ON SOME INFINITE SERIES INVOLVING THE ZEROS OF BESSEL FUNCTIONS OF THE FIRST KIND

by IAN N. SNEDDON

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1. In this paper we shall be concerned with the derivation of simple expressions for the sums of some infinite series involving the zeros of Bessel functions of the first kind. For instance, if we denote by $\gamma_{\nu,n}$ $(n=1, 2, 3, \ldots)$ the positive zeros of $J_\nu(z)$, then, in certain physical applications, we are interested in finding the values of the sums

$$S_{2m,\nu} = \sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}^{2m}} \qquad \ldots\ldots(1)$$

and

$$T_{2m,\nu} = \sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}^{2m-\nu+1} J_{\nu+1}(\gamma_{\nu,n})}, \qquad \ldots\ldots(2)$$

where m is a positive integer. In § 4 of this paper we shall derive a simple recurrence relation for $S_{2m,\nu}$ which enables the value of any sum to be calculated as a rational function of the order ν of the Bessel function. Similar results are given in § 5 for the sum $T_{2m,\nu}$.

From the recurrence relation $J_{\nu+1}(z)+J_{\nu-1}(z) = 2\nu J_\nu(z)/z$, we see that

$$J_{\nu-1}(\gamma_{\nu,n}) = -J_{\nu+1}(\gamma_{\nu,n})$$

so that we may write

$$T_{2m,\nu} = -\sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}^{2m-\nu+1} J_{\nu-1}(\gamma_{\nu,n})}.$$

Analogous to these sums we have the sums

$$S'_{2m,\nu} = \sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}'^{2m}(\gamma_{\nu,n}'^{2}-\nu^2)}, \qquad \ldots\ldots(3)$$

and

$$T'_{2m,\nu} = \sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}'^{2m-\nu}(\gamma_{\nu,n}'^{2}-\nu^2) J_\nu(\gamma'_{\nu,n})}, \qquad \ldots\ldots(4)$$

taken over the positive zeros $\gamma'_{\nu,n}$ $(n=1, 2, 3, \ldots)$ of the function $J'_\nu(z)$. These sums are discussed in §§ 6, 7 of this paper.

In § 8 we consider the determination of sums of the kind

$$U_{2m,\nu} = \sum_{n=1}^{\infty} \frac{1}{\lambda_{\nu,n}^{2m}(k^2\lambda_{\nu,n}^2+h^2-k^2\nu^2)}, \qquad \ldots\ldots(5)$$

where $\lambda_{\nu,n}$ $(n=1, 2, 3, \ldots)$ are the positive roots of the equation

$$hJ_\nu(\lambda)+k\lambda J'_\nu(\lambda)=0, \qquad \ldots\ldots(6)$$

with $\nu \geqslant -\frac{1}{2}$ and $(h/k)+\nu > 0$.

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The method employed here depends on simple properties of Fourier-Bessel series and Dini series and can obviously be extended to the determination of more complicated sums. This is indicated briefly in § 9.

The summation of series of this kind seems to have been considered systematically first by Forsyth [**1**], although isolated results were given earlier by Nielson [**2**]. By expressing $J_n(z)$, with n integral, as an infinite product and identifying the coefficient of z^{2r+n} in that representation with that of z^{2r+n} in the series expansion of $J_n(z)$, Forsyth made some observations about the sum $S_{2m,n}$ and found explicit forms for the sums $S_{2m,0}$, $S_{2m,1}$ for $m=1$, 2 and 3.

More recently Buchholz [**3**] has derived partial fraction expressions for the differences of the products of two linearly independent Bessel functions with unequal arguments multiplied by the quotient of two Bessel functions, and from them has deduced expressions for $S'_{2m,\nu}$ ($m = 0, 1, 2, 3, 4$). The expressions derived in equations (74) to (77) below are in agreement with these results.

These earlier investigations are conveniently summarized in [**4**], and since this book looks like becoming the standard book of reference on the subject we shall use here the notation adopted there.

The advantages of the method outlined here are that it depends essentially on well-known results in the theory of Bessel functions and requires no elaborate analytical apparatus, and that it is capable of generalization to the consideration of more complicated sums.

2. It is a well-known result of the theory of Bessel functions [**5**, p. 591] that, if the function $f(x)$ is defined arbitrarily in the open interval (0, 1) and is such that the integral

$$\int_0^1 x^{\frac{1}{2}} f(x)\,dx$$

exists and is absolutely convergent, and such that $f(x)$ has limited total fluctuation in the interval (a, b), where $0 < a < b < 1$, then the series

$$\sum_{n=1}^{\infty} a_n J_\nu(\gamma_{\nu,n} x),$$

in which $\gamma_{\nu,n}$ ($n=1, 2, 3, \ldots$) are the positive roots of the equation

$$J_\nu(\gamma) = 0, \qquad \text{(7)}$$

and

$$a_n = \frac{2}{J^2_{\nu+1}(\gamma_{\nu,n})} \int_0^1 x f(x) J_\nu(\gamma_{\nu,n} x)\,dx,$$

converges to the sum $\frac{1}{2}[f(x+0)+f(x-0)]$. If, in addition, the function $f(x)$ is continuous in the interval $0 < x < 1$, then we may write

$$\tfrac{1}{2} f(x) = \sum_{n=1}^{\infty} \frac{J_\nu(\gamma_{\nu,n} x)}{J^2_{\nu+1}(\gamma_{\nu,n})} \int_0^1 t f(t) J_\nu(\gamma_{\nu,n} t)\,dt. \qquad \text{(8)}$$

