Bull. Aust. Math. Soc. **83** (2011), 420–434
doi:10.1017/S0004972711002048

BOUNDS IN TERMS OF GÂTEAUX DERIVATIVES FOR THE WEIGHTED f-GINI MEAN DIFFERENCE IN LINEAR SPACES

S. S. DRAGOMIR

(Received 21 June 2010)

Abstract

Some bounds in terms of Gâteaux lateral derivatives for the weighted f-Gini mean difference generated by convex and symmetric functions in linear spaces are established. Applications for norms and semi-inner products are also provided.

2010 *Mathematics subject classification*: primary 94A17; secondary 26D15.

Keywords and phrases: convex functions, Jensen's inequality, Gâteaux lateral derivatives, norms, semi-inner products, means, weighted f-Gini mean difference.

1. Introduction

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \ldots, p_n)$ a probability sequence, meaning that $p_i \geq 0$ $(i \in \{1, \ldots, n\})$ and $\sum_{i=1}^n p_i = 1$, define the *r-weighted Gini mean difference*, for $r \in [1, \infty)$, by the formula [1, p. 291]:

$$G_r(\mathbf{p}, \mathbf{a}) := \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j |a_i - a_j|^r = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|^r. \tag{1.1}$$

For the uniform probability distribution $\mathbf{p} = (1/n, \ldots, 1/n)$ we denote

$$G_r(\mathbf{a}) := G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i - a_j|^r = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|^r.$$

For $r = 1$ we have the *weighted Gini mean difference* $G(\mathbf{p}, \mathbf{a})$, where

$$G(\mathbf{p}, \mathbf{a}) := \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j |a_i - a_j| = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|, \tag{1.2}$$

which becomes, for the uniform probability distribution $\mathbf{p} = (1/n, \ldots, 1/n)$, the *Gini mean difference*

$$G(\mathbf{a}) := \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|.$$

For various properties of this and the *Gini index*

$$R(\mathbf{a}) = \frac{1}{\bar{a}} G(\mathbf{a}), \quad \text{where } \bar{a} := \frac{1}{n} \sum_{i=1}^{n} a_i \neq 0,$$

see [1, 11–13].

Now, if we define $\Delta := \{(i, j) \mid i, j \in \{1, \ldots, n\}\}$, then we can simply write from (1.1) that

$$G_r(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^r, \quad r \geq 1. \tag{1.3}$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p}, \mathbf{a})$ may be stated (see [2]).

THEOREM 1.1. *For any $p_i \in (0, 1)$, $i \in \{1, \ldots, n\}$, with $\sum_{i=1}^{n} p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, we have the inequalities*

$$\begin{aligned} \frac{1}{2} \max_{(i,j)\in\Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^{r-1}}{(1 - p_i p_j)^{r-1}} |a_i - a_j|^r \right\} &\leq G_r(\mathbf{p}, \mathbf{a}) \\ &\leq \frac{1}{2} \max_{(i,j)\in\Delta} |a_i - a_j|^r, \end{aligned} \tag{1.4}$$

where $r \in [1, \infty)$.

REMARK 1.2. The case $r = 2$ is of interest, since

$$G_2(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2,$$

for which we can obtain from Theorem 1.1 the following bounds:

$$\frac{1}{2} \max_{(i,j)\in\Delta} \left\{ \frac{p_i p_j}{1 - p_i p_j} (a_i - a_j)^2 \right\} \leq G_2(\mathbf{p}, \mathbf{a}) \leq \frac{1}{2} \max_{(i,j)\in\Delta} (a_i - a_j)^2. \tag{1.5}$$

REMARK 1.3. Since the function

$$h_r(t) := \frac{t^r + t(1 - t)^{r-1}}{(1 - t)^{r-1}} = t + t^r (1 - t)^{1-r},$$

defined for $t \in [0, 1)$ and $r > 1$, is strictly increasing on $[0, 1)$ from Theorem 1.1 we can obtain a coarser but perhaps a more useful lower bound for the r-weighted Gini mean difference, namely (see [2]),

$$G_r(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}} \max_{(i,j)\in\Delta} |a_i - a_j|^r, \tag{1.6}$$

where p_m is defined by $p_m := \min_{i\in\{1,\ldots,n\}}\{p_i\} > 0$. For $r = 2$, we then have

$$G_2(\mathbf{p}, \mathbf{a}) \geq \frac{1}{2} \frac{p_m^2}{1 - p_m^2} \max_{(i,j)\in\Delta} (a_i - a_j)^2. \tag{1.7}$$

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

For other results related to the above, see [2]. For various inequalities concerning $G_2(\mathbf{p}, \mathbf{a})$, see [6] and the references therein.

