ON THE FUNCTIONAL EQUATIONS $|f(x+iy)| = |f(x)+f(iy)|$ AND $|f(x+iy)| = |f(x)-f(iy)|$ AND ON IVORY'S THEOREM

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1. Introduction. Considering Cauchy's functional equation

$$f(z_1+z_2) = f(z_1)+f(z_2),$$

where $f(z)$ is an entire function of z, we have the following functional equation:

$$(1) \qquad |f(x+iy)| = |f(x)+f(iy)|,$$

where x and y are real.

R. M. Robinson [1] proved the following interesting theorem: If $f(z)$ is regular for $|z| < r$, and satisfies the functional equation (1) for all real values of x and y, then

$$f(z) = Az, \quad f(z) = A\sin(bz), \quad \text{or} \quad f(z) = A\sinh(bz),$$

where A and b are constants and b is real.

In § 2 we shall give a new proof of the above theorem.

In § 3 we shall solve the following functional equation by using the method of R. M. Robinson [1]:

$$(2) \qquad |f(x+iy)| = |f(x)-f(iy)|,$$

where x and y are real and $f(z)$ is an entire function of z.

In a previous paper "On Ivory's Theorem" ([2], cf. [3]) we discussed the functional equation $|f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|$ connected with Ivory's Theorem. In § 4 we

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shall solve two functional equations connected with Ivory's Theorem the second of which shall be solved by applying the result of § 3.

2. New proof of the Theorem of R.M. Robinson.

Proof. We may assume that $f(z) \not\equiv$ constant. Using the power series $f(z) = \sum_{n=0}^{+\infty} a_{p+n} z^{p+n}$ $(a_p \neq 0)$ in $|z| < r$ and equating the terms of degree $2p$ with respect to x and y in (1), we have $p = 1$.

Putting $g(z) = \frac{f(z)}{a_1}$ in $|z| < r$, we have in $|z| < r$

$$g(z) = z + b_2 z^2 + b_3 z^3 + \dots + b_n z^n + \dots \quad . \tag{3}$$

We have $b_1 = 1$. We use induction to prove that b_n $(n = 1, 2, 3, \dots)$ are real. The result is true for the integer 1. Suppose that it is true for $1, 2, 3, \dots, m$.

Since $g(z) = \frac{f(z)}{a_1}$, by (1) we have in $|x+iy| < r$

$$|g(x+iy)|^2 = |g(x)+g(iy)|^2 . \tag{4}$$

By (3) we have in $|x+iy| < r$

$$g(x+iy) = (x+iy) + b_2(x+iy)^2 + b_3(x+iy)^3 + \dots + b_m(x+iy)^m + b_{m+1}(x+iy)^{m+1} + \dots, \tag{5}$$

$$\begin{aligned} \overline{g(x+iy)} &= (x-iy) + \bar{b}_2(x-iy)^2 + \bar{b}_3(x-iy)^3 + \dots \\ &\quad + \bar{b}_m(x-iy)^m + \bar{b}_{m+1}(x-iy)^{m+1} + \dots \\ &= (x-iy) + b_2(x-iy)^2 + b_3(x-iy)^3 + \dots \\ &\quad + b_m(x-iy)^m + \bar{b}_{m+1}(x-iy)^{m+1} + \dots , \end{aligned} \tag{6}$$

where by inductive hypothesis $\bar{b}_2 = b_2$, $\bar{b}_3 = b_3$, ..., $\bar{b}_m = b_m$.

By (5), (6) the coefficient of $x^{m+1}y$ in the left side of (4) is

$$(7) \qquad i\binom{m+1}{1} b_{m+1} - ib_{m+1} - i\binom{m+1}{1}\bar{b}_{m+1} + i\bar{b}_{m+1}.$$

Since $|g(x) + g(iy)|^2 = (g(x) + g(iy))(\overline{g(x)} + \overline{g(iy)})$, the coefficient of $x^{m+1}y$ in the right side of (4) is

$$(8) \qquad -ib_{m+1} + i\bar{b}_{m+1}.$$

Since (7) = (8), we have $\bar{b}_{m+1} = b_{m+1}$.

The result now follows by induction.

If x and y are real, then by the above result we have in $|x+iy| < r$

$$(9) \qquad \overline{g(x+iy)} = g(x-iy),$$

$$(10) \qquad \overline{g(x)} = g(x),$$

$$(11) \qquad \overline{g(iy)} = g(-iy).$$

By (4), (9), (10), (11) we have in $|x+iy| < r$

$$(12) \qquad g(x+iy)g(x-iy) = g^2(x)+g(x)g(-iy)+g(iy)g(x)+g(iy)g(-iy).$$

Differentiating (12) twice with respect to y and putting $y = 0$, by $g(0) = 0$ we have in $|x| < r$

$$(13) \qquad g(x)g''(x) - g'^2(x) = g''(0)g(x) - g'^2(0).$$

Equating the coefficients of xy^2 of both sides of (4), we have $b_2 = \frac{g''(0)}{2!} = 0$. Hence by (13) we have in $|x| < r$

$$(14) \qquad g(x)g''(x)-g'^2(x) = -g'^2(0).$$

Since $g(z)$ is a regular function of z, by the identity theorem and (14) we have in $|z| < r$

$$g(z)g''(z)-g'^2(z) = -g'^2(0).$$

Solving this differential equation, the theorem is proved.