If, now, we multiply both sides of this equation by $x^{\nu+1}$ and integrate from 0 to x, we find, on making use of the result

$$\int_0^x x^{\nu+1} J_\nu(\xi x)\,dx = \frac{x^{\nu+1}}{\xi} J_{\nu+1}(\xi x), \qquad \text{(9)}$$

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that

$$\frac{1}{2}\int_0^x u^{\nu+1}f(u)\,du = \sum_{n=1}^{\infty} \frac{x^{\nu+1}J_{\nu+1}(\gamma_{\nu,n}x)}{\gamma_{\nu,n}J_{\nu+1}^2(\gamma_{\nu,n})} \int_0^1 tf(t)J_\nu(\gamma_{\nu,n}t)\,dt. \quad \text{.........(10)}$$

Multiplying both sides of this equation by $x^{-2\nu-1}$ and making use of the result

$$\int_x^1 x^{-\nu}J_{\nu+1}(\gamma_{\nu,n}x)\,dx = \gamma_{\nu,n}^{-1}\, x^{-\nu}J_\nu(\gamma_{\nu,n}x)$$

we obtain the equation

$$\frac{1}{2}\int_x^1 y^{-2\nu-1}\,dy\int_0^y u^{\nu+1}f(u)\,du = \sum_{n=1}^{\infty} \frac{x^{-\nu}J_\nu(\gamma_{\nu,n}x)}{\gamma_{\nu,n}^2 J_{\nu+1}^2(\gamma_{\nu,n})} \int_0^1 tf(t)J_\nu(\gamma_{\nu,n}t)\,dt. \quad \text{.........(11)}$$

Putting $x=1$ in equation (9) we obtain the result

$$\sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}J_{\nu+1}(\gamma_{\nu,n})} \int_0^1 xf(x)J_\nu(\gamma_{\nu,n}x)\,dx = \frac{1}{2}\int_0^1 x^{\nu+1}f(x)\,dx\,; \quad \text{.........(12)}$$

letting $x \to 0$ in equation (11) and using the fact that

$$\lim_{z\to 0} z^{-\nu}J_\nu(z) = \frac{1}{2^\nu\Gamma(\nu+1)},$$

we obtain the identity

$$\sum_{n=1}^{\infty} \frac{1}{\gamma_{\nu,n}^{2-\nu}J_{\nu+1}^2(\gamma_{\nu,n})} \int_0^1 xf(x)J_\nu(\gamma_{\nu,n}x)\,dx = 2^{\nu-1}\Gamma(\nu+1)\int_0^1 y^{-2\nu-1}\,dy\int_0^y u^{\nu+1}f(u)\,du. \quad \text{......(13)}$$

Similarly, if $\nu > 0$, and $\gamma'_{\nu,n}\,(n=1, 2, 3, \ldots)$ are the positive roots of the equation

$$J'_\nu(z) = 0, \quad \text{.........(14)}$$

we know that, if $f(x)$ satisfies the same conditions as before,

$$f(x) = \sum_{n=1}^{\infty} a'_n J_\nu(\gamma'_{\nu,n}x), \quad \text{.........(15)}$$

where

$$a'_n = \frac{2}{[J_\nu(\gamma'_{\nu,n})]^2}\cdot\frac{\gamma'^2_{\nu,n}}{\gamma'^2_{\nu,n}-\nu^2}\int_0^1 xf(x)J_\nu(\gamma'_{\nu,n}x)\,dx\,; \quad \text{.........(16)}$$

using equation (9), the recurrence relation

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x) \quad \text{.........(17)}$$

and equation (14), we find that

$$\int_0^1 x^{\nu+1}J_\nu(\gamma'_{\nu,n}x)\,dx = \frac{\nu}{\gamma'^2_{\nu,n}}J_\nu(\gamma'_{\nu,n}x).$$

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Hence, if we multiply equation (15) by $x^{\nu+1}$, integrate and use this last result, we find that

$$\sum_{n=1}^{\infty} \frac{1}{(\gamma_{\nu,n}'^2 - \nu^2) J_\nu(\gamma_{\nu,n}')} \int_0^1 x f(x) J_\nu(\gamma_{\nu,n}' x)\, dx = \frac{1}{2\nu} \int_0^1 x^{\nu+1} f(x)\, dx. \quad \text{.............. (18)}$$

From the result

$$\int_0^x u^{\nu+1} f(u)\, du = \sum_{n=1}^{\infty} a_n' \frac{x^{\nu+1}}{\gamma_{\nu,n}'} J_{\nu+1}(\gamma_{\nu,n}' x)$$

we obtain the identity

$$\int_x^1 y^{-2\nu-1}\, dy \int_0^y u^{\nu+1} f(u)\, du = \sum_{n=1}^{\infty} \frac{a_n' x^{-\nu}}{\gamma_{\nu,n}'^2} J_\nu(\gamma_{\nu,n}' x).$$