The main purpose of the present paper is to provide some bounds in terms of Gâteaux lateral derivatives for the *weighted f-Gini mean difference* generated by convex and symmetric functions in linear spaces that has been introduced in the recent work [3] and briefly recalled in the next section. Applications for norms and semi-inner products are also provided.

2. Some preliminary results

2.1. Weighted f-Gini mean difference. Let $f : X \to \mathbb{R}$ be a convex function on the linear space X. Assume that $f(0) = 0$ and f is symmetric, that is, $f(x) = f(-x)$ for any $x \in X$. In these circumstances,

$$f(x) = \frac{f(x) + f(-x)}{2} \geq f\left(\frac{x - x}{2}\right) = f(0) = 0,$$

showing that f is nonnegative on the entire space X.

For $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ we define the *weighted f-Gini mean difference* of the n-tuple $\mathbf{x}$ with the probability distribution $\mathbf{p}$ as the positive quantity

$$G_f(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j f(x_i - x_j) = \sum_{1 \leq i < j \leq n} p_i p_j f(x_i - x_j) \geq 0. \tag{2.1}$$

For the uniform distribution $\mathbf{u} = (1/n, \ldots, 1/n) \in \mathbf{P}^n$ we have the *f-Gini mean difference* defined by

$$G_f(\mathbf{x}) := \frac{1}{2n^2} \sum_{i,j=1}^{n} f(x_i - x_j) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} f(x_i - x_j).$$

A natural example of such *f-Gini mean difference* can be provided by the convex function $f(x) := \|x\|^r$, with $r \geq 1$, defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$G_r(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r = \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^r.$$

We now need to consider another quantity that is naturally related to the *f-Gini mean difference*. For a convex function $f : X \to \mathbb{R}$ defined on the linear space X with the property that $f(0) = 0$, define the *mean f-deviation* of an n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ by the nonnegative quantity

$$K_f(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i f\left(x_i - \sum_{k=1}^{n} p_k x_k\right). \tag{2.2}$$

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

The fact that $K_f(\mathbf{p}, \mathbf{x})$ is nonnegative follows by Jensen's inequality, namely

$$K_f(\mathbf{p}, \mathbf{x}) \geq f\left(\sum_{i=1}^{n} p_i\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\right) = f(0) = 0.$$

For other Jensen's type inequalities, see [4, 7–9].

A natural example of such deviations can be provided by the convex function $f(x) := \|x\|^r$, with $r \geq 1$, defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$K_r(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r \tag{2.3}$$

and call it the *mean r-absolute deviation* of the n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$.

The following connection between the f-Gini mean difference and the mean f-deviation holds true [10].

THEOREM 2.1. *If $f : X \to [0, \infty)$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ we have the inequalities*

$$G_f(\mathbf{p}, \mathbf{x}) \geq \tfrac{1}{2} K_f(\mathbf{p}, \mathbf{x}) \geq G_f(\mathbf{p}, \tfrac{1}{2}\mathbf{x}). \tag{2.4}$$

Both inequalities in (2.4) are sharp and the constant $\frac{1}{2}$ best possible.

The following particular case for norms is of interest due to its natural generalization for the scalar case that is used in applications.

COROLLARY 2.2. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ we have*

$$G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} K_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2^r} G_r(\mathbf{p}, \mathbf{x}) \tag{2.5}$$

or, equivalently,

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r \geq \frac{1}{2^r} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r, \tag{2.6}$$

for any $r \geq 1$.

