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RESEARCH ARTICLE

Noninjectivity of the cycle class map in continuous ℓ-adic cohomology

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Abstract
Jannsen asked whether the rational cycle class map in continuous ℓ-adic cohomology is injective, in every codimension for all smooth projective varieties over a field of finite type over the prime field. As recently pointed out by Schreieder, the integral version of Jannsen's question is also of interest. We exhibit several examples showing that the answer to the integral version is negative in general. Our examples also have consequences for the coniveau filtration on Chow groups and the transcendental Abel-Jacobi map constructed by Schreieder.

Contents

1. Introduction

Let k be a field, $k_s \subset \overline{k}$ be a separable and an algebraic closure of k, respectively, ℓ be a prime number invertible in k and X be a smooth projective k-variety. For all integers i and j, we denote by $CH^i(X)$ the Chow group of codimension i cycles modulo rational equivalence and by $H^i(X, \mathbb{Q}_\ell(j))$ the continuous ℓ-adic cohomology defined by Jannsen [17] (or equivalently, the pro-étale cohomology defined by Bhatt and Scholze [3]). Motivated by the Bloch–Beilinson conjecture on the existence of a certain functorial filtration on $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and its relation to the conjectural theory of mixed motives, Jannsen [18, Question 2.8] asked the following question.

Question 1.1 (Jannsen). Suppose that k is of finite type over its prime field. Is the ℓ-adic cycle class map

$$\mathrm{cl}\colon CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2i}(X, \mathbb{Q}_\ell(i))$$

injective?

A positive answer to Question 1.1 would imply the Bloch–Beilinson conjecture [18, Conjecture 2.1] over k. More precisely, consider the Hochschild-Serre spectral sequence in continuous ℓ-adic cohomology [17, Corollary 3.4]:

$$E_2^{p,q} = H^p(k, H^q(X_{k_s}, \mathbb{Q}_\ell(i))) \Rightarrow H^{p+q}(X, \mathbb{Q}_\ell(i)).$$

The spectral sequence degenerates at the E_2 page, and so gives a filtration

$$\{0\} = F^{i+1} \subset F^i \subset \cdots \subset F^1 \subset F^0 = H^{2i}(X, \mathbb{Q}_\ell(i)),$$

where $F^p/F^{p+1} \simeq H^p(k, H^{2i-p}(X_{k_s}, \mathbb{Q}_\ell(i)))$ for all $p \geq 0$. If Question 1.1 had a positive answer, then the inverse image of $F^\cdot$ would be a filtration on $CH^i(X)$ with all properties predicted by Bloch and Beilinson, proving the Bloch–Beilinson conjecture (see [18, Lemma 2.7]).

Of course, there is no reason to expect Question 1.1 to have an affirmative answer over an arbitrary field. For example, if k is algebraically closed, the kernel of the cycle class map is the group of homologically trivial cycles modulo rational equivalence, and it is often nontrivial: in particular, the cycle class map factors through algebraic equivalence. However, the situation when k is of finite type over its prime field is very different. Indeed, in this case, Jannsen observed that, as a consequence of the Mordell-Weil theorem, the *integral* codimension 1 cycle class map $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$ is injective (see [17, Remark 6.15 (a)]). This naturally leads to the following variant of Jannsen's question.

Question 1.2. Suppose that k is of finite type over its prime field. Is the ℓ-adic cycle class map

$$\mathrm{cl}\colon CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^{2i}(X, \mathbb{Z}_\ell(i))$$

injective?

As noted by Jannsen, Question 1.2 has an affirmative answer for $i = 1$. Question 1.2 is also implicit in work of Saito [36], who obtained some positive results for $i = 2$. Colliot-Thélène–Sansuc–Soulé [12] showed that the ℓ-adic cycle class map is injective on torsion when $i = 2$ and the field k is finite.