Letting $x \to 0$ we find that

$$\int_0^1 y^{-2\nu-1}\, dy \int_0^y u^{\nu+1} f(u)\, du = \sum_{n=1}^{\infty} \frac{a_n'}{\gamma_{\nu,n}'^{2-\nu}},$$

which is equivalent to the identity

$$\sum_{n=1}^{\infty} \frac{\gamma_{\nu,n}'^{\nu}}{(\gamma_{\nu,n}'^2 - \nu^2) J_\nu^2(\gamma_{\nu,n}')} \int_0^1 x f(x) J_\nu(\gamma_{\nu,n}' x)\, dx = \frac{1}{2} \int_0^1 y^{-2\nu-1}\, dy \int_0^y u^{\nu+1} f(u)\, du. \quad \text{......... (19)}$$

Corresponding to the Fourier-Bessel expansion (8) we have the Dini expansion [**5**, p. 596]

$$f(x) = \sum_{n=1}^{\infty} b_n J_\nu(\lambda_{\nu,n} x), \quad \text{... (20)}$$

where $\lambda_{\nu,n}$ $(n = 1, 2, 3, \ldots)$ are the positive roots of the equation

$$h J_\nu(\lambda) + k\lambda J_\nu'(\lambda) = 0, \quad \text{...................................... (21)}$$

in which h and k are constants, $\nu \geqslant -\frac{1}{2}$ and we confine our attention to the case

$$\frac{h}{k} + \nu > 0. \quad \text{.. (22)}$$

The coefficients b_n are given by the equation

$$b_n = \frac{2k^2 \lambda_{\nu,n}^2}{J_\nu^2(\lambda_{\nu,n})(k^2\lambda_{\nu,n}^2 + h^2 - k^2\nu^2)} \int_0^1 x f(x) J_\nu(\lambda_{\nu,n} x)\, dx. \quad \text{..................... (23)}$$

Using equation (9), the recurrence relation (17) and the equation (21), we find that

$$\int_0^1 x^{\nu+1} J_\nu(\lambda_{\nu,n} x)\, dx = \frac{k\nu + h}{k\lambda_{\nu,n}^2} J_\nu(\lambda_{\nu,n}), \quad \text{............................. (24)}$$

from which it follows that

$$\sum_{n=1}^{\infty} \frac{\int_0^1 x f(x) J_\nu(\lambda_{\nu,n} x)\, dx}{(k^2\lambda_{\nu,n}^2 + h^2 - k^2\nu^2) J_\nu(\lambda_{\nu,n})} = \frac{1}{2k(h + k\nu)} \int_0^1 x^{\nu+1} f(x)\, dx. \quad \text{................. (25)}$$

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3. We shall now derive recurrence formulae for some definite integrals whose values we require in making particular applications of the identities (12), (13), (18), (19) and (25).

Consider the integral

$$\int_0^1 x^{2m+\nu+1} J_\nu(\xi x)\, dx \quad (m > 0,\ \nu > 0). \quad \ldots\ldots(26)$$

If we make use of the result (9) and integrate by parts we find that the integral can be written in the form

$$\frac{J_{\nu+1}(\xi)}{\xi} - \frac{2m}{\xi}\int_0^1 x^{2m+2\nu} \,.\, x^{-\nu} J_{\nu+1}(\xi x)\, dx.$$

This second integral can be reduced in turn by writing

$$x^{-\nu} J_{\nu+1}(\xi x) = -\frac{1}{\xi}\frac{\partial}{\partial x}[x^{-\nu} J_\nu(\xi x)]$$

and integrating by parts. In this way we find that the integral (26) takes the form

$$\frac{J_{\nu+1}(\xi)}{\xi} + \frac{2m J_\nu(\xi)}{\xi^2} - \frac{4m(m+\nu)}{\xi^2}\int_0^1 x^{2m+\nu-1} J_\nu(\xi x)\, dx. \quad \ldots\ldots(27)$$

Hence, if we write

$$I_m(\nu, n) = \frac{1}{\gamma_{\nu,n} J_{\nu+1}(\gamma_{\nu,n})}\int_0^1 x^{\nu+2m+1} J_\nu(\gamma_{\nu,n} x)\, dx, \quad \ldots\ldots(28)$$

where m and n are positive integers and $\gamma_{\nu,n}\,(n=1, 2, 3, \ldots)$ are the roots of equation (7), we find that

$$I_m(\nu, n) = \frac{1}{\gamma_{\nu,n}^2} - \frac{4m(m+\nu)}{\gamma_{\nu,n}^2} I_{m-1}(\nu, n). \quad \ldots\ldots(29)$$

From equation (9) we have the result

$$I_0(\nu, n) = \frac{1}{\gamma_{\nu,n}^2}, \quad \ldots\ldots(30)$$

so that, combining this equation with the recurrence formula (29), we find that, if m is a positive integer,

$$I_m(\nu, n) = \frac{1}{\gamma_{\nu,n}^2}\sum_{r=0}^{m}\frac{m!\,\Gamma(m+\nu+1)}{(m-r)!\,\Gamma(m+\nu-r+1)}\left(-\frac{4}{\gamma_{\nu,n}^2}\right)^r. \quad \ldots\ldots(31)$$

