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RESOLVING AN OLD PROBLEM ON THE PRESERVATION OF THE IFR PROPERTY UNDER THE FORMATION OF *K*-OUT-OF-*N* SYSTEMS WITH DISCRETE DISTRIBUTIONS

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Abstract

More than half a century ago, it was proved that the increasing failure rate (IFR) property is preserved under the formation of k-out-of-n systems (order statistics) when the lifetimes of the components are independent and have a common absolutely continuous distribution function. However, this property has not yet been proved in the discrete case. Here we give a proof based on the log-concavity property of the function $f(e^x)$. Furthermore, we extend this property to general distribution functions and general coherent systems under some conditions.

Keywords: Coherent systems; ageing classes; failure rate; log-concavity; order statistics

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1. Introduction

A k-out-of-n system with n components functions if at least k components work. Parallel and series systems are particular cases of such systems corresponding to $k = 1$ and $k = n$, respectively. These systems play an important role in reliability theory and life testing with several practical applications. The lifetime of such a system is described by the $(n - k + 1)$th-order statistic in a sample of size n (or the corresponding ordered component lifetime). For many years, different properties of such systems have been studied assuming that the lifetimes of the components are independent and identically distributed (i.i.d.) and continuously distributed. For thorough details of these concepts, we refer the readers to [5], [17], [18], [20], [26] and the references therein.

It is well known that the monotonicity property for the failure (or hazard) rate function of a life distribution plays an important role in modeling failure time data since it describes the ageing process. Therefore the identification and properties of increasing failure rate (IFR) and decreasing failure rate (DFR) distributions have been extensively studied in the literature. In the continuous case, the IFR property is equivalent to possessing ordered residual lifetimes, in which the residual lifetime of a younger used unit is more reliable than the residual lifetime of any older one. Hence the preservation of this natural property in systems is to be expected.

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The preservation of the IFR ageing class under the formation of k-out-of-n systems with i.i.d. components with a common absolutely continuous distribution function can be traced back to the 1963 paper by Esary and Proschan [9]. However, the DFR class is not preserved except in the case of series systems (see [20, p. 122]).

The preservation of the IFR class in k-out-of-n systems with i.i.d. components cannot be extended to coherent systems with different structures. In the i.i.d. continuous case, this preservation will depend on the structure of the system. In this sense, a sufficient condition for that preservation based on the signature of the system was obtained by Samaniego [25]. More recently, a necessary and sufficient condition was obtained in [21] (see also Theorem 4.1 of [20]). Surprisingly, there exist coherent systems with i.i.d. components that do not preserve the IFR property (see e.g. Example 4.1 of [20]). The condition provided in [21] can also be applied to systems with dependent and identically distributed (i.d.) component lifetimes. In this case, the preservation will also depend on the copula that describes the dependence between the component lifetimes.

As the IFR property was not preserved in all the coherent systems with i.i.d. components, another ageing class was considered, the new better than used (NBU) property, which just assumes that new units are always more reliable than used units (of any age). In this sense Esary *et al.* [10] (see also [5, p. 85] or [20, p. 131]) proved that the NBU ageing class is preserved in all the coherent systems with independent components. The same holds for the class of increasing failure rate average (IFRA) distributions. Also, Block and Savits [6] proved that the IFRA class is closed under convolution. Another relevant ageing class is the decreasing mean residual life (DMRL) property. Abouammoh and El-Neweihi [1] proved that this class is preserved under the formation of parallel systems with i.i.d. components. This result was extended in [19] given sufficient conditions for the preservation of DMRL/IMRL classes in systems with i.d. components. Preservation properties for NBUE/NWUE classes can be seen in [15] and [16]. Recent preservation results for systems under the exponential distribution were obtained in [24].

In another direction, discrete failure rates arise in several common situations in reliability theory where clock time is not the best scale to describe lifetimes. For example, in weapons reliability, the number of rounds fixed until failure is more important than the age at the time of failure. However, for the discrete case, the preservation of the IFR and DFR properties is not straightforward because of the complexity of the discrete failure rate. Some properties on ageing notions and order statistics in the discrete case can be found in [2], [7], [8], [11] and [14].

In the discrete case, Roy and Gupta [23] proved that IFRA ageing class is preserved in coherent systems with independent components. This property also holds in the discrete case for the NBU ageing class (see Proposition 4.1 of [20]). However, the IFR closure property has not yet been proved in the discrete case.