Question 1.2 fits into a constellation of conjectural integral refinements of well-known rational cycle conjectures. These questions go back at least to Totaro [46], who suggested that certain Lefschetz-hyperplane properties for Chow groups, originally conjectured rationally by Hartshorne, Nori and Paranjape, should also hold for integral Chow groups. Totaro also showed that Nori connectivity for Chow groups fails on torsion cycles. Later, Soulé–Voisin [42] showed that Voevodsky's smash nilpotence conjecture fails integrally.

In contrast to these negative results, Schreieder [38] recently proved that some aspects of the rational conjectures hold in fact integrally. For example, Schreieder proved a torsion analogue of a certain conjecture of Jannsen, asserting that cycles in the kernel of the Abel-Jacobi map have coniveau one (see [38, Corollary 1.3]). In his talk at the conference "Géométrie Algébrique en l'honneur de Claire Voisin," held in May 2022 in Paris, he used this result to motivate the general and natural question of to which extent rational cycle conjectures hold integrally, and in particular, Question 1.2.

The purpose of the present work is to show that Question 1.2 has a negative answer in general. We offer examples of very different natures: topological (Atiyah–Hirzebruch-style approximations of classifying spaces), geometric (products of a Kummer threefold and an elliptic curve) and arithmetic (quadrics, norm varieties). As we explain below, our examples exhibit new and interesting behaviour of the coniveau filtration on Chow groups and of Schreieder's transcendental Abel-Jacobi map over finitely generated fields.

Theorem 1.3 (Theorem 2.3). *There exist a finite field (respectively, a number field) k and a smooth complete intersection $Y \subset \mathbb{P}^N_k$ of dimension 15 with a free action of a finite 2-group G, such that, letting $X := Y/G$, the cycle class map*

$$\mathrm{cl}\colon CH^3(X)[2] \to H^6(X, \mathbb{Z}_2(3))$$

is not injective.

The aforementioned result of Colliot-Thélène–Sansuc–Soulé [12] shows that 3 is the least possible codimension in which one can find a torsion counterexample over a finite field.

The dimension of the examples of Theorem 1.3 is quite large. The following theorem yields examples of smaller dimension over a number field.

Theorem 1.4 (Theorem 4.3). *There exist a number field k and a fourfold product $X = Y \times E$ over k, where Y is a Kummer threefold and E is an elliptic curve, such that the cycle class map*

$$\mathrm{cl}: CH^3(X)[2] \to H^6(X, \mathbb{Z}_2(3))$$

is not injective.

The examples of Theorem 1.4 are the counterexamples of smallest dimension that we could find over number fields. Over a field of transcendence degree 1 over $\mathbb{Q}$, we provide examples of one dimension lower, in one codimension lower. Recall that a number field is said to be totally imaginary if it admits no real places.

Theorem 1.5 (Theorem 6.3). *Let k be a totally imaginary number field and $k(t)$ be a purely transcendental extension of k of transcendence degree 1. There exists a smooth quadric hypersurface $X \subset \mathbb{P}^4_{k(t)}$, such that the cycle class map*

$$\mathrm{cl}\colon CH^2(X)[2] \to H^4(X, \mathbb{Z}_2(2))$$

is not injective.

We also show that, if ℓ is an odd prime invertible in k, there exists a norm variety X of dimension $\ell^2 - 1$ over $k(t)$, such that $\mathrm{cl}\colon CH^2(X)[\ell] \to H^4(X, \mathbb{Z}_\ell(2))$ is not injective. Thus, Question 1.2 has a negative answer for all prime numbers ℓ.

We now explain the relation of our examples to Schreieder's results on the coniveau filtration on Chow groups. By now, we have a good understanding of the filtration over the complex numbers, especially for codimension ≤ 3 (see, for example, [38, Corollary 1.2]). Our examples show that this filtration is still interesting when k is of finite type over its prime field. We also relate our examples to the transcendental Abel-Jacobi map on torsion cycles constructed by Schreieder [38, Section 7.5].

Remark 1.6. We denote by $N^{\cdot}CH^i(X)$ the coniveau filtration on Chow groups [38, Section 1.1], and by $H^i_{j,\mathrm{nr}}(X, -)$ Schreieder's refined unramified cohomology [38, Section 5], which for $j = 0$ coincides with the ordinary unramified cohomology: $H^i_{0,\mathrm{nr}} = H^i_{\mathrm{nr}}$. In the following, we assume that k is of finite type over its prime field.