Similarly, if we write

$$I'_m(\nu, n) = \frac{1}{J_\nu(\gamma'_{\nu,n})}\int_0^1 x^{\nu+2m+1} J_\nu(\gamma'_{\nu,n} x)\, dx, \quad \ldots\ldots(32)$$

we find that

$$I'_m(\nu, n) = \frac{2m}{\gamma'^2_{\nu,n}} + \frac{J_{\nu+1}(\gamma'_{\nu,n})}{\gamma'_{\nu,n} J_\nu(\gamma'_{\nu,n})} - \frac{4m(m+\nu)}{\gamma'^2_{\nu,n}} I'_{m-1}(\nu, n).$$

Now, putting $x=\gamma'_{\nu,n}$ in equation (17), we see that

$$\frac{J_{\nu+1}(\gamma'_{\nu,n})}{\gamma'_{\nu,n} J_\nu(\gamma'_{\nu,n})} = \frac{\nu}{\gamma'^2_{\nu,n}},$$

so that

$$I'_m(\nu, n) = \frac{2m+\nu}{\gamma'^2_{n,\nu}} - \frac{4m(m+\nu)}{\gamma'^2_{\nu,n}} I'_{m-1}(\nu, n). \qquad \text{(33)}$$

Also, from equation (9), we know that

$$I'_0(\nu, n) = \frac{\nu}{\gamma'^2_{\nu,n}}. \qquad \text{(34)}$$

From equations (33) and (34) we find that, if m is a positive integer,

$$I'_m(\nu, n) = \frac{1}{\gamma'^2_{\nu,n}} \sum_{r=0}^{m} (2m - 2r + \nu) \frac{m!\,\Gamma(m+\nu+1)}{(m-r)!\,\Gamma(m+\nu-r+1)} \left(-\frac{4}{\gamma'^2_{\nu,n}}\right)^r. \qquad \text{(35)}$$

Another integral we require is

$$H_m(\nu, n) = \frac{k}{J_\nu(\lambda_{\nu,n})} \int_0^1 x^{\nu+2m+1} J_\nu(\lambda_{\nu,n} x)\, dx, \qquad \text{(36)}$$

where $\lambda_{\nu,n}$ is a root of the equation (21). From the expression (27) with $\xi = \lambda_{\nu,n}$ with the help of equations (17) and (21), we find that

$$H_m(\nu, n) = \frac{1}{\lambda^2_{\nu,n}} \sum_{r=0}^{m} [h + (2m - 2r + \nu)k] \frac{m!\,\Gamma(m+\nu+1)}{(m-r)!\,\Gamma(m+\nu-r+1)} \left(-\frac{4}{\lambda^2_{\nu,n}}\right)^r. \qquad \text{(37)}$$

In particular, we have

$$H_0(\nu, n) = \frac{h + k\nu}{\lambda^2_{\nu,n}}. \qquad \text{(38)}$$

4. In this section we shall make use of the results derived in the last two sections to derive a recurrence formula for $S_{2m,\nu}$ and then to study some particular cases.

If we put $f(x) = x^{\nu+2m}$ in equation (12), we obtain the relation

$$\sum_{m=1}^{\infty} I_m(\nu, n) = \frac{1}{4(\nu+m+1)},$$

where $I_m(\nu, n)$ is defined by equation (28). Substituting the form (31) for $I_m(\nu, n)$ and interchanging the order of the summations, we find that

$$\sum_{r=0}^{m} \frac{m!\,\Gamma(m+\nu+1)}{(m-r)!\,\Gamma(m+\nu-r)} (-4)^r S_{2r+2,\nu} = \frac{1}{4(\nu+m+1)},$$

where $S_{2m,\nu}$ is defined by equation (1). From this last equation we can easily show that the sum $S_{2m,\nu}$ satisfies the recurrence formula

$$S_{2m+2,\nu} = \frac{1}{4} \sum_{r=1}^{m} \left(-\frac{1}{4}\right)^{m-r} \frac{S_{2r,\nu}}{(m-r+1)!\,(\nu+1)_{m-r+1}} + (-1)^m \left(\frac{1}{4}\right)^{m+1} \frac{1}{m!\,(\nu+1)_{m+1}}, \qquad \text{(39)}$$

where we have introduced the symbol

$$(a)_s = a(a+1)\ldots(a+s-1).$$

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Furthermore, if we put $f(x)=x^\nu$ in equation (12) and make use of equation (30) we find that

$$S_{2,\nu} = \frac{1}{4(\nu+1)}. \qquad (40)$$

Using equations (39) and (40), we can obtain the value of $S_{2m,\nu}$ for any integral value of m. For instance, we have the formulae

$$S_{4,\nu} = \frac{1}{16(\nu+1)^2(\nu+2)}, \qquad (41)$$

$$S_{6,\nu} = \frac{1}{32(\nu+1)^3(\nu+2)(\nu+3)}, \qquad (42)$$

and

$$S_{8,\nu} = \frac{5\nu+11}{256(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}. \qquad (43)$$

Special Cases. Special cases of these formulae are of interest and so we outline the results for them.

