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MINIMALITY AND STABILITY OF MINIMAL HYPERSURFACES IN $\mathbb{R}^N$

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In this paper we show that the hypercone over $S^2 \times S^4$ is strictly area-minimizing in $\mathbb{R}^8$. We also show the existence of smooth embedded stable hypersurfaces in $\mathbb{R}^8$ which are not area-minimizing.

1. Introduction

Given a regular minimal hypercone C in $\mathbb{R}^{n+2}$ (that is $C = 0 \times \Sigma$

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for some smoothly embedded minimal hypersurface Σ of S^{n+1}) , we say that C is strictly area-minimizing if there exists a constant $\theta > 0$ such that

$$(*)\ M(C_1) \le M(T) - \theta\, \varepsilon^{n+1}$$

for $T \in I_{n+1}(\mathbb{R}^{n+2})$, where $C_1 = C \llcorner B_1(0)$, whenever $\varepsilon \in (0,1)$, $\partial T = \partial C_1$ and Spt $(T) \cap B_\varepsilon(0) = \emptyset$.

Let E_+, E_- be the two connected components of $\mathbb{R}^{n+2} \sim C$. Then we say that C is one-sided strictly area minimizing in $\bar{E}_+$ (respectively, in $\bar{E}_-$) if $(*)$ holds for all such T above satisfying, in addition, the condition spt$(T) \subseteq \bar{E}_+$ (spt $(T) \subseteq \bar{E}_-$, respectively).

The aim of this note is to prove the following:

THEOREM. *Let* $\Sigma = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$ *where* $n \ge 2m$ *and either* $n \ge 6$, $m \ge 2$ *or* $n \ge 7$, $m \ge 1$. *Then* $C = 0 \times \Sigma$ *is strictly area minimizing in* $\mathbb{R}^{n+2}$. *If* $\Sigma = S^1\left(\sqrt{\frac{1}{6}}\right) \times S^5\left(\sqrt{\frac{5}{6}}\right)$, *then* $C = 0 \times \Sigma$ *is one-sided strictly area minimizing in* $\bar{E} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : |y| \le 5^{\frac{1}{2}} |x|\}$.

The strictly area minimality of $C(1,5) = 0 \times \Sigma$, $\Sigma = S^1\left(\sqrt{\frac{1}{6}}\right) \times S^5\left(\sqrt{\frac{5}{6}}\right)$, in $\bar{E}$ implies that $C(1,5)$ is stable (see [5]). In fact, it is strictly stable by [2] and [6] . Moreover, we have the following:

COROLLARY. $E = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : |y| < 5^{\frac{1}{2}} |x|\}$ *is foliated by smoothly embedded minimal hypersurfaces. Each of these hypersurfaces is one-sided area minimizing (hence stable) but not globally area minimizing.*

The above corollary solves the open problem [1.6] of [1].

2. Proofs

First we recall some results and notation from the recent work of Hardt and Simon [3]. They show that, if C is area-minimizing, then there exist minimal hypersurfaces $S_\pm \subset E_\pm$ which coincide near infinity with

$$\{x \pm V_\pm(x)\, \nu_C(x) : x \in C\}$$

where $V_\pm$ are functions on C and ν_C is an orienting unit normal vector field for C. Let $\gamma_\pm$ denote the characteristic exponents of the O.D.E. obtained by separating variables in the Jacobi field equation for C. By [3], we have the following alternative characterizations of strict minimality:

(i) $V_\pm$ both have the slower decay at infinity. That is

$$\liminf_{|x| \to \infty} |x|^{\gamma-} V_\pm(x) > 0 \quad \text{in the case that } \Gamma_+ > \gamma_-$$

$$\liminf\, (\log |x|^{-1})\, |x|^{(n-1)/2}\, V_\pm(x) > 0 \quad \text{in the case that}$$

$$\gamma_+ = \gamma_- = (n-1)/2 \,.$$

(ii) There are a closed, homothetically invariant $K \subset \mathbb{R}^{n+2}$ with H^{n+1}-measure zero and a C^1-vector field X on $\mathbb{R}^{n+2} \sim K$ such that $X = \nu_C$ on $C \sim K$ and $|X| \le 1$, $\pm \operatorname{div} X \ge 0$ on $E_\pm$, and at least one of these inequalities is strict in at least one point $x_+ \in E_+ \sim K$ and at least one point $x_- \in E_- \sim K$.

By (ii) and the construction of Lawson [4], we see that all known examples of minimizing hypercones, except the case

$\Sigma = S^2\left(\sqrt{\frac{1}{3}}\right) \times S^4\left(\sqrt{\frac{2}{3}}\right)$, are strictly area minimizing.

Our theorem is, actually, a directly consequence of the characterization (i) and the O.D.E. results due to Simoes [7].