In this article we prove that the IFR class is preserved in k-out-of-n systems with i.i.d. components with a common discrete distribution. The proof is based on a characterization of log-concavity of the function $f(\mathrm{e}^x)$. This method motivated us to extend the result to general coherent systems having i.d. components with discrete distributions or with general distributions.

The rest of the paper is organized as follows. In Section 2 we include definitions and some preliminary results. The proof for the discrete case is in Section 3. The extension for arbitrary distributions and systems is presented in Section 4.

2. Preliminaries

Throughout the paper, increasing and decreasing mean non-decreasing and non-increasing, respectively. Furthermore, ratios, derivatives, and conditional distributions are assumed to be well-defined whenever they are used.

In this section we review some notions related to our main results (see [5]). First we give the following definition extracted from [3].

Definition 1. Let $f:\mathbb{R}\longmapsto\mathbb{R}_+$ be a Lebesgue-measurable function. Suppose

$$\{x: f(x)>0\}=(l,u)\subseteq\mathbb{R}.$$

The function f is said to be log-concave in (l, u) if

$$f(\alpha x+(1-\alpha)y)\geq[f(x)]^{\alpha}[f(y)]^{1-\alpha}$$

for all $x, y\in(l, u)$ and $\alpha\in(0, 1)$.

With this definition, f is log-concave if and only if $\log f$ is concave. Moreover, it is known (see e.g. [3]) that if the function f is continuous, then this definition is equivalent to

$$f(x_1+\delta)f(x_2)\geq f(x_1)f(x_2+\delta) \tag{1}$$

for all $\delta\geq 0$ and all $l\leq x_1\leq x_2\leq x_2+\delta\leq u$.

Definition 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the real line $\mathbb{R}$. A function $f:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ is said to be totally positive of order 2 (TP_2) if

$$f(x_1, y_1)f(x_2, y_2)-f(x_1, y_2)f(x_2, y_1)\geq 0 \tag{2}$$

for all $x_1, x_2\in\mathcal{X}$ and all $y_1, y_2\in\mathcal{Y}$ such that $x_1\leq x_2$ and $y_1\leq y_2$.

In the next lemma we give a useful characterization for log-concavity of the function $f(\mathrm{e}^x)$.

Lemma 1. *Let* $f:\mathbb{R}_+\longmapsto\mathbb{R}_+$ *be a Lebesgue-measurable function. Suppose*

$$\{x: f(x)>0\}=(l,u)\subseteq\mathbb{R}_+.$$

Then $f(\mathrm{e}^x)$ *is log-concave in* $(\log l, \log u)\subseteq\mathbb{R}$ *if and only if*

$$f(x)f(y)\geq f(x/\epsilon)f(y\epsilon) \tag{3}$$

for all $\epsilon\geq 1$ *and all* $l\leq x\leq y\leq y\epsilon\leq u$.

Proof. (Necessity) In [12] it is pointed out that $f(\mathrm{e}^x)$ is log-concave if and only if $f(s/t)$ is TP_2 in $(s, t)\in\mathbb{R}_+\times\mathbb{R}_+$. So, from (2), $f(\mathrm{e}^x)$ is log-concave in $(\log l, \log u)$ if and only if

$$f\left(\frac{s_1}{t_1}\right)f\left(\frac{s_2}{t_2}\right)\geq f\left(\frac{s_1}{t_2}\right)f\left(\frac{s_2}{t_1}\right)$$

for any $0<s_1\leq s_2$ and $0<t_1\leq t_2$ such that these values are inside the interval $(\log l, \log u)$. Now, for any $\epsilon\geq 1$ such that $l\leq x\leq y\leq y\epsilon\leq u$, consider the following values: $s_1=x$, $s_2=\epsilon y$, $t_1=1$, and $t_2=\epsilon$. Thus the desired result follows since $s_1=x\leq y\leq\epsilon y=s_2$ and $t_1=1\leq\epsilon=t_2$.