REMARK 2.3. By symmetry we have

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r = 2 \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^r,$$

and since

$$\sum_{1 \leq i < j \leq n} p_i p_j = \frac{1}{2}\left(\sum_{i,j=1}^{n} p_i p_j - \sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\left(1 - \sum_{i=1}^{n} p_i^2\right) = \frac{1}{2} \sum_{i=1}^{n} p_i(1 - p_i),$$

we may state from (2.6) the simpler inequality

$$\begin{aligned}\sum_{i=1}^{n} p_i(1-p_i)\max_{1\le i<j\le n}\|x_i-x_j\|^r &\ge \sum_{i=1}^{n} p_i\Big\|x_i-\sum_{k=1}^{n} p_k x_k\Big\|^r\\ &\ge \frac{1}{2^r}\sum_{i=1}^{n} p_i(1-p_i)\min_{1\le i<j\le n}\|x_i-x_j\|^r.\end{aligned}\tag{2.7}$$

2.2. The Gâteaux derivatives of convex functions. Assume that $f : X \to \mathbb{R}$ is a *convex function* on the real linear space X. Since for any vectors $x, y \in X$ the function $g_{x,y} : \mathbb{R} \to \mathbb{R}$, $g_{x,y}(t) := f(x+ty)$ is convex, it follows that the limits

$$\nabla_{+(-)} f(x)(y) := \lim_{t\to 0+(-)} \frac{f(x+ty)-f(x)}{t}$$

exist, and they are called the *right (left) Gâteaux derivatives* of the function f at the point x in the direction y.

It is obvious that, for any $t > 0 > s$, we have

$$\begin{aligned}\frac{f(x+ty)-f(x)}{t} &\ge \nabla_+ f(x)(y) = \inf_{t>0}\Big[\frac{f(x+ty)-f(x)}{t}\Big]\\ &\ge \sup_{s<0}\Big[\frac{f(x+sy)-f(x)}{s}\Big] = \nabla_- f(x)(y)\\ &\ge \frac{f(x+sy)-f(x)}{s}\end{aligned}\tag{2.8}$$

for any $x, y \in X$ and, in particular,

$$\nabla_- f(u)(u-v) \ge f(u)-f(v) \ge \nabla_+ f(v)(u-v) \tag{2.9}$$

for any $u, v \in X$. We call this *the gradient inequality* for the convex function f. It will be used frequently in the following in order to obtain various results related to Jensen's inequality.

The following properties are also of importance:

$$\nabla_+ f(x)(-y) = -\nabla_- f(x)(y), \tag{2.10}$$

and

$$\nabla_{+(-)} f(x)(\alpha y) = \alpha \nabla_{+(-)} f(x)(y) \tag{2.11}$$

for any $x, y \in X$ and $\alpha \ge 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, that is,

$$\nabla_+ f(x)(y+z) \le \nabla_+ f(x)(y) + \nabla_+ f(x)(z) \tag{2.12}$$

and

$$\nabla_- f(x)(y+z) \ge \nabla_- f(x)(y) + \nabla_- f(x)(z) \tag{2.13}$$

for any $x, y, z \in X$.

Some natural examples can be provided by the use of normed spaces. Assume that $(X, \|\cdot\|)$ is a real normed linear space. The function $f : X \to \mathbb{R}$, $f(x) := \frac{1}{2}\|x\|^2$ is a convex function which generates the *superior* and *inferior semi-inner products*

$$\langle y, x\rangle_{s(i)} := \lim_{t\to 0+(-)} \frac{\|x+ty\|^2 - \|x\|^2}{t}.$$

For a comprehensive study of the properties of these mappings in the geometry of Banach spaces see [5].

For the convex function $f_p : X \to \mathbb{R}$, $f_p(x) := \|x\|^p$ with $p > 1$,

$$\nabla_{+(-)} f_p(x)(y) = \begin{cases} p\|x\|^{p-2}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for any $y \in X$. If $p = 1$, then we have

$$\nabla_{+(-)} f_1(x)(y) = \begin{cases} \|x\|^{-1}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0, \\ +(-)\|y\| & \text{if } x = 0, \end{cases}$$

for any $y \in X$. This class of functions will be used to illustrate the inequalities obtained in the general case of convex functions defined on an entire linear space.