(i) $\nu=\pm\frac{1}{2}$. The positive zeros of $J_{\frac{1}{2}}(z)$ are those of $\sin z$, so that they are $\gamma_{\frac{1}{2},n} = n\pi$. Thus we find that $S_{2m,\frac{1}{2}} = s_{2m}\pi^{-2m}$ where

$$s_{2m} = \sum_{n=1}^{\infty} \frac{1}{n^{2m}}. \qquad (44)$$

Putting $\nu=\frac{1}{2}$ in the above formulae we find that

$$s_2 = \frac{\pi^2}{6}, \quad s_4 = \frac{\pi^4}{90}, \quad s_6 = \frac{\pi^6}{945}, \quad s_8 = \frac{\pi^8}{9450}.$$

Higher values are obtained from the formula

$$s_{2m+2} = \sum_{r=1}^{\infty} \frac{(-1)^{m-r}}{(m-r+1)!}\cdot\frac{s_{2r}}{(\frac{3}{2})_{m-r+1}}\cdot(\tfrac{1}{4}\pi^2)^{m-r+1} + \frac{(-1)^m}{m!}(\tfrac{1}{4}\pi^2)^{m+1}\frac{1}{(\frac{3}{2})_{m+1}}. \qquad (45)$$

Similar results hold for $\nu=-\frac{1}{2}$. Since the positive zeros of $J_{-\frac{1}{2}}(z)$ are those of $\cos z$, we see that $\gamma_{-\frac{1}{2}n}=(n-\frac{1}{2})\pi$ and that we can write $S_{2m,-\frac{1}{2}}=(\frac{1}{2}\pi)^{-2m}s^*_{2m}$, where

$$s^*_{2m} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2m}}. \qquad (46)$$

It follows that

$$s^*_2 = \frac{\pi^2}{8}, \quad s^*_4 = \frac{\pi^4}{96}, \quad s^*_6 = \frac{\pi^6}{960}, \dots$$

the higher values being given by the formula

$$s^*_{2m+2} = \sum_{r=1}^{m} \frac{(-1)^{m-r}s^*_{2r}}{(m-r+1)!(\frac{1}{2})_{m-r+1}}\left(\frac{\pi^2}{16}\right)^{m-r+1} + \frac{(-1)^m}{m!}\left(\frac{\pi^2}{16}\right)^{m+1}\frac{1}{(\frac{1}{2})_{m+1}}. \qquad (47)$$

(ii) $\nu = \frac{3}{2}$ (the roots of $\tan z = z$). If we let $\nu = \frac{3}{2}$ we obtain for $\gamma_{\frac{3}{2},n}$ the positive roots, σ_n say, of the equation $\tan z = z$. Putting $\nu = \frac{3}{2}$ in the above formulae we find that

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$$\sum_{n=1}^{\infty}\frac{1}{\sigma_n^2}=\frac{1}{10},\quad \sum_{n=1}^{\infty}\frac{1}{\sigma_n^4}=\frac{1}{350},\quad \sum_{n=1}^{\infty}\frac{1}{\sigma_n^6}=\frac{1}{7875},\quad \sum_{n=1}^{\infty}\frac{1}{\sigma_n^8}=\frac{37}{6063750}. \qquad (48)$$

For integral values of m greater than 4 we make use of the recurrence relation

$$\Sigma_{2m+2}=\sum_{r=1}^{\infty}(-1)^{m-r}\left(\frac{1}{4}\right)^{m-r+1}\frac{\Sigma_{2r}}{(m-r+1)!\,(\frac{5}{2})_{m-r+1}}+(-1)^m\left(\frac{1}{4}\right)^{m+1}\frac{1}{m!\,(\frac{5}{2})_{m+1}} \qquad (49)$$

to obtain values of the sum

$$\Sigma_{2m}=\sum_{n=1}^{\infty}\frac{1}{\sigma_{2m}^{n}}. \qquad (50)$$

(iii) $\nu=0$. If we denote the zeros of $J_0(z)$ by κ_n, $(n=1, 2, 3, \ldots)$, we find on putting $\nu=0$ in the general formulae, that

$$\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}=\frac{1}{4},\quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^4}=\frac{1}{32},\quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^6}=\frac{1}{192},\quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^8}=\frac{11}{12288},\quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^{10}}=\frac{19}{122880}.$$

Values of the sum

$$K_{2m}=\sum_{n=1}^{\infty}\frac{1}{\kappa_n^{2m}} \qquad (51)$$

for higher values of m are given by the recurrence formula

$$K_{2m+2}=\sum_{r=1}^{m}(-1)^{m-r}\left(\frac{1}{4}\right)^{m-r+1}\frac{K_{2r}}{[(m-r+1)!]^2}+(-1)^m\left(\frac{1}{4}\right)^{m+1}\frac{1}{m!\,(m+1)!}. \qquad (52)$$

(iv) $\nu=1$. If we denote the positive zeros of $J_1(z)$ by $\mu_n (n=1, 2, 3, \ldots)$, we find that

$$\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}=\frac{1}{8},\quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^4}=\frac{1}{192},\quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^6}=\frac{1}{3072},\quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^8}=\frac{1}{46080}.$$

For integral values of m greater than 4, values of the sum

$$M_{2m}=\sum_{n=1}^{\infty}\frac{1}{\mu_n^{2m}}$$

are given by the recurrence formula

$$M_{2m+2}=\sum_{r=1}^{\infty}(-1)^{m-r}\left(\frac{1}{4}\right)^{m-r+1}\frac{M_{2r}}{(m-r+1)!\,(m-r+2)!}+(-1)^m\left(\frac{1}{4}\right)^{m+1}\frac{1}{m!\,(m+2)!}. \qquad (53)$$