(a) We have $N^{i-1}CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = 0$ for all smooth projective k-varieties X by Jannsen's result (see [38, Lemma 7.5(2)]). The examples of Theorem 1.5 show that $N^{i-2}CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ can be nonzero for $i = 2$ (in this case, $N^0CH^2(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is exactly the kernel of the cycle class map). One can further analyse the torsion part of the stage of the filtration on the examples using the transcendental Abel-Jacobi map, and [38, Corollary 9.5, Proposition 7.16] yields

$$N^1H^3(X, \mathbb{Q}_2/\mathbb{Z}_2(2))_{\mathrm{div}}/N^1H^3(X, \mathbb{Q}_2(2)) \otimes \mathbb{Q}_2/\mathbb{Z}_2 \neq 0.$$

In other words, there is a cohomology class in $H^3(X, \mathbb{Q}_2/\mathbb{Z}_2(2))$ of coniveau 1, which lifts to a rational class but not to a rational class of coniveau 1.

(b) By [38, Theorem 1.8], the kernel of the cycle class map is given by

$$H^{2i-1}_{i-2,\mathrm{nr}}(X, \mathbb{Z}_\ell(i))/H^{2i-1}(X, \mathbb{Z}_\ell(i)).$$

Question 1.1 asks whether this group is torsion. Our examples of Theorems 1.3, 1.4 and 1.5 show that it can be nonzero for $i = 2, 3$. In the case $i = 2$, we get an explicit statement on ordinary unramified cohomology: the examples of Theorem 1.5 have an unramified class of degree 3 which does not extend to a class on all of X. In fact, using a restriction-corestriction argument, one sees that in this case, the inclusion

$$H^3(k,\mathbb{Z}_2(2)) \subset H^3_{\mathrm{nr}}(X,\mathbb{Z}_2(2))$$

has cokernel of finite torsion order > 1, a phenomenon that does not seem to have been observed before (in contrast, $H^3_{\mathrm{nr}}(X,\mathbb{Z}_2(2))$ is torsion-free and $H^3(k,\mathbb{Z}_2(2))$ is a direct summand of $H^3_{\mathrm{nr}}(X,\mathbb{Z}_2(2))$ if $X(k) \neq \emptyset$).

(c) In our setting, Schreieder's transcendental Abel-Jacobi map is of the form: $\lambda_{\mathrm{tr}}\colon CH^i_0(X)\{\ell\} \to H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))$, where $CH^i_0(X)\{\ell\}$ is the kernel of $\mathrm{cl}\colon CH^i(X)\{\ell\} \to H^{2i}(X,\mathbb{Z}_\ell(i))$. For $i = 2$, the transcendental Abel-Jacobi map is injective by [38, Corollary 9.5]. In particular, the torsion cycles in the examples of Theorem 1.5 do not lie in the kernel of λ_{tr}. This also shows that λ_{tr} can be nonzero. In contrast, Theorems 1.3 and 1.4 provide examples where λ_{tr} is not injective for $i = 3$ (see Remark 3.3).

We now comment on the proofs of the main theorems. In view of the discussion around Question 1.1, it is natural to approach Question 1.2 by considering the filtration $F^\cdot$ on $H^{2i}(X,\mathbb{Z}_\ell(i))$ induced by the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(k, H^q(X_{k_s},\mathbb{Z}_\ell(i))) \Rightarrow H^{p+q}(X,\mathbb{Z}_\ell(i)).$$

We start with a nonzero torsion cycle $\alpha \in CH^i(X)$ (producing such examples is generally quite difficult) and try to show that $\mathrm{cl}(\alpha) \in F^p$ for all $p \geq 0$. To show that $\mathrm{cl}(\alpha) \in F^1$, we only need to show that $\mathrm{cl}(\alpha)$ is geometrically trivial, but the subsequent steps of the filtration are more difficult because the groups appearing in the spectral sequence are typically huge and the image of $\mathrm{cl}(\alpha) \in F^p/F^{p+1}$ often seems hard to compute (see [19] for the case $p = 2$). In the examples used to prove Theorems 1.3 and 1.5, we get around this by showing that all F^p/F^{p+1} are torsion free, which forces $\mathrm{cl}(\alpha) = 0$.