The following result for the general case of convex functions holds (see [3]).

THEOREM 2.4. *Let $f : X \to \mathbb{R}$ be a convex function. Then, for any $x, y \in X$ and $t \in [0, 1]$, we have*

$$\begin{aligned} t(1-t)&[\nabla_- f(y)(y-x) - \nabla_+ f(x)(y-x)] \\ &\geq tf(x) + (1-t)f(y) - f(tx + (1-t)y) \\ &\geq t(1-t)[\nabla_+ f(tx + (1-t)y)(y-x) \\ &\quad - \nabla_- f(tx + (1-t)y)(y-x)] \geq 0. \end{aligned} \tag{2.14}$$

The following particular case for norms may be stated.

COROLLARY 2.5. *If x and y are two vectors in the normed linear space $(X, \|\cdot\|)$ such that $0 \notin [x, y] := \{(1-s)x + sy, s \in [0, 1]\}$, then, for any $p \geq 1$, we have the inequalities*

$$\begin{aligned} pt(1-t)&[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s] \\ &\geq t\|x\|^p + (1-t)\|y\|^p - \|tx + (1-t)y\|^p \\ &\geq pt(1-t)\|tx + (1-t)y\|^{p-2}[\langle y-x, tx + (1-t)y\rangle_s \\ &\quad - \langle y-x, tx + (1-t)y\rangle_i] \geq 0 \end{aligned} \tag{2.15}$$

for any $t \in [0, 1]$. If $p \geq 2$ the inequality holds for any x and y.

REMARK 2.6. If the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{R}$, then inequality (2.15) can be written as

$$\begin{aligned} pt(1-t)&\{[y-x, \|y\|^{p-2}y] - [y-x, \|x\|^{p-2}x]\} \\ &\geq t\|x\|^p + (1-t)\|y\|^p - \|tx + (1-t)y\|^p \end{aligned} \tag{2.16}$$

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

for any $t \in [0, 1]$. Moreover, if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, then (2.16) becomes

$$
\begin{aligned}
&pt(1-t)\langle y - x, \|y\|^{p-2}y - \|x\|^{p-2}x\rangle \\
&\quad \geq t\|x\|^p + (1-t)\|y\|^p - \|tx + (1-t)y\|^p
\end{aligned}
\tag{2.17}
$$

for any $t \in [0, 1]$.

3. Bounds in terms of Gâteaux derivatives

In the following result we provide some upper and lower bounds for the nonnegative quantity

$$G_f(\mathbf{p}, \mathbf{x}) - \tfrac{1}{2}K_f(\mathbf{p}, \mathbf{x})$$

considered in Theorem 2.1.

THEOREM 3.1. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$, we have the inequalities*

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&\quad \geq G_f(\mathbf{p}, \mathbf{x}) - \frac{1}{2}K_f(\mathbf{p}, \mathbf{x}) \\
&\quad \geq \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \geq 0.
\end{aligned}
\tag{3.1}
$$

PROOF. Utilizing the gradient inequality (2.9), we have

$$
\begin{aligned}
\nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) &\geq f(x_i - x_j) - f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) \\
&\geq \nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - x_j\right)
\end{aligned}
\tag{3.2}
$$

for any $i, j \in \{1, \ldots, n\}$. By the symmetry of the function f and the subadditivity of the Gâteaux derivative $\nabla_+ f(\cdot)(\cdot)$ in the second variable, we also have

$$
\begin{aligned}
&\nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&\quad = \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \\
&\quad \geq \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)(x_j) - \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(\sum_{k=1}^{n} p_k x_k\right)
\end{aligned}
\tag{3.3}
$$

for any $i, j \in \{1, \ldots, n\}$.

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

Utilizing (3.2) and (3.3), we may state that

$$\begin{aligned}\nabla_- f(x_i - x_j)\Big(\sum_{k=1}^n p_k x_k - x_j\Big) &\geq f(x_i - x_j) - f\Big(x_i - \sum_{k=1}^n p_k x_k\Big)\\ &\geq \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)\Big(x_j - \sum_{k=1}^n p_k x_k\Big)\\ &\geq \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)(x_j) - \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)\Big(\sum_{k=1}^n p_k x_k\Big)\end{aligned} \tag{3.4}$$

for any $i, j \in \{1, \ldots, n\}$.