(Sufficiency) Let us assume now that (3) holds and let $g(x) = \log(f(e^x))$. Since f is Lebesgue-measurable, hence so is g. Letting $x = y$ in inequality (3), we have

$$(f(y))^2 \geq f(y/\epsilon)f(y\epsilon)$$

for all $\epsilon \geq 1$ such that $l \leq y \leq y\epsilon \leq u$. Taking $y = e^x$ and $\epsilon = e^c$, we get

$$2g(x) \geq g(x-c) + g(x+c)$$

for all real numbers x and c such that $c > 0$ and $\log l \leq x - c \leq x + c \leq \log u$. Let $a = x - c$ and $b = x + c$. Then

$$g\left(\frac{a+b}{2}\right) \geq \frac{g(a)+g(b)}{2}$$

for all real numbers a and b. So, g is mid-concave (for the definition and properties of mid-concavity, we refer the readers to [22]). As g is mid-concave and Lebesgue-measurable, then it is continuous (see e.g. [22, p. 221]). Now, since mid-concavity with continuity implies concavity, g is concave and thus $f(e^x)$ is log-concave. □

Let $X_1, \ldots, X_n$ be a random sample of size n formed with i.i.d. random variables with a common cumulative distribution function (CDF) $F(x) = \mathbb{P}(X \leq x)$ and survival (or reliability) function (SF) $\bar{F}(x) = \mathbb{P}(X > x)$. We denote the SF of its order statistics $X_{1:n} \leq \cdots \leq X_{n:n}$ by $\bar{F}_{k:n}$ for $k = 1, \ldots, n$. From expression (2.2.15) in [4], the SF of the kth-order statistic can be written as $\bar{F}_{k:n}(x) = G_{k:n}(\bar{F}(x))$, where

$$G_{k:n}(x) = k\binom{n}{k}\int_0^x y^{n-k}(1-y)^{k-1}\mathrm{d}y \tag{4}$$

is the CDF of a beta distribution. To get the proof of our main result, we would also need the following technical lemma.

Lemma 2. *If $G_{k:n}$ is defined as in (4), then $G_{k:n}(e^x)$ is log-concave in $(-\infty, 0)$ for $k = 1, \ldots, n$.*

Proof. The function $G_{k:n}(e^x)$ is log-concave in $(-\infty, 0)$ if and only if $\psi(x) = \log G_{k:n}(e^x)$ is concave in $(-\infty, 0)$. This property holds if

$$\psi'(x) = \frac{e^x g_{k:n}(e^x)}{G_{k:n}(e^x)} = \alpha(e^x)$$

is decreasing in $(-\infty, 0)$, where

$$g_{k:n}(x) = G'_{k:n}(x) = k\binom{n}{k}x^{n-k}(1-x)^{k-1} \tag{5}$$

and $\alpha(u) = ug_{k:n}(u)/G_{k:n}(u)$. A straightforward calculation shows that the function α is decreasing in (0,1) for $k = 1, \ldots, n$. □

Finally we include here the basic definitions and properties of systems needed for our results. For more properties we refer the reader to [5] and [20]. A *system* with n components is a Boolean function $\phi : \{0, 1\}^n \to \{0, 1\}$, where $x_i = 1$ (resp. 0) indicates that the ith component works (does not work) and where the state of the system $\phi(x_1, \ldots, x_n)$ just depends on the state of these components. A system is *semi-coherent* if ϕ is increasing, $\phi(0, \ldots, 0) = 0$ and

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$\phi(1, \ldots, 1) = 1$. A system is *coherent* if ϕ is increasing and it is strictly increasing in each variable for at least one point (see [5]).

Another well-known property is that the lifetime T of a system is also a function of the lifetimes $X_1, \ldots, X_n$ of its components. Moreover, if these component lifetimes have a common SF function $\bar{F}(t) = \mathbb{P}(X_i > t)$, then the system SF $\bar{F}_T(t) = \mathbb{P}(T > t)$ can be written as (see e.g. Theorem 2.11 of [20])

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)) \tag{6}$$

for all t, where $\bar{q} : [0, 1] \to [0, 1]$ is a distortion function, that is, it is continuous, increasing, and satisfies $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$. This function $\bar{q}$ does not depend on $\bar{F}$. In Theorem 4.1 of [20] it is proved that if $\bar{F}$ is absolutely continuous and $\bar{q}$ is differentiable in (0,1), then the IFR (DFR) class is preserved if and only if the function $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ is decreasing (increasing) in (0,1). It can be proved that this alpha function is decreasing for $T = X_{k:n}$ and i.i.d. components, that is, the IFR class is preserved under the formation of k-out-of-n in the i.i.d. continuous case. However, the DFR class is not preserved in these systems except in the case of series systems.