5. We now turn our attention to the evaluation of $T_{2m,\nu}$. If we put $f(x)=x^{\nu+2m}$ in equation (13) we obtain the identity

$$\sum_{n=1}^{\infty}\frac{I_m(\nu, n)}{\gamma_{\nu,n}^{1-\nu}J_{\nu+1}(\gamma_{\nu,n})}=\frac{2^{\nu-3}\Gamma(\nu+1)}{(m+1)(m+\nu+1)}.$$

Substituting for $I_m(\nu, n)$ from equation (31) and interchanging the order of the summations, we find that

$$\sum_{r=0}^{m}\frac{(-4)^r m!\,\Gamma(m+\nu+1)}{(m-r)!\,\Gamma(m+\nu-r+1)}T_{2r+2,\nu}=\frac{2^{\nu-3}\Gamma(\nu+1)}{(m+1)(m+\nu+1)},$$

from which we can deduce the recurrence relation

$$T_{2m+2,\nu} = \frac{1}{4}\sum_{r=1}^{m}\left(-\frac{1}{4}\right)^{m-r}\frac{T_{2r,\nu}}{(m-r+1)!\,(\nu+1)_{m-r+1}} + (-1)^m\left(\frac{1}{2}\right)^{2m+3-\nu}\frac{[\Gamma(\nu+1)]^2}{(m+1)!\,(m+\nu+2)} \quad \ldots(54)$$

for the sum $T_{2m,\nu}$. Also, if we put $f(x)=x^\nu$ in equation (13) and make use of equation (30), we find that

$$T_{2,\nu} = \left(\frac{1}{2}\right)^{3-\nu}\frac{\Gamma(\nu+1)}{\nu+1}. \quad \ldots\ldots(55)$$

From equations (54) and (55) we obtain the results

$$T_{4,\nu} = \left(\frac{1}{2}\right)^{6-\nu}\frac{(\nu+3)\Gamma(\nu+1)}{(\nu+1)^2(\nu+2)} \quad \ldots\ldots(56)$$

and

$$T_{6,\nu} = \left(\frac{1}{2}\right)^{8-\nu}\frac{(\nu^2+8\nu+19)\Gamma(\nu+1)}{3(\nu+1)^3(\nu+2)(\nu+3)}. \quad \ldots\ldots(57)$$

Special Cases. (i) $\nu = \pm\frac{1}{2}$. Putting $\gamma_{\frac{1}{2},n} = n\pi$, $J_{\frac{3}{2}}(\gamma_{\frac{1}{2},n}) = (-1)^{n+1}2^{\frac{1}{2}}/(\pi n^{\frac{1}{2}})$, we see that we may write

$$T_{2m,\frac{1}{2}} = 2^{-\frac{1}{2}}\pi^{-2m+\frac{1}{2}}t_{2m}, \quad \ldots\ldots(58)$$

where

$$t_{2m} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2m}}. \quad \ldots\ldots(59)$$

Making the substitution $\nu=\frac{1}{2}$ in equations (55) to (57) and using equations (58) and (59), we find that

$$t_2 = \frac{\pi^2}{12},\quad t_4 = \frac{7\pi^4}{720},\quad t_6 = \frac{31\pi^6}{30240}, \quad \ldots\ldots(60)$$

and in general that

$$t_{2m+2} = \tfrac{1}{4}\pi^2\sum_{r=1}^{m}\frac{(-\frac{1}{4}\pi^2)^{m-r}t_{2r}}{(m-r+1)!\,(\frac{3}{2})_{m-r+1}} + \tfrac{1}{2}(-1)^m\frac{(\frac{1}{4}\pi^2)^{m+1}}{(m+1)!\,(\frac{3}{2})_{m+1}}. \quad \ldots\ldots(61)$$

Similarly, in the case $\nu=-\frac{1}{2}$, we write

$$T_{2m,-\frac{1}{2}} = \left(\frac{2}{\pi}\right)^{2m+\frac{1}{2}}t^*_{2m+1}, \quad \ldots\ldots(62)$$

where

$$t^*_{2m+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2m+1}}. \quad \ldots\ldots(63)$$

We then find, on putting $\nu=-\frac{1}{2}$ in equations (54) to (57), that

$$t^*_3 = \frac{\pi^3}{32},\quad t^*_5 = \frac{5\pi^5}{1536},\quad t^*_7 = \frac{61\pi^7}{184320}, \quad \ldots\ldots(64)$$

and that values of the sum corresponding to higher values of the suffix are given by the recurrence formula

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$$t^*_{2m+3} = \frac{\pi^2}{16}\sum_{r=1}^{m}\left(-\frac{\pi^2}{16}\right)^{m-r}\frac{t^*_{2r+1}}{(m-r+1)!\,(\frac{1}{2})_{m-r+1}} + (-1)^m\frac{\pi^{2m+3}}{m!\,(\frac{3}{2})_m}. \qquad (65)$$