Now, if we multiply inequality (3.4) by $p_j \geq 0$ and sum over j from 1 to n, we get

$$\begin{aligned}\sum_{j=1}^n p_j \nabla_- f(x_i - x_j)\Big(\sum_{k=1}^n p_k x_k - x_j\Big) &\geq \sum_{j=1}^n p_j f(x_i - x_j) - f\Big(x_i - \sum_{k=1}^n p_k x_k\Big)\\ &\geq \sum_{j=1}^n p_j \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)\Big(x_j - \sum_{k=1}^n p_k x_k\Big)\\ &\geq \sum_{j=1}^n p_j \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)(x_j)\\ &\quad - \nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)\Big(\sum_{k=1}^n p_k x_k\Big)\\ &\geq 0,\end{aligned} \tag{3.5}$$

where the last inequality follows by the subadditivity of the function

$$\nabla_+ f\Big(\sum_{k=1}^n p_k x_k - x_i\Big)(\cdot) \quad \text{with } i \in \{1, \ldots, n\}.$$

Finally, if we multiply inequality (3.5) by $p_i \geq 0$ and sum over i from 1 to n, we get the desired result (3.1). □

The following particular case for norms holds.

COROLLARY 3.2. *Let* $(X, \|\cdot\|)$ *be a normed space. Then for an* n*-tuple of vectors* $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ *and the probability distribution* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$,

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

we have the inequalities

$$
\begin{aligned}
r\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \|x_l - x_j\|^{r-2}\Big\langle \sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j\Big\rangle_i
&\geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^r - \sum_{l=1}^{n} p_l \Big\| x_l - \sum_{k=1}^{n} p_k x_k\Big\|^r \\
&\geq r\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \Big\|\sum_{k=1}^{n} p_k x_k - x_l\Big\|^{r-2} \\
&\quad\times \Big\langle x_j - \sum_{k=1}^{n} p_k x_k, \sum_{k=1}^{n} p_k x_k - x_l\Big\rangle_s \geq 0.
\end{aligned} \tag{3.6}
$$

If $r \geq 2$ then we have no restriction on $\mathbf{x}$ and $\mathbf{p}$. If $r \in [1, 2)$ then we need to assume that $x_l - x_j \neq 0$ and $\sum_{k=1}^{n} p_k x_k - x_l \neq 0$ for all $l, j \in \{1, \ldots, n\}$.

REMARK 3.3. The case $r = 2$ produces the simpler inequality

$$
\begin{aligned}
2\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \Big\langle \sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j\Big\rangle_i
&\geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^2 - \sum_{l=1}^{n} p_l \Big\| x_l - \sum_{k=1}^{n} p_k x_k\Big\|^2 \\
&\geq 2\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \Big\langle x_j - \sum_{k=1}^{n} p_k x_k, \sum_{k=1}^{n} p_k x_k - x_l\Big\rangle_s \geq 0,
\end{aligned} \tag{3.7}
$$

which holds for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$.

REMARK 3.4. If the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{R}$ (see, for instance, [5]), then inequality (3.7) can be written as

$$
\begin{aligned}
2\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \Big[\sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j\Big]
&\geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^2 - \sum_{l=1}^{n} p_l \Big\| x_l - \sum_{k=1}^{n} p_k x_k\Big\|^2 \geq 0.
\end{aligned} \tag{3.8}
$$

In what follows we provide upper and lower bounds for the nonnegative quantity considered in the second part of Theorem 2.1, namely,

$$\tfrac{1}{2}K_f(\mathbf{p}, \mathbf{x}) - G_f(\mathbf{p}, \tfrac{1}{2}\mathbf{x}).$$

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

THEOREM 3.5. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then, for any n-tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{P}^n$, we have the inequalities*