3. Preservation of the IFR class in the discrete case

Let us suppose in this section that X is a discrete random variable with ordered support $S_X = \{x_i\}_{i\in I}$, with $x_i < x_{i+1}$ for all $i, i+1 \in I$ and probability mass function (PMF) $p(x) = \mathbb{P}(X = x)$. The failure rate of a discrete distribution is (see [13, p. 45])

$$h(x_i) = \mathbb{P}(X = x_i \mid X \geq x_i) = \frac{\mathbb{P}(X = x_i)}{\mathbb{P}(X \geq x_i)} = \frac{p(x_i)}{\bar{F}(x_{i-1})} = 1 - \frac{\bar{F}(x_i)}{\bar{F}(x_{i-1})}, \quad i \in I.$$

Then we say that X is IFR (DFR) if and only if $h(x_i)$ is increasing (decreasing) in i, which is equivalent to $\bar{F}(x_i)/\bar{F}(x_{i-1})$ is decreasing (increasing) in i. This property holds if and only if

$$(\bar{F}(x_i))^2 \geq (\leq) \bar{F}(x_{i-1})\bar{F}(x_{i+1}) \quad \text{for all } i-1, i, i+1 \in I.$$

Let us now study the preservation of the IFR/DFR ageing classes in k-out-of-n systems. First we note that it is not difficult to see that the DFR class is not preserved under the formation of k-out-of-n systems in the discrete case by constructing a parallel system with two components having a common geometric distribution which are both DFR and IFR. Both IFR and DFR classes are closed under the formation of series systems with n i.i.d. components because their survival functions are $(\bar{F}(x))^n$ for $n = 1, 2, \ldots$.

Now we focus on the preservation of the IFR property in the discrete case.

Theorem 1. *Let X be a discrete random variable with ordered support $\{x_i\}_{i\in I}$. Let $X_1, \ldots, X_n$ be independent random variables with the same distribution as X. If X is IFR, then $X_{k:n}$ is IFR for $k = 1, \ldots, n$.*

Proof. We first note that X is IFR if and only if

$$\bar{F}(x_i)\bar{F}(x_i) \geq \bar{F}(x_{i-1})\bar{F}(x_{i+1}) \tag{7}$$

for all $i-1, i, i+1 \in I$. Analogously, $X_{k:n}$ is IFR if and only if

$$G_{k:n}(\bar{F}(x_i))G_{k:n}(\bar{F}(x_i)) \geq G_{k:n}(\bar{F}(x_{i-1}))G_{k:n}(\bar{F}(x_{i+1})) \tag{8}$$

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for all $i-1, i, i+1 \in I$, where $G_{k:n}$ is the function defined in (4). Then, from Lemmas 1 and 2, we have that

$$G_{k:n}(x)G_{k:n}(y) \geq G_{k:n}(x/\epsilon)G_{k:n}(y\epsilon) \tag{9}$$

holds for any $\epsilon \geq 1$ and any $0 < x \leq y \leq y\epsilon < 1$. Hence, from (9) with $x = y = \bar{F}(x_i)$ and $\epsilon = \bar{F}(x_i)/\bar{F}(x_{i+1}) \geq 1$, we get

$$G_{k:n}(\bar{F}(x_i))G_{k:n}(\bar{F}(x_i)) \geq G_{k:n}(\bar{F}(x_{i+1}))G_{k:n}\left(\frac{\bar{F}^2(x_i)}{\bar{F}(x_{i+1})}\right).$$

Finally, by using (7) and the fact that $G_{k:n}$ is non-negative and increasing, we deduce that (8) holds, which concludes the proof. □

The same technique could be applied to other system structures with i.i.d. components or to the case of dependent i.d. components when the common distribution is a discrete distribution $\bar{F}$ with ordered support $\{x_i\}_{i\in I}$ by using (6). Thus it can be proved that the IFR class is preserved when the function α defined in the preceding section (by using the distortion function of the system) is decreasing. We include a general result in this sense and an example in the following section.

4. General case

First we give a general definition for the IFR ageing class.