(ii) $\nu=0$. With the notation of § 4 (iii) we obtain the formulae

$$\sum_{n=7}^{\infty}\frac{1}{\kappa_n^3 J_1(\kappa_n)} = \frac{1}{8}, \quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^5 J_1(\kappa_n)} = \frac{3}{128}, \quad \sum_{n=1}^{\infty}\frac{1}{\kappa_n^7 J_1(\kappa_n)} = \frac{19}{4608}, \qquad (66)$$

higher sums being given by the recurrence formula

$$K^*_{2m+1} = \frac{1}{4}\sum_{r=1}^{m}\left(-\frac{1}{4}\right)^{m-r+1}\frac{K^*_{2r}}{[(m-r+1)!]^2} + (-1)^m\left(\frac{1}{2}\right)^{2m+3}\frac{1}{[(m+1)!]^2}, \qquad (67)$$

where

$$K^*_{2m} = \sum_{n=1}^{\infty}\frac{1}{\kappa_n^{2m+1}J_1(\kappa_n)}. \qquad (68)$$

(iii) $\nu=1$. With the notation of § 4 (iv) we obtain the formulae

$$\sum_{n=1}^{\infty}\frac{1}{\mu_n^3 J_2(\mu_n)} = \frac{1}{16}, \quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^5 J_2(\mu_n)} = \frac{1}{96}, \quad \sum_{n=1}^{\infty}\frac{1}{\mu_n^7 J_2(\mu_n)} = \frac{7}{73728}. \qquad (69)$$

Putting

$$M^*_{2m+2} = \sum_{n=1}^{\infty}\frac{1}{\mu_n^{2m+1}J_2(\mu_n)} = -\sum_{n=1}^{\infty}\frac{1}{\mu_n^{2m+1}J_0(\mu_n)}, \qquad (70)$$

we have

$$M^*_{2m+2} = \frac{1}{4}\sum_{r=1}^{m}\left(-\frac{1}{4}\right)^{m-r+1}\frac{M^*_{2r}}{(m-r+1)!\,(m-r+2)!} + (-1)^m\left(\frac{1}{4}\right)^{m+1}\frac{1}{(m+1)!\,(m+2)!}, \qquad (71)$$

from equation (54).

6. The sums $S'_{2m,\nu}$ may be evaluated by a similar procedure ($\nu>0$). If we put $f(x)=x^{\nu+2m}$ in the identity (18), we find that

$$\sum_{n=1}^{\infty}\frac{I'_m(\nu,n)}{\gamma'^2_{\nu,n}-\nu^2} = \frac{1}{4\nu(\nu+m+1)}, \qquad (72)$$

where $I'_m(\nu,n)$ is defined by equation (32). Substituting the value (35) for the integral $I'_m(\nu,n)$ and interchanging the order of the summations, we find that

$$\sum_{r=0}^{m}(2m-2r+\nu)\frac{m!\,\Gamma(m+\nu+1)(-4)^r}{(m-r)!\,\Gamma(m+\nu-r+1)}S'_{2r+2,\nu} = \frac{1}{4\nu(\nu+m+1)},$$

from which it follows that

$$S'_{2m+2,\nu} = \frac{1}{4\nu}\sum_{r=1}^{m}\left(-\frac{1}{4}\right)^{m-r}\frac{(2m-2r+2+\nu)}{(m-r+1)!\,(\nu+1)_{m-r+1}}S'_{2r,\nu} + \frac{(-1)^m}{\nu^2}\left(\frac{1}{4}\right)^{m+1}\frac{1}{m!\,(\nu+1)_{m+1}}, \qquad (73)$$

where $S'_{2r,\nu}$ is defined by equation (3) of § 1.

Also, putting $f(x)=x^\nu$ in (18) and using (34), we find that

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$$S'_{2,\nu} = \frac{1}{4\nu^2(\nu+1)}, \qquad \text{(74)}$$

and hence that

$$S'_{4,\nu} = \frac{3\nu+4}{16\nu^3(\nu+1)^2(\nu+2)}, \qquad \text{(75)}$$

$$S'_{6,\nu} = \frac{5\nu^2+16\nu+12}{32\nu^4(\nu+1)^2(\nu+2)(\nu+3)}, \qquad \text{(76)}$$

and

$$S'_{8,\nu} = \frac{35\nu^4+273\nu^3+768\nu^2+896\nu+392}{256\nu^5(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}, \qquad \text{(77)}$$

in agreement with the expressions obtained by Buchholz. Sums corresponding to higher values of the suffix m can be obtained from the recurrence formula (73).

7. In this section we outline the method of deriving the values of the sums $T'_{2m,\nu}$. Making the substitution $f(x)=x^{\nu+2m}$ in the relation (19), we obtain the identity

$$\sum_{n=1}^{\infty} \frac{\gamma'^{\nu}_{\nu,n} I'_m(\nu,n)}{(\gamma'^2_{\nu,n}-\nu^2)J_\nu(\gamma'_{\nu,n})} = \frac{1}{8(m+1)(m+\nu+1)}, \qquad \text{(78)}$$

where the integral $I'_m(\nu, n)$ is defined by equation (32). If we insert the expression (33) for $I'_m(\nu, n)$ and interchange the order of the summations, we obtain the relation

$$\sum_{r=0}^{m} \frac{(2m-2r+\nu)m!\,\Gamma(m+\nu+1)(-4)^r}{(m-r)!\,\Gamma(m+\nu-r+1)} T'_{2r+2,\nu} = \frac{1}{8(m+1)(m+\nu+1)}, \qquad \text{(79)}$$

from which we deduce the recurrence formula

$$T'_{2m+2,\nu} = \frac{1}{4\nu}\sum_{r=1}^{m}\left(-\frac{1}{4}\right)^{m-r}\frac{(2m-2r+2+\nu)}{(m-r+1)!\,(\nu+1)_{m-r+1}}T'_{2r,\nu} + \frac{(-1)^m}{\nu}\left(\frac{1}{2}\right)^{2m+3}\frac{1}{(m+1)!\,(\nu+1)_{m+1}} \qquad \text{(80)}$$

for the sum $T'_{2m,\nu}$ defined by equation (4) of § 1.