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\\
&\quad \geq \frac{1}{2}K_f(\mathbf{p}, \mathbf{x}) - G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right)\\
&\quad \geq \frac{1}{4}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right.\\
&\qquad \left. - \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \geq 0.
\end{aligned} \tag{3.9}$$

PROOF. Consider inequality (2.14) for $t = \frac{1}{2}$ to get

$$\begin{aligned}
\frac{1}{4}[\nabla_- f(y)(y - x) - \nabla_+ f(x)(y - x)] &\geq \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\\
&\geq \frac{1}{4}\left[\nabla_+ f\left(\frac{x + y}{2}\right)(y - x) - \nabla_- f\left(\frac{x + y}{2}\right)(y - x)\right] \geq 0
\end{aligned} \tag{3.10}$$

for any $x, y \in X$. Now, if in (3.10) we choose

$$x = x_i - \sum_{k=1}^{n} p_k x_k \quad \text{and} \quad y = \sum_{k=1}^{n} p_k x_k - x_j$$

with $i, j \in \{1, \dots, n\}$ and take into account the symmetry of the function f, then

$$\begin{aligned}
&\frac{1}{2}\left[\nabla_- f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right.\\
&\quad \left. - \nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right]\\
&\qquad \geq \frac{1}{2}\left[f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) + f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\right] - f\left(\frac{1}{2}(x_i - x_j)\right)\\
&\qquad \geq \frac{1}{2}\left[\nabla_+ f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right.\\
&\qquad\quad \left. - \nabla_- f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \geq 0
\end{aligned} \tag{3.11}$$

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

for any $i, j \in \{1, \ldots, n\}$. Furthermore, if we multiply (3.11) by $p_i p_j \geq 0$ and sum over i and j from 1 to n, we deduce that

$$\begin{aligned} &\frac{1}{4}\Bigg[\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\ &\quad - \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\Bigg] \\ &\quad \geq \frac{1}{2} K_f(\mathbf{p}, \mathbf{x}) - G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right) \\ &\quad \geq \frac{1}{4}\Bigg[\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\ &\quad\quad - \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\Bigg] \geq 0. \end{aligned} \tag{3.12}$$

By the symmetry of the function and the symmetry of summation,

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\ &\quad = \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\ &\quad = \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\frac{x_i + x_j}{2} - \sum_{k=1}^{n} p_k x_k\right) \\ &\quad = -\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right). \end{aligned} \tag{3.13}$$

Finally, on utilizing the relations (3.12) and (3.13), we deduce the desired result (3.9). □

The following particular case for norms can be stated.

COROLLARY 3.6. *Let $(X, \|\cdot\|)$ be a normed space. Then, for an n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and the probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$, we have the inequalities*

$$\begin{aligned} &r \sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \left\|\sum_{k=1}^{n} p_k x_k - x_j\right\|^{r-2}\left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j\right\rangle_i \\ &\quad \geq \sum_{i=1}^{n} p_i \left\|x_i - \sum_{k=1}^{n} p_k x_k\right\|^r - \frac{1}{2^r}\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \end{aligned}$$

$$\geq \frac{1}{2^r} r \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \|x_l - x_j\|^{r-2} \left[\left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_s - \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_i \right] \geq 0. \tag{3.14}$$

If $r \geq 2$ then we have no restriction on $\mathbf{x}$ *and* $\mathbf{p}$. *If $r \in [1, 2)$ then we need to assume that $x_l - x_j \neq 0$ and $\sum_{k=1}^{n} p_k x_k - x_j \neq 0$ for all $l, j \in \{1, \ldots, n\}$.*

REMARK 3.7. The case $r = 2$ is of interest since it produces a much simpler inequality,

$$\begin{aligned} 2 \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j \right\rangle_i \\ \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^2 - \frac{1}{4} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \\ \geq \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left[\left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_s \right. \\ \left. - \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_i \right] \geq 0, \end{aligned} \tag{3.15}$$

which holds for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$.

REMARK 3.8. If the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{R}$, then inequality (3.15) can be written as

$$\begin{aligned} 2 \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left[\sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j \right] \\ \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^2 - \frac{1}{4} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \geq 0. \end{aligned} \tag{3.16}$$

4. Other bounds

In [3] we also established the following upper bound for the weighted f-Gini mean difference.