3. <u>On the functional equation</u> $|f(x+iy)| = |f(x)-f(iy)|$.

In this section we shall solve the functional equation

$$|f(x+iy)| = |f(x)-f(iy)|, \tag{2}$$

where x and y are real and $f(z)$ is a regular function of z.

THEOREM 1. If $f(z)$ is regular for $|z| < r$, and satisfies the functional equation (2) for all real values of x and y, then the solutions of (2) are the following and only these:

$$f(z) = Az+Bz^2,$$

$$\text{or } f(z) = A\sin\alpha z + B\cos\alpha z - B,$$

$$\text{or } f(z) = A\sinh\alpha z + B\cosh\alpha z - B,$$

where A, B are arbitrary complex constants and α is an arbitrary real constant.

<u>Proof.</u> By (2) we have in $|z| < r$

$$|f(x+iy)|^2 = |f(x)-f(iy)|^2, \tag{15}$$

where x, y are real. We may assume that $f(z) \not\equiv 0$.

Using the power series $f(z) = \sum_{n=0}^{+\infty} a_{p+n} z^{p+n}$ ($a_p \neq 0$ where p is a natural number) in $|z| < r$ and equating the terms of degree $2p$ with respect to x and y in (15), we have $p = 1$. Putting $g(z) = \frac{f(z)}{a_1}$ in $|z| < r$, we have in $|z| < r$

$$g(z) = z + b_2 z^2 + b_3 z^3 + \ldots + b_n z^n + \ldots . \tag{16}$$

By (15) we have in $|z| < r$

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$$(17) \qquad g(z)\overline{g(z)} = \left(g\left(\frac{z+\bar z}{2}\right) - g\left(\frac{z-\bar z}{2}\right)\right)\overline{\left(g\left(\frac{z+\bar z}{2}\right) - g\left(\frac{z-\bar z}{2}\right)\right)} .$$

Substituting (16) in (17) and equating the coefficients of $z^3\bar z$ of both sides, we have

$$b_3 = \frac{3}{4} b_3 + \frac{1}{4}\bar b_3 .$$

Hence we have $\bar b_3 = b_3$. Hence b_3 is real.

Substituting (16) in (17) and equating the coefficients of $z^n\bar z$ of both sides for $n > 3$, we have

$$\frac{2^{n-1}-n}{2^{n-1}} b_n = P(b_2, b_3, b_4, \ldots, b_{n-1}, \bar b_2, \bar b_3, \bar b_4, \ldots, \bar b_{n-1}),$$

where $n(> 3)$ is even and P is a polynomial in the earlier coefficients $b_2, b_3, b_4, \ldots, b_{n-1}, \bar b_2, \bar b_3, \bar b_4, \ldots, \bar b_{n-1}$, and

$$\frac{2^{n-1}-n}{2^{n-1}} b_n - \frac{1}{2^{n-1}}\bar b_n = P(b_2, b_3, b_4, \ldots, b_{n-1}, \bar b_2, \bar b_3, \bar b_4, \ldots, \bar b_{n-1}),$$

where $n(> 3)$ is odd, and P is a polynomial in the earlier coefficients $b_2, b_3, b_4, \ldots, b_{n-1}, \bar b_2, \bar b_3, \bar b_4, \ldots, \bar b_{n-1}$.

Since $2^{n-1}-n \neq 0$ (> 0), $2^{n-1}-(n+1) \neq 0$ (> 0) and $2^{n-1}-(n-1) \neq 0$ (> 0) for $n > 3$ the remaining coefficients $b_n (n > 3)$ are uniquely determined in terms of b_2, b_3 where b_3 is real.

On the other hand

$$g(z) = \frac{1}{\sqrt{-6b_3}} \sin \sqrt{-6b_3}\, z + \frac{b_2}{3b_3} \cos \sqrt{-6b_3}\, z - \frac{b_2}{3b_3}$$

$$= z + b_2 z^2 + b_3 z^3 + \ldots ,$$

$$\text{or} \quad g(z) = \frac{1}{\sqrt{6b_3}} \sinh \sqrt{6b_3}\, z + \frac{b_2}{3b_3} \cosh \sqrt{6b_3}\, z - \frac{b_2}{3b_3}$$

$$= z + b_2 z^2 + b_3 z^3 + \dots ,$$

$$\text{or} \quad g(z) = z + b_2 z^2 ,$$

respectively, are normalized solutions of the functional equation $|g(x+iy)| = |g(x) - g(iy)|$, if b_3 is negative or positive or 0.