The simplest case is obtained by putting $f(x)=x^\nu$ in equation (19) and making use of (34). We find that

$$T'_{2,\nu} = \frac{1}{8\nu(\nu+1)}. \qquad \text{(81)}$$

Using this result and the recurrence relation (80), we find that

$$T'_{4,\nu} = \frac{\nu^2+7\nu+8}{64\nu^2(\nu+1)^2(\nu+2)}, \qquad \text{(82)}$$

$$T'_{6,\nu} = \frac{\nu^4+14\nu^3+91\nu^2+210\nu+144}{768\nu^3(\nu+1)^3(\nu+2)(\nu+3)}, \qquad \text{(83)}$$

and so on.

8. We conclude our discussion of the sums listed in § 1 by considering $U_{2m,\nu}$. Putting $f(x)=x^{\nu+2m}$ in the identity (25), we find that

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$$\sum_{n=1}^{\infty} \frac{H_m(\nu, n)}{k^2\lambda_{\nu,n}^2 + h^2 - k^2\nu^2} = \frac{1}{4(h+k\nu)(\nu+m+1)}, \qquad (84)$$

where $H_m(\nu, n)$ is the integral defined by equation (36). Inserting the expression (37) for this integral and interchanging the order of the summations we obtain the identity

$$\sum_{r=0}^{m} [h+(2m-2r+\nu)k] \frac{m!\,\Gamma(m+\nu+1)(-4)^r U_{2r+2,\nu}}{(m-r)!\,\Gamma(m+\nu-r+1)} = \frac{1}{4(h+k\nu)(\nu+m+1)}, \qquad (85)$$

where the sum $U_{2m,\nu}$ is defined by equation (5) of § 1. From this last equation we find that

$$U_{2m+2,\nu} = \frac{1}{4(h+k\nu)} \sum_{r=1}^{m} [h+(2m-2r+2+\nu)k] \left(-\frac{1}{4}\right)^{m-r} \frac{U_{2r,\nu}}{(m-r+1)!\,(\nu+1)_{m-r+1}}$$
$$+(-1)^m \left(\frac{1}{4}\right)^{m+1} \frac{1}{(h+k\nu)^2 m!\,(\nu+1)_{m+1}}. \qquad (86)$$

If we put $f(x) = x^\nu$ in equation (25) and make use of the result (38), we find that

$$U_{2,\nu} = \frac{1}{4(h+k\nu)^2(\nu+1)}. \qquad (87)$$

From this result and the recurrence relation (86) we deduce the expressions

$$U_{4,\nu} = \frac{h+(3\nu+4)k}{16(h+k\nu)^3(\nu+1)^2(\nu+2)}, \qquad (88)$$

$$U_{6,\nu} = \frac{h^2+2(2\nu+3)hk+(5\nu^2+16\nu+2)k^2}{32(h+k\nu)^4(\nu+1)^3(\nu+2)(\nu+3)}, \qquad (89)$$

and so on.

9. Finally we consider briefly how the method used in the previous sections may be extended to the evaluation of sums other than those defined in § 1.

We evaluated the sums in the previous sections by putting $f(x) = x^{2m+\nu}$ in the fundamental identities (12), (13), (18), (19) and (25). By choosing other forms for the function $f(x)$ we can obtain the sums of series of a different kind.

For example, if we put $\nu = 0$, $f(x) = (1-x^2)^{-\frac{1}{2}}$ in equation (12) and use the integral

$$\int_0^1 \frac{x J_0(\xi x)\, dx}{\sqrt{(1-x^2)}} = \frac{\sin \xi}{\xi},$$

we obtain the result

$$\sum_{n=1}^{\infty} \frac{\sin \kappa_n}{\kappa_n^2 J_1(\kappa_n)} = \frac{1}{2}, \qquad (90)$$

where the κ_n $(n = 1, 2, 3, \ldots)$ are the positive zeros of $J_0(x)$.

Similarly, from equation (13), we find that

$$\sum_{n=1}^{\infty} \frac{\sin \kappa_n}{\kappa_n^3 J_1^2(\kappa_n)} = \tfrac{1}{2}(1-\log 2), \qquad (91)$$

and from equation (25) that

$$\sum_{n=1}^{\infty} \frac{\sin \lambda_n}{(k^2\lambda_n^2+h^2)J_0(\lambda_n)} = \frac{1}{2hk}, \quad \text{.....................................(92)}$$

where $\lambda_1, \lambda_2, \ldots$, are the roots of the equation

$$hJ_0(\lambda) = k\lambda J_1(\lambda), \quad \text{..(93)}$$

it being assumed that h/k is finite and positive.

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