Definition 3. Let X be a random variable with SF $\bar{F}$ and left-hand end point of the support $\ell = \inf\{x : \bar{F}(x) < 1\}$. Then we say that X (or $\bar{F}$) is IFR if

$$\bar{F}(z+s)\bar{F}(t) \geq \bar{F}(z+t)\bar{F}(s) \quad \text{for all } \ell \leq s \leq t,\, z \geq 0. \tag{10}$$

Note that if $\bar{F}(t) > 0$, then the condition in (10) implies that $X_s \geq_{\text{ST}} X_t$ for all $s \leq t$, where $\geq_{\text{ST}}$ denotes the usual stochastic order (see Chapter 1 of [26]) and where $X_z = (X - z \mid X > z)$ represents the residual lifetime of X at age z. This is the property used in Theorem 1.A.30 of [26] to characterize the IFR class in the continuous case. Thus the meaning of the IFR property is clear: the younger units are ST-better (more reliable) than the older ones. It also implies that if X has a discrete distribution, then its hazard rate function is increasing in the mass points (see Section 3). If X has an absolutely continuous distribution, then it implies that $\bar{F}$ is log-concave and that there exists a probability density function such that its hazard rate function is increasing. Note that the condition '$\bar{F}$ is log-concave' stated in [26, p. 1] cannot be used in discrete models. However, the general definition given in (10) could be applied to any kind of model (for example, in mixtures of discrete and continuous models). It may also be applied to random variables that can take negative values.

However, we must note that the DFR ageing class cannot be extended to the general case by reversing the inequality in (10). The reason is that if X has a discrete distribution and x_i and x_{i+1} are two consecutive mass points, then X_t is ST-strictly decreasing in t in the interval (x_i, x_{i+1}). For example, the geometric distribution does not satisfy this reverse inequality. So the reverse inequality in (10) can only be used in the continuous case.

Recall from (6) that if T is the lifetime of a semi-coherent system with i.d. components, then the system SF function can be obtained as $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}$ is a distortion function. If $\bar{q}$ is strictly increasing and differentiable, then we can state the following general result for

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the preservation of the IFR class in semi-coherent systems with i.d. components. Note that this result can also be applied to general distorted distributions and for random variables with negative values.

Theorem 2. *Let T be the lifetime of a semi-coherent system with i.d. component lifetimes $X_1, \ldots, X_n$. Let us assume that its distortion function $\bar{q}$ in (6) is differentiable in* (0,1) *and let $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ for $u \in (0, 1)$. Then the following conditions are equivalent.*

(i) *The IFR ageing property is preserved (that is, if X_1 is IFR, then T is IFR).*

(ii) *The function α is decreasing in* (0,1).

Proof. (i) $\Rightarrow$ (ii) If X_1 has a standard exponential model, then it is absolutely continuous and IFR and, from (*i*), so is T. Hence its hazard rate h_T is increasing. As the system SF function is $\bar{F}_T(t) = \bar{q}(\bar{F}(t)) = \bar{q}(\mathrm{e}^{-t})$ for all $t \geq 0$, then its probability density function can be obtained as

$$f_T(t) = -\bar{F}_T'(t) = \bar{q}'(\bar{F}(t))f(t) = \bar{q}'(\mathrm{e}^{-t})\mathrm{e}^{-t}$$

for all $t \geq 0$, where $f(t) = -\bar{F}'(t) = \mathrm{e}^{-t}$ is the probability density function of the standard exponential. Hence the system hazard rate function is

$$h_T(t) = \frac{f_T(t)}{\bar{F}_T(t)} = \alpha(\mathrm{e}^{-t})$$

for all $t \geq 0$. Then $\alpha(u) = h_T(-\log(u))$ for $u \in (0, 1)$. Therefore, as h_T is increasing in $(0, \infty)$, α is decreasing in (0,1).

(i) $\Leftarrow$ (ii) Now we assume that α is decreasing. Then we consider the function $\psi(x) = \log \bar{q}(\mathrm{e}^x)$ for $x \in (-\infty, 0)$. Its derivative is

$$\psi'(x) = \frac{\mathrm{e}^x\bar{q}'(\mathrm{e}^x)}{\bar{q}(\mathrm{e}^x)} = \alpha(\mathrm{e}^x).$$

Hence ψ' is decreasing in $(-\infty, 0)$. Therefore ψ is concave and $\bar{q}(\mathrm{e}^x)$ is log-concave. So we can apply Lemma 2 to the function $\bar{q}$ in the interval (0,1), that is,

$$\bar{q}(x)\bar{q}(y) \geq \bar{q}(x/\epsilon)\bar{q}(y\epsilon) \tag{11}$$

for any $\epsilon \geq 1$ such that $0 < x \leq y \leq y\epsilon < 1$. If X is IFR with left-hand end point of the support ℓ, $\ell \leq s \leq t$ and $z \geq 0$, then

$$\bar{F}(z+s)\bar{F}(t) \geq \bar{F}(z+t)\bar{F}(s).$$

From (6), as $\bar{q}$ is strictly increasing, we find that the left-hand end point of the support of T satisfies $\ell_T = \ell$.