THEOREM 4.1. *Assume that $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If x and y are two vectors and $t \in [0, 1]$ with $(1 - t)x + ty = 0$, then, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ with the property that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, \ldots, n\}$, we have the inequality*

$$\tfrac{1}{2}[(1 - t) f(x) + t f(y)] \geq G_f(\mathbf{p}, \mathbf{x}), \tag{4.1}$$

for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$.

It is thus natural to ask for an upper bound for the positive quantity

$$\tfrac{1}{2}[(1-t)f(x)+tf(y)]-G_f(\mathbf{p},\mathbf{x}).$$

The following result holds.

THEOREM 4.2. *Assume that $f: X\to\mathbb{R}$ is a symmetric convex function with $f(0)=0$. If x and y are two vectors and $t\in[0,1]$ with $(1-t)x+ty=0$ then, for any n-tuple of vectors $\mathbf{x}=(x_1,\ldots,x_n)\in X^n$ with the property that $x_i-x_j\in[x,y]$ for all $i,j\in\{1,\ldots,n\}$, we have the inequality*

$$\begin{aligned}0&\le \tfrac{1}{2}[(1-t)f(x)+tf(y)]-G_f(\mathbf{p},\mathbf{x})\\&\le \tfrac{1}{8}[\nabla_-f(y)(y-x)-\nabla_+f(x)(y-x)],\end{aligned}\tag{4.2}$$

for any probability distribution $\mathbf{p}=(p_1,\ldots,p_n)\in\mathbf{P}^n$.

PROOF. Since $x_i-x_j\in[x,y]$ for $i,j\in\{1,\ldots,n\}$, then there exist $t_{ij}\in[0,1]$ such that $x_i-x_j=(1-t_{ij})x+t_{ij}y$ for $i,j\in\{1,\ldots,n\}$.

Let $\mathbf{p}=(p_1,\ldots,p_n)\in\mathbf{P}^n$. Then by the above equality we get that

$$p_ip_j(x_i-x_j)=(1-t_{ij})p_ip_jx+t_{ij}p_ip_jy$$

for any $i,j\in\{1,\ldots,n\}$. If we sum over i,j from 1 to n, then we get

$$\begin{aligned}0&=\sum_{i,j=1}^n p_ip_j(x_i-x_j)=\sum_{i,j=1}^n[(1-t_{ij})p_ip_jx+t_{ij}p_ip_jy]\\&=\bigg(1-\sum_{i,j=1}^n t_{ij}p_ip_j\bigg)x+\bigg(\sum_{i,j=1}^n t_{ij}p_ip_j\bigg)y.\end{aligned}\tag{4.3}$$

Now, due to the fact that $(1-t)x+ty=0$ and the representation is unique, we get that $t=\sum_{i,j=1}^n t_{ij}p_ip_j$.

On the other hand, we have (see Theorem 2.4)

$$\begin{aligned}t_{ij}(1-t_{ij})&[\nabla_-f(y)(y-x)-\nabla_+f(x)(y-x)]\\&\ge t_{ij}f(x)+(1-t_{ij})f(y)-f[t_{ij}x+(1-t_{ij})y]\\&=t_{ij}f(x)+(1-t_{ij})f(y)-f(x_i-x_j).\end{aligned}\tag{4.4}$$

Now, if we multiply (4.4) by $p_ip_j\ge 0$, sum over i and j from 1 to n and divide by 2, then we get

$$\begin{aligned}\frac{1}{2}&[\nabla_-f(y)(y-x)-\nabla_+f(x)(y-x)]\sum_{i,j=1}^n p_ip_jt_{ij}(1-t_{ij})\\&\ge\frac{1}{2}[(1-t)f(x)+tf(y)]-G_f(\mathbf{p},\mathbf{x}),\end{aligned}\tag{4.5}$$

which is an interesting inequality in itself provided that one knows the parameters t_{ij} for any $i,j\in\{1,\ldots,n\}$.

https://doi.org/10.1017/S0004972711002048 Published online by Cambridge University Press

In the case that these are not known, since $t_{ij}(1-t_{ij}) \leq \frac{1}{4}$ for any $i, j \in \{1, \ldots, n\}$, then

$$\sum_{i,j=1}^{n} p_i p_j t_{ij}(1-t_{ij}) \leq \frac{1}{4},$$

which together with (4.5) provides the desired result (4.2). □

The following particular case for norms is of interest.