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Proof of Theorem. For $\Sigma = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$, $|A_\Sigma|^2 =$ the square of the length of the second fundamental form of $\Sigma = n$, see [6]. Since $\gamma_+ \geq \gamma_-$ are the roots of the characteristic equation: $\gamma^2 - (n-1)\gamma + n = 0$, we have that

$$\gamma_\pm = \tfrac{1}{2}(n-1 \pm [(n-1)^2-4n]^{\frac{1}{2}}) = \tfrac{1}{2}(n-1 \pm (n^2-6n+1)^{\frac{1}{2}})$$

Now for $n \geq 6$, $n \geq 2m$ and $m \geq 1$, we have, by [7, Theorem 2.9.3], on S_+ the following:

(a) $$\text{Lim}\,[\arctan(dv/du) - \tfrac{\pi}{4}]/[\arctan(V/U) = \tfrac{\pi}{4}] = -\gamma_- ,$$

where $v = |y|$, $u = 5^{1/2}\ |x|$ and $U > V$; and S_+ denotes the leaf of the global foliation (see [3], [7]) in $U > V$, which passes through the point $U = 1$ and $V = 0$.

Then (a) is equivalent to

(a') $$\lim_{u \to +\infty}(dY/du)/(Y/u) = -\gamma_-$$

where $Y = u - v > 0$.

The latter implies that

$$u - v = u^{-\gamma_-} + o(u^{-\gamma_-}) \quad \text{as} \quad u \to +\infty .$$

Similarly, for $n \geq 6$, $n \geq 2m \geq 4$; or $n \geq 7$, $n \geq 2m \geq 2$; by [7, Theorem 2.9.4], we have on S_- the following:

(b) $$\lim\,[\arctan(dv/du) - \pi/4]/[\arctan v/u - \pi/4] = -\gamma_-$$

where $v > u$ and S_- denotes the leaf of the global foliation in $v > u$ (see [2], [6]), which passes through the point $u = 0$, $v = 1$.

Hence

$$v - u = u^{-\gamma_-} + O(u^{-\gamma_-})$$

as before.

Thus $S_\pm$ both decay to the cone $C : u = v$ at the slower rate, because $V_{\pm}(X) \simeq |X|^{-\gamma_-}$ follows from the fact that $|u - v| \simeq |X|^{-\gamma_-}$, where $X = (x,y)$. Thus we conclude, by (ii) that for $n \geq 2m$ and either $n \geq 6$, $m \geq 2$ or $n \geq 7$, $m \geq 1$ the corresponding minimal hypercones C are strictly minimizing. We also obtain that, when $n = 6$, $m = 1$, C is one-sided strictly area minimizing in $\overline{E}$. □

Proof of Corollary. Using the same technique as [2], one concludes that $\overline{E} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : |y| \leq \sqrt{5}\, |X|\}$ is foliated by $S_\lambda = \mu_\lambda \times S_+$, $0 \leq \lambda < \infty$. Each S_λ will be a smoothly embedded one-sided area minimizing hypersurface, for $0 < \lambda < \infty$, hence stable (see [3]). But S_λ cannot be area minimizing in $\mathbb{R}^8$, since $C(1,5)$ is not area minimizing in $\mathbb{R}^8$, and $C(1,5)$ is the tangent cone of S_λ at infinity. □

3. An open problem

The following problem, which was raised by Simon, remains open.

(P) Is there an example (other than $\mathbb{R}^2$ in $\mathbb{R}^3$) of a minimal hypercone C in $\mathbb{R}^n$ which is minimizing but not strictly minimizing?

The candidate $S^2 \times S^4$ is now ruled out by our result.

References

[1] *Geometric measure theory and the calculus of variations.* (Proc. Sympos. Pure Math. 44, W.K. Allard and F.J. Almgren, Jr., Eds. American Mathematical Society, Providence, Rhode Island, 1986.)

[2] L. Caffarelli, R. Hardt, L. Simon, "Minimal surfaces with isolated singularities", *Manuscript Math.* 48 (1984), 1-18.

[3] R. Hardt, L. Simon, "Minimizing hypersurfaces with isolated singularities", (preprint).

[4] B. Lawson, "The equivalent Plateau problem and interior regularity", *Trans. Amer. Math. Soc.* 173 (1972), 231-246.

[5] F.H. Lin, "Approximation by smoothly embedded hypersurfaces with positive mean curvature", (preprint).

[6] R. Osserman, "Minimal varieties", *Bull. Amer. Math. Soc.* 75 (1969), 1092-1120.

[7] P. Simoes, "On a class of minimal cones in $\mathbb{R}^n$", *Bull. Amer. Math. Soc.* 80 (1974), 488-489.

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