Then we have two options. If $t \geq z + s$, then $\bar{F}(t) \leq \bar{F}(z+s)$. Hence we apply inequality (11) with $x = \bar{F}(t)$, $y = \bar{F}(z+s)$, and $\epsilon = \bar{F}(s)/\bar{F}(z+s) \geq 1$, obtaining

$$\bar{q}(\bar{F}(t))\bar{q}(\bar{F}(z+s)) \geq \bar{q}\left(\frac{\bar{F}(t)\bar{F}(z+s)}{\bar{F}(s)}\right)\bar{q}(\bar{F}(s))$$

for s such that $\bar{F}(s) > 0$.

The other option is when $t < z + s$, then $\bar{F}(t) \geq \bar{F}(z+s)$. Hence we apply (11) with $y = \bar{F}(t)$, $x = \bar{F}(z+s)$, and $\epsilon = \bar{F}(s)/\bar{F}(t) \geq 1$, obtaining

$$\bar{q}(\bar{F}(t))\bar{q}(\bar{F}(z+s)) \geq \bar{q}\left(\frac{\bar{F}(t)\bar{F}(z+s)}{\bar{F}(s)}\right)\bar{q}(\bar{F}(s))$$

for s such that $\bar{F}(s) > 0$.

Finally, by using the fact that X is IFR and $\bar{q}$ is increasing, in both options we get

$$\bar{q}(\bar{F}(t))\bar{q}(\bar{F}(z+s)) \geq \bar{q}(\bar{F}(z+t))\bar{q}(\bar{F}(s))$$

for all $\ell \leq s \leq t$ and $z \geq 0$. Note that this inequality also holds if $\bar{F}(s) = 0$, since it implies that $\bar{F}(z+s) = 0$ for all $z \geq 0$. □

Note that the preceding result can be applied to systems with dependent or independent components when they have a common distribution. This distribution can be continuous, discrete, or a mixture. In the preceding section we mentioned that the function α is always decreasing for k-out-of-n systems with i.i.d. components. Hence, from the preceding theorem, the IFR class is always preserved in k-out-of-n systems with i.i.d. components having a common general reliability function $\bar{F}$.

However, it should be noted that the IFR property is not preserved in k-out-of-n systems when the components are dependent and i.d. Thus Example 4.3 of [20] proves that the IFR class is not preserved in $X_{1:2}$ when the components are dependent with a Clayton copula since α is strictly increasing.

Example 4.2 of [20] proves that the IFR class can be preserved in other systems (i.e. systems without a k-out-of-n structure). It also shows that there exist systems where neither the IFR nor the DFR classes are preserved.

To conclude, the following example shows how to apply Theorem 2 to a general SF $\bar{F}$ in a system that does not have a k-out-of-n structure.

Example 1. Let us consider the system with lifetime

$$T = \min(X_1, \max(X_2, X_3))$$

and let us assume that the component lifetimes are i.i.d. with any IFR distribution. Then the distortion function in (6) is $\bar{q}(u) = 2u^2 - u^3$ and the associated alpha function $\alpha(u) = (4 - 3u)/(2-u)$ is decreasing in (0,1) (see [20, p. 124]). Then the system is also IFR. This result holds for continuous distributions (Theorem 4.1 of [20]), discrete distributions with ordered support (preceding section), or in the general case by using Definition 3 and Theorem 2. For example, we can consider the SF

$$\bar{F}(t) = \begin{cases} 1 & \text{for } t < 0, \\ pe^{-t} & \text{for } t \geq 0, \end{cases}$$

for $0 < p < 1$. It is a mixture of a discrete distribution with mass $1-p$ at time $t = 0$ and a continuous exponential distribution with mass p in $(0, \infty)$. A straightforward calculation shows that $\bar{F}$ is IFR according to Definition 3. Hence so is the system SF $\bar{F}_T$.

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References

[1] ABOUAMMOH, A. AND EL-NEWEIHI, E. (1986). Closure of NBUE and DMRL under the formation of parallel systems. *Statist. Prob. Lett.* **4**, 223–225.