Since the remaining coefficients b_n $(n > 3)$ are uniquely determined in terms of b_2, b_3, there can be no other normalized solutions. Thus the theorem is proved.

4. On Ivory's Theorem. In a previous paper "On Ivory's Theorem" ([2], cf. [3]) we discussed the functional equation $|f(x+y) - f(x-y)| = |f(x+\bar{y}) - f(x-\bar{y})|$ connected with Ivory's Theorem. In this section we shall solve two functional equations connected with Ivory's Theorem.

LEMMA. If $H(z)$ is one-valued and regular in $|z| < \delta$ where δ is a positive constant and $A(t) = |H(te^{i\varphi})|^2$ where t, φ (φ fixed) are real, then we have

$$A^{(4)}(0) = 2\mathrm{Re}(e^{4i\varphi} H^{(4)}(0)\overline{H(0)} + 4e^{2i\varphi} H'''(0)\overline{H'(0)}) + 6|H''(0)|^2 .$$

Proof. Since this is easy, we omit it.

THEOREM 2. Let us assume that $ABCD$ is an arbitrary rectangle (whose sides are parallel to the coordinate axes) with the constant intersecting angle 2φ $(0 < \varphi < \frac{\pi}{2})$ of the two diagonals. We put $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ where $W = f(z)$ is an entire function of z. If $\overline{A'C'} = \overline{B'D'}$ in the W-plane for all such A, B, C, D, then $f(z) = az^2 + bz + c$ or $f(z) = a\sin(\alpha z) + b\cos(\alpha z) + c$ or $f(z) = a\sinh(\alpha z) + b\cosh(\alpha z) + c$ where a, b, c are arbitrary complex constants and α is an arbitrary real constant, and only these.

Proof. By the condition $\overline{A'C'} = \overline{B'D'}$ we have the following functional equation:

$$(18) \qquad |f(x+te^{i\varphi}) - f(x-te^{i\varphi})| = |f(x+te^{-i\varphi}) - f(x-te^{-i\varphi})| ,$$

where x is an arbitrary complex number and t is an arbitrary real number. (φ is a real constant with $0 < \varphi < \frac{\pi}{2}$.)

By (18) we have

$$(19) \qquad |f(x+te^{i\varphi}) - f(x-te^{i\varphi})|^2 = |f(x+te^{-i\varphi}) - f(x-te^{-i\varphi})|^2 .$$

Differentiating (19) four times with respect to t and putting $t = 0$, by the above Lemma we have

$$2\mathrm{Re}(16e^{2i\varphi} f'''(x)\overline{f'(x)}) = 2\mathrm{Re}(16e^{-2i\varphi} f'''(x)\overline{f'(x)}) .$$

Hence, by $\sin 2\varphi \neq 0$ we have

$$(20) \qquad \mathrm{Im}(f'''(x)\overline{f'(x)}) = 0.$$

We may assume that $f(x) \not\equiv 0$. Then there exists a vicinity V where $f(x) \neq 0$. Hence, by (20) we have in V

$$\mathrm{Im}\left(\frac{f'''(x)}{f'(x)}\right) = 0.$$

Hence, we have in V

$$(21) \qquad \frac{f'''(x)}{f'(x)} = K,$$

where K is a real constant.

Solving this differential equation (21), the theorem is proved.

THEOREM 3. Let us assume that $ABCD$ is an arbitrary rectangle (whose sides are parallel to the coordinate axes) and one of four vertices A, B, C, D is a fixed point which represents a complex constant γ. We put $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ where $W = f(z)$ is an entire function of z. If $\overline{A'C'} = \overline{B'D'}$ in the W-plane for all such A, B, C, D, then $f(z) = az^2 + bz + c$ or $f(z) = a\sin\alpha z + b\cos\alpha z + c$ or $f(z) = a\sinh\alpha z + b\cosh\alpha z + c$ where a, b, c are arbitrary complex constants and α is an arbitrary real constant, and only these.

Proof. By the condition $\overline{A'C'} = \overline{B'D'}$ we have the following functional equation:

$$(22) \qquad |f(\gamma + x + iy) - f(\gamma)| = |f(\gamma + x) - f(\gamma + iy)| ,$$

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where x, y are real. Putting $g(z) = f(\gamma + z) - f(\gamma)$, by (22) we have

$$|g(x+iy)| = |g(x) - g(iy)|, \tag{23}$$

where x, y are real. By (23) and Theorem 1 in § 3 the theorem is proved.

REFERENCES

1. R.M. Robinson, A Curious Trigonometric Identity. Amer. Math. Monthly 64, (1957), pages 83-85.

2. H. Haruki, On Ivory's Theorem. Mathematica Japonicae, Vol. 1, No. 4, page 151, (1949).

3. H. Haruki, Studies on Certain Functional Equations from the Standpoint of Analytic Function Theory. Sci. Rep. College of General Education, Osaka Univ., Vol. 14, No. 1, page 32, (1965).

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