Theorem 1.4 lies deeper. A key result (Proposition 3.1), relating injectivity of the ℓ-adic cycle class map to that of Bloch's map, reduces Theorem 1.4 to finding fourfold examples defined over a number field where the Deligne cycle class map is not injective on torsion in codimension 3. We then achieve this in two steps: a result of Bloch–Esnault yields examples defined over a number field with nonvanishing fourth unramified cohomology group $H^4_{\mathrm{nr}}(X_{\mathbb{C}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(3))$, where, with extra care, one can find such examples with a small Chow group of zero-cycles; then using the Bloch–Kato conjecture and a result of Voisin and Ma relating $H^4_{\mathrm{nr}}(X_{\mathbb{C}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(3))$ to the kernel of the Deligne cycle class map on torsion in codimension 3, one deduces the desired noninjectivity. The construction is inspired by the work of Diaz.

Our work leads us to the following questions.

Question 1.7. (a) Is there a smooth projective d-dimensional variety X over a field of finite type over its prime field, such that the ℓ-adic map $\mathrm{cl}\colon CH^d(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H^{2d}(X,\mathbb{Z}_\ell(d))$ is not injective?[1]

(b) Let i be either 2 or 3. Is there a smooth projective threefold over a number field k, such that the ℓ-adic map $\mathrm{cl}\colon CH^i(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H^{2i}(X,\mathbb{Z}_\ell(i))$ is not injective? What happens over $k = \mathbb{Q}$?

The paper is organised as follows. In Section 2, we prove Theorem 1.3. In Section 3, we prove a key result (Proposition 3.1), relating the injectivity of the ℓ-adic cycle class map to that of Bloch's map, which is useful in Sections 4 and 5. As first application, we give a second proof of Theorem 1.3. In Section 4, we prove Theorem 1.4. In Section 5, we construct further examples in codimension 3 using nontorsion type counterexamples to the integral Hodge and Tate conjectures. Finally, in Section 6, we prove Theorem 1.5.

[1]After the first version of this manuscript was posted on arXiv, Alexandrou and Schreieder annouced a construction of such d-folds for all $d \geq 3$ (see [1, Corollary 1.4]). Later, Colliot-Thélène and the first author [13] found an example for $d = 2$.

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Notation

If k is a field, we write $H^i(k,-)$ for continuous Galois cohomology. If X is a smooth projective k-variety, we write $H^i(X,-)$ for the continuous étale cohomology, as defined by Jannsen [17], $CH^i(X)$ for the Chow group of codimension i cycles modulo rational equivalence and cl for the cycle class map in continuous ℓ-adic cohomology; when k is algebraically closed, we write λ for Bloch's map. If $k = \mathbb{C}$, we denote by $H^i_{\mathcal{D}}(X,\mathbb{Z}(j))$ the Deligne cohomology group and by $\mathrm{cl}_{\mathcal{D}}$ the Deligne cycle class map; for $A \in \{\mathbb{Z}, \mathbb{Q}/\mathbb{Z}, \mathbb{Z}/2\}$, we denote by $H^i_{\mathrm{nr}}(X, A)$ the i-th unramified cohomology group.

For an abelian group A, an integer $n \geq 1$ and a prime number ℓ, we denote $A[n] := \{a \in A \mid na = 0\}$, by $A\{\ell\}$ the subgroup of ℓ-primary torsion elements of A, by A_{tors} the subgroup of torsion elements of A and $A_{\mathrm{tf}} := A/A_{\mathrm{tors}}$.

2. Proof of Theorem 1.3

In order to prove Theorem 1.3, we will make use of a construction due to Totaro [46]. Totaro's construction is stated over the complex numbers but works over an arbitrary field of characteristic zero. It has been generalised to fields of characteristic not 2 by Quick [32].

Let k_0 be a field of characteristic different from 2. Let H be the Heisenberg group of order 32 (see [46, Section 5]), and set $G := H \times \mathbb{Z}/2$. We have a group homomorphism

$$\varphi \colon G \xrightarrow{\mathrm{pr}_1} H \hookrightarrow \mathrm{SO}_4,$$

where the map on the right is the Heisenberg representation of H (see [46, Section 5]) (Totaro works in characteristic zero, but as observed during the proof of [32, Theorem 7.2], the Heisenberg representation is defined over any field of characteristic different from 2). Let $A : \mathrm{SO}_4 \longrightarrow \mathrm{GL}_3$ be the representation given by the composition

$$\mathrm{SO}_4 \twoheadrightarrow \mathrm{SO}_4/\mu_2 \xrightarrow{\sim} \mathrm{SO}_3 \times \mathrm{SO}_3 \xrightarrow{\mathrm{pr}_1} \mathrm{SO}_3 \hookrightarrow \mathrm{GL}_3$$

and $B : \mathrm{SO}_4 \longrightarrow \mathrm{GL}_4$ be the natural 4-dimensional representation of SO_4. Define

$$C := c_2(A \circ \varphi) - c_2(B \circ \varphi) \in CH^2(BG),$$

let $c_1 \in CH^1(BG)$ be the pullback along the second projection $\mathrm{pr}_2 : G \longrightarrow \mathbb{Z}/2$ of the first Chern-class of the nontrivial character of $\mathbb{Z}/2$, and set

$$\alpha := Cc_1 \in CH^3(BG).$$

We have $2\alpha = 0$ because $2c_1 = 0$.

Finally, let V be a G-representation of finite dimension over k_0, $U \subset V$ be a G-invariant open subscheme of V, such that G acts freely on U and the codimension of $V - U$ in V is at least 4.

Lemma 2.1. *Let $(k_0)_s$ be a separable closure of k_0.*

(a) *We have* $\mathrm{cl}(\alpha_{(k_0)_s}) = 0$ *in* $H^6((U/G)_{(k_0)_s}, \mathbb{Z}_2(3))$.
(b) *There exists a finite field subextension $k_0 \subset k \subset (k_0)_s$, such that* $\mathrm{cl}(\alpha_k) = 0$ *in* $H^6((U/G)_k, \mathbb{Z}_2(3))$.

Proof. Since the codimension of $V - U$ in V is at least 4, we have

$$CH^3(U/G) = CH^3(BG)$$

(see [47, Definition 1.2]).

(a) By the invariance of étale cohomology under purely inseparable field extensions, it suffices to show that $\mathrm{cl}(\alpha_{\overline{k}_0}) = 0$ in $H^6((U/G)_{\overline{k}_0}, \mathbb{Z}_2(3))$, where $\overline{k}_0$ is an algebraic closure of k_0 containing $(k_0)_s$. If $k_0 = \mathbb{C}$, the map $U/G \longrightarrow BG$ corresponding to the principal G-bundle $U \longrightarrow U/G$ induces an

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isomorphism $H^6((U/G)_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{\sim} H^6(BG, \mathbb{Z})$, and the cycle class of α in $H^6(BG, \mathbb{Z})$ is zero as stated in [46, p. 485], hence, the cycle class of α in $H^6((U/G)_{\mathbb{C}}, \mathbb{Z})$ vanishes. Since Artin's comparison isomorphism is compatible with cycle classes in singular and ℓ-adic cohomology, this implies that (a) holds for $k_0 = \mathbb{C}$. If k_0 is an arbitrary field of characteristic zero, then (a) follows from the case $k_0 = \mathbb{C}$ and the invariance of ℓ-adic cohomology under extensions of algebraically closed fields. Finally, if k_0 is an arbitrary field of characteristic different from 2, the arguments of Totaro have been adapted by Quick using étale cobordism (see the proof of [32, Proposition 5.3]). One could also argue more directly via a specialisation argument from the characteristic zero case. This completes the proof of (a).

(b) The morphism $U_{(k_0)_s} \to (U/G)_{(k_0)_s}$ is a Galois G-cover, hence, we have the Hochschild-Serre spectral sequence in ℓ-adic cohomology

$$E_2^{i,j} = H^i(G, H^j(U_{(k_0)_s}, \mathbb{Z}_2(3))) \Rightarrow H^{i+j}((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)). \tag{2.1}$$

Here, $H^i(G, -)$ denotes group cohomology. Since U is an open subscheme of a vector space whose complement has codimension ≥ 4, we have $H^0(U_{(k_0)_s}, \mathbb{Z}_2(3)) = \mathbb{Z}_2(3)$ and $H^j(U_{(k_0)_s}, \mathbb{Z}_2(3)) = 0$ for all $1 \leq j \leq 6$. We deduce that the natural map $H^i(G, \mathbb{Z}_2(3)) \to H^i((U/G)_{(k_0)_s}, \mathbb{Z}_2(3))$ is an isomorphism for all $1 \leq i \leq 6$. Since the group G is finite, the group

$$H^i(G, \mathbb{Z}_2(3)) \simeq H^i(G, \mathbb{Z}_2) \simeq H^i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$$

is finite for all $i \geq 1$, hence

$$H^i((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)) \text{ is finite for all } 1 \leq i \leq 6. \tag{2.2}$$

For every finite field subextension $k_0 \subset k \subset (k_0)_s$, the Hochschild-Serre spectral sequence in continuous ℓ-adic cohomology

$$E_2^{i,j} = H^i(k, H^j((U/G)_{(k_0)_s}, \mathbb{Z}_2(3))) \Rightarrow H^{i+j}((U/G)_k, \mathbb{Z}_2(3)) \tag{2.3}$$

yields a filtration

$$\{0\} = F^7 \subset F^6 \subset \cdots \subset F^1 \subset F^0 = H^6((U/G)_k, \mathbb{Z}_2(3)),$$

where F^i/F^{i+1} is a subquotient of $H^i(k, H^{6-i}((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)))$. When $i = 0, 1$, F^i/F^{i+1} is even a submodule of $H^i(k, H^{6-i}((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)))$.

It is a consequence of [41, I.2.2, Corollary 1] that for all $i \geq 1$ and all finite continuous $\mathrm{Gal}((k_0)_s/k)$-modules M, any element of $H^i(k, M)$ is killed by passage to a suitable finite extension of k. Thus, (2.2) implies that for all $1 \leq i \leq 6$, any element of $H^i(k, H^{6-i}((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)))$ vanishes after base change to a suitable finite extension of k. By (a), we know that $\mathrm{cl}(\alpha_{(k_0)_s}) = 0$, that is, $\mathrm{cl}(\alpha) \in F^1$. Using the fact that F^i/F^{i+1} is a subquotient of $H^i(k, H^{6-i}((U/G)_{(k_0)_s}, \mathbb{Z}_2(3)))$, we may now construct finite field extensions

$$k_0 \subset k_1 \subset \cdots \subset k_6 \subset k_7,$$

such that $\mathrm{cl}(\alpha_{k_i}) \in F^i$ for all i. In particular, $\mathrm{cl}(\alpha_{k_7}) = 0$, hence, $k = k_7$ satisfies the conclusion of the lemma. □

Remark 2.2. It is important to note that continuous ℓ-adic cohomology does not commute with inverse limits of schemes, so (b) is not a formal consequence for (a).

Here is a generalised version of Theorem 1.3.

Theorem 2.3. *Let k_0 be a field of characteristic different from 2. There exist a finite 2-group G, a smooth complete intersection $Y \subset \mathbb{P}^N_{k_0}$ of dimension 15 with a free G-action and finite extension k/k_0, such that,*

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letting $X := Y/G$, the cycle class map

$$\mathrm{cl}\colon CH^3(X_k)[2] \to H^6(X_k, \mathbb{Z}_2(3))$$

is not injective.

Proof. Let Y be a smooth complete intersection of dimension 15 over k_0 on which $G := H \times \mathbb{Z}/2$ acts freely, and set $X := Y/G$: (see [40, Proposition 15]). Letting G act diagonally on $Y \times U$, the projections of $Y \times U$ onto its factors are G-equivariant: we write $\pi_1\colon (Y \times U)/G \to X$ and $\pi_2\colon (Y \times U)/G \to U/G$ for the induced morphisms. We have a commutative diagram

$$\begin{array}{ccccc} CH^3(X_k) & \xrightarrow{\pi_1^*} & CH^3(((Y\times U)/G)_k) & \xleftarrow{\pi_2^*} & CH^3((U/G)_k) \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^6(X_k, \mathbb{Z}_2(3)) & \xrightarrow{\pi_1^*} & H^6(((Y\times U)/G)_k, \mathbb{Z}_2(3)) & \xleftarrow{\pi_2^*} & H^6((U/G)_k, \mathbb{Z}_2(3)). \end{array}$$

The projection $Y \times V \to Y$ is a G-equivariant vector bundle and the G-action on Y is free, therefore, by descent and Grothendieck's version of Hilbert's Theorem 90 (see [28, Proposition III.4.9]), the induced morphism $(Y \times V)/G \to X$ is also a vector bundle. Since $Y \times U \to Y$ is a G-invariant dense open subscheme of the G-equivariant vector bundle $Y \times V \to Y$, π_1 is a dense open subscheme of a vector bundle. Moreover, since $V - U$ has codimension ≥ 4 in V, the codimension of the complement $(Y \times U)/G$ inside $(Y \times V)/G$ is also ≥ 4, hence, by [17, Theorem 3.23] and homotopy invariance, the maps π_1^* are isomorphisms. We get a well-defined element

$$\beta := (\pi_1^*)^{-1}(\pi_2^*(\alpha_k)) \in CH^3(X_k)[2].$$

By Lemma 2.1(b), we have $\mathrm{cl}(\beta) = 0$. In order to complete the proof, it remains to show that $\beta \neq 0$.

Suppose first that $k = \mathbb{C}$. Then Totaro showed in [46] that the class of β in the complex cobordism group $MU^6(X) \otimes_{MU^*(X)} \mathbb{Z}$ is not zero, hence, $\beta \neq 0$. If k is a field of characteristic zero, the rigidity of the 2-torsion subgroup of the Chow group [25] implies $\beta_{\overline{k}} \neq 0$, hence, $\beta \neq 0$. If k has positive characteristic (different from 2), the arguments of Totaro have been adapted by Quick (see the proof of [32, Proposition 5.3(b)]). We conclude that $\beta \neq 0$, as desired. □

3. ℓ-adic cycle class map and Bloch's map

In this section, we explain the relation between the cycle class map in continuous ℓ-adic cohomology and a certain map defined by Bloch. The main result of this section (Proposition 3.1) will be used to produce counterexamples to Question 1.2 in Sections 4 and 5.

Let k_0 be a field, $i \geq 0$ be an integer, ℓ be a prime number invertible in k_0 and X be a smooth projective k_0-variety. For every finite extension k/k_0, we have the cycle class map $\mathrm{cl}_k\colon CH^i(X_k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^{2i}(X_k, \mathbb{Z}_\ell(i))$ and the Bockstein homomorphism

$$\beta_k\colon H^{2i-1}(X_k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^{2i}(X_k, \mathbb{Z}_\ell(i)).$$

It will be important for us that $CH^i(X_{\overline{k}_0}) = \varinjlim_{k/k_0} CH^i(X_k)$ and

$$H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = \varinjlim_{k/k_0} H^{2i-1}(X_k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

where the direct limits are over all finite extensions k/k_0 contained in $\overline{k}_0/k_0$. Finally, recall that Bloch [4] (also see [10]) defined a map

$$\lambda\colon CH^i(X_{\overline{k}_0})\{\ell\} \to H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

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which, for $\overline{k}_0 = \mathbb{C}$, coincides with the Deligne cycle class map on torsion [4, Proposition 3.7]. Note that λ is rigid, that is, it does not change under algebraically closed field extensions, because the rigidity property holds for the torsion part of Chow groups [25] and for étale cohomology with torsion coefficients.

Proposition 3.1. *The composition*

$$CH^i(X_{\overline{k}_0})\{\ell\} \xrightarrow{\lambda} H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\varinjlim_{k/k_0} \beta_k} \varinjlim_{k/k_0} H^{2i}(X_k, \mathbb{Z}_\ell(i))$$

coincides with $\varinjlim_{k/k_0} \mathrm{cl}_k$ *on torsion. If* k_0 *is of finite type over its prime field,* $\varinjlim_{k/k_0} \beta_k$ *induces an isomorphism*

$$H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\sim} \left(\varinjlim_{k/k_0} H^{2i}(X_k, \mathbb{Z}_\ell(i))\right)\{\ell\},$$

hence, $\varinjlim_{k/k_0} \mathrm{cl}_k$ *is injective on torsion if and only if* λ *is injective.*

Remark 3.2. For $i \in \{1, 2, \dim X\}$, λ is injective: the case of $i = 1$ is elementary using the Kummer sequence [4, Proposition 3.6], the case of $i = \dim X$ is due to Rojtman [33] (see also [4, Theorem 4.2]) and the case of $i = 2$ is a consequence of a theorem of Merkurjev–Suslin [27, Section 18]. In these cases, if k_0 is of finite type over its prime field, $\varinjlim_{k/k_0} \mathrm{cl}_k$ is injective on torsion by Proposition 3.1. For $i = 1$, this is also a direct consequence of the observation of Jannsen [17, Remark 6.15 (a)] that $\mathrm{cl}_k : CH^1(X_k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^2(X_k, \mathbb{Z}_\ell(1))$ is injective. Remarkably, the kernel of $\mathrm{cl}_k : CH^2(X_k)\{\ell\} \to H^4(X_k, \mathbb{Z}_\ell(2))$ might be nonzero, as we will see in Section 6.

For $3 \leq i \leq \dim X - 1$, there are several known examples [34, 37, 39, 42, 46, 49] where λ is not injective; among them, [34, 37, 39, 49] even showed that the kernel of λ may be infinite. Note that fields of definition for [34, 37, 39, 42, 49] have positive transcendence degree over $\mathbb{Q}$, while Totaro's 15-dimensional examples in [46] may be defined over $\mathbb{Q}$ or $\mathbb{F}_p$ with $p \neq 2$. In Section 4, we exhibit the first fourfold examples defined over $\mathbb{Q}$ where λ is not injective over $\overline{\mathbb{Q}}$ (4 is the least possible dimension in which one can find such an example). In Section 5, we give further instances of noninjectivity of λ in relation to the integral Hodge and Tate conjectures. Using Proposition 3.1 and the rigidity property of λ, all of these provide counterexamples to Question 1.2 over all sufficiently large finite extensions of fields of definition.

Proof of Proposition 3.1. The second assertion follows by observing that if k_0 is of finite type over its prime field, then for every finite extension k/k_0 contained in $\overline{k}_0/k_0$, the map $H^{2i-1}(X_k, \mathbb{Q}_\ell(i)) \to H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is zero, because it factors through $H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}_0/k)}$ which vanishes by weight reasons.

It remains to show the first assertion. By construction, λ fits into the commutative diagram:

$$\begin{array}{ccc} H^{i-1}(X_{\overline{k}_0}, \mathcal{H}(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))) & \xrightarrow{f} & CH^i(X_{\overline{k}_0})\{\ell\} \\ \downarrow g & \swarrow{\scriptstyle -\lambda} & \\ H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)), & & \end{array}$$

where f is the surjection given in [12, Proposition 1], $H^{i-1}(X_{\overline{k}_0}, \mathcal{H}(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)))$ is the $E_2^{i-1,i}$ term of the Bloch-Ogus spectral sequence [7] and g is the edge homomorphism. Hence, the proof will follow once we show the anticommutativity of the following diