f)(z)| < d/2\},$$

where $d = \sup_{z\in K} |h(z)|$. Thus, there exists an integer N large enough, such that, for every $n \geq N$, $\lambda_n \notin A$. Set the sequence $\tilde{\lambda} = (\lambda_N, \lambda_{N+1}, \dots)$. Clearly $\overline{\text{dens}}(\tilde{\lambda}) = 1$. So the inclusion $A \subset \mathbb{N} \setminus \tilde{\lambda}$ implies

$$\underline{\text{dens}}(A) \leq \underline{\text{dens}}(\mathbb{N} \setminus \tilde{\lambda}).$$

But it is easy to check that $\underline{\text{dens}}(\mathbb{N} \setminus \tilde{\lambda}) = 1 - \overline{\text{dens}}(\tilde{\lambda}) = 0$. This gives the conclusion.

□

REMARK 3.5. It is easy to check that Theorems 3.3 and 3.4 remain true for universal Taylor series, where we replace $\mathbb{D}$ by a simply connected domain $\Omega \subset \mathbb{C}$ (see [**3**] and the references therein for the definitions) provided that the complement of Ω contains a family of circle arcs which goes to infinity.

Finally the proof of Theorem 3.4 gives implicitly a more general result. Let X, Y be Hausdorff topological vector spaces and $T_n : X \to Y$ continuous linear operators. Then, we call (T_n) *universal* (resp. *1-upper frequently universal*) if there is some $x \in X$ such that for any $y \in Y$ there is an increasing sequence (n_k) (resp. of upper density 1) such that $T_{n_k}x \to y$, as $k \to +\infty$. Such an element x is said to be *universal* (resp. *1-upper frequently universal*) for (T_n). Then, we have:

PROPOSITION 3.6. *Let (T_n) be a 1-upper frequently sequence of operators such that all the universal elements are 1-upper frequently universal. Then, (T_n) cannot be frequently universal.*

Proof. Let $x \in X$ be an universal element. By hypothesis x is 1-upper frequently universal for (T_n) too. Let U, V be disjoint non-empty open sets in Y. Choose $y \in V$. Let (n_k) be a corresponding sequence as above. Then, there is some $K \geq 1$ such that $T_{n_k}x \in V$ fo all $k \geq K$. Since $\overline{\text{dens}}\{n_k\ :\ k \geq K\} = 1$ we necessarily have that

$$\underline{\text{dens}}\{n\ :\ T_nx \in U\} = 0,$$

which implies the fact that (T_n) cannot be frequently universal. □

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