COROLLARY 4.3. *Let $(X, \|\cdot\|)$ be a normed space. If x and y are two nonzero vectors and $t \in [0, 1]$ with $(1-t)x + ty = 0$ then, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ with the property that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, \ldots, n\}$, we have the inequality*

$$\begin{aligned} 0 &\leq \tfrac{1}{2}[(1-t)\|x\|^r + t\|y\|^r] - G_r(\mathbf{p}, \mathbf{x}) \\ &\leq \tfrac{1}{8} r[\langle y - x, y\rangle_i \|y\|^{r-2} - \langle y - x, x\rangle_s \|x\|^{r-2}], \end{aligned} \tag{4.6}$$

for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ and $r \geq 1$.

REMARK 4.4. We observe that if $(X, \langle\cdot, \cdot\rangle)$ is an inner product space, then inequality (4.7) has a simpler form, namely,

$$0 \leq \tfrac{1}{2}[(1-t)\|x\|^r + t\|y\|^r] - G_r(\mathbf{p}, \mathbf{x}) \leq \tfrac{1}{8} r\langle y - x, \|y\|^{r-2} y - \|x\|^{r-2} x\rangle, \tag{4.7}$$

for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}^n$ and $r \geq 1$.

Acknowledgement

The author would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

References

[1] P. Cerone and S. S. Dragomir, 'Bounds for the Gini mean difference of an empirical distribution', *Appl. Math. Lett.* **19** (2006), 283–293.

[2] P. Cerone and S. S. Dragomir, 'Bounds for the r-weighted Gini mean difference of an empirical distribution', *Math. Comput. Modelling* **49**(1–2) (2009), 180–188.

[3] S. S. Dragomir, 'Inequalities in terms of the Gâteaux derivatives for convex functions in linear spaces with applications', *Bull. Aust. Math. Soc.* (2011), doi: 10.1017/S0004972710002054.

[4] S. S. Dragomir, 'Some refinements of Jensen's inequality', *J. Math. Anal. Appl.* **168**(2) (1992), 518–522.

[5] S. S. Dragomir, *Semiinner Products and Applications* (Nova Science Publishers, New York, 2004).

[6] S. S. Dragomir, *Discrete Inequalities of the Cauchy–Bunyakovsky–Schwarz Type* (Nova Science Publishers, New York, 2004).

[7] S. S. Dragomir, 'Bounds for the normalised Jensen functional', *Bull. Aust. Math. Soc.* **74**(3) (2006), 471–478.

[8] S. S. Dragomir, 'A refinement of Jensen's inequality with applications for f-divergence measures', *Taiwanese J. Math.* **14**(1) (2010), 153–164.

[9] S. S. Dragomir, 'Superadditivity of some functionals associated with Jensen's inequality for convex functions on linear spaces with applications', *Bull. Aust. Math. Soc.* **82**(1) (2010), 44–61.
[10] S. S. Dragomir, 'Weighted f-Gini mean difference for convex and symmetric functions in linear spaces', *Comput. Math. Appl.* **60** (2010), 734–743.
[11] G. M. Giorgi, 'Bibliographic portrait of the Gini concentration ratio', *Metron* **48** (1990), 183–221.
[12] G. M. Giorgi, Il rapporto di concentrazione di Gini, Liberia Editrice Ticci, Siena, 1992.
[13] G. A. Koshevoy and K. Mosler, 'Multivariate Gini indices', *J. Multivariate Anal.* **60** (1997), 252–276.

S. S. DRAGOMIR, Mathematics, School of Engineering and Science,
Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
and
School of Computational and Applied Mathematics, University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg, South Africa
e-mail: sever.dragomir@vu.edu.au