[2] ALIMOHAMMADI, M., ALAMATSAZ, M. H. AND CRAMER, E. (2015). Discrete strong unimodality of order statistics. *Statist. Prob. Lett.* **103**, 176–185.

[3] AN, M. Y. (1998). Logconcavity versus logconvexity: a complete characterization. *J. Econom. Theory* **80**, 350–369.

[4] ARNOLD, B. C., BALAKRISHNAN, N. AND NAGARAJA, H. N. (2008). *A First Course in Order Statistics*, 2nd edn. John Wiley, New York.

[5] BARLOW, R. E. AND PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart & Winston, New York.

[6] BLOCK, H. W. AND SAVITS, T. H. (1976). The IFRA closure problem. *Ann. Prob.* **4**, 1030–1032.

[7] BRACQUEMOND, C., GAUDOIN, O., ROY, D. AND XIE, M. (2001). On some discrete notions of aging. In *System and Bayesian Reliability: Essays in Honor of Professor Richard E. Barlow on His 70th Birthday*, ed. Y. Hayakawa *et al.*, pp. 185–197. World Scientific.

[8] DEMBIŃSKA, A. (2008). Discrete order statistics. In *Encyclopedia of Statistical Sciences*, ed. S. Kotz *et al.* John Wiley, Hoboken, NJ.

[9] ESARY, J. AND PROSCHAN, F. (1963). Relationship between system failure rate and component failure rates. *Technometrics* **5**, 183–189.

[10] ESARY, J. D., MARSHALL, A. W. AND PROSCHAN, F. (1970). Some reliability applications of the hazard transform. *SIAM J. Appl. Math.* **18**, 849–860.

[11] GUPTA, P. L., GUPTA, R. C. AND TRIPATHI, R. C. (1997). On the monotone properties of discrete failure rates. *J. Statist. Planning Infer.* **65**, 255–268.

[12] HU, T., NANDA, A. K., XIE, H. AND ZHU, Z. (2004). Properties of some stochastic orders: a unified study. *Naval Res. Logistics* **51**, 193–216.

[13] JOHNSON, N. L., KEMP, A. W. AND KOTZ, S. (2005). *Univariate Discrete Distributions*, 3rd edn. John Wiley, New York.

[14] KEMP, A. W. (2004). Classes of discrete life distributions. *Commun. Statist. Theory Meth.* **33**, 3069–3093.

[15] LI, C. AND LI, X. (2020). Weak aging properties for coherent systems with statistically dependent component lifetimes. *Naval Res. Logistics* **67**, 559–572.

[16] LINDQVIST, B. H. AND SAMANIEGO, F. J. (2019). Some new results on the preservation of the NBUE and NWUE aging classes under the formation of coherent systems. *Naval Res. Logistics* **66**, 430–438.

[17] LINDQVIST, B. H., SAMANIEGO, F. J. AND HUSEBY, A. B. (2016). On the equivalence of systems of different sizes, with applications to system comparisons. *Adv. Appl. Prob.* **48**, 332–348.

[18] MARSHALL, A. W. AND OLKIN, I. (2007). *Life Distributions*. Springer, New York.

[19] NAVARRO, J. (2018). Preservation of DMRL and IMRL aging classes under the formation of order statistics and coherent systems. *Statist. Prob. Lett.* **137**, 264–268.

[20] NAVARRO, J. (2022). *Introduction to System Reliability Theory*. Springer.

[21] NAVARRO, J., DEL ÁGUILA, Y., SORDO, M. A. AND SUÁREZ-LLORENS, A. (2014). Preservation of reliability classes under the formation of coherent systems. *Appl. Stoch. Models Business Industry* **30**, 444–454.

[22] ROBERTS, A. W. AND VARBERG, D. E. (1973). *Convex Functions*. Academic Press, New York and London.
[23] ROY, D. AND GUPTA, R. P. (1992). Classifications of discrete lives. *Microelectron. Reliab.* **32**, 1459–1473.
[24] RYCHLIK, T. AND SZYMKOWIAK, M. (2021). Properties of system lifetime in the classical model with I.I.D. exponential component lifetimes. In *Advances in Statistics: Theory and Applications*, ed. I. Ghosh *et al.*, pp. 43–66. Springer.
[25] SAMANIEGO, F. J. (1985). On closure of the IFR class under formation of coherent systems. IEEE Trans. Reliab. **R-34**, 69–72.
[26] SHAKED, M. AND SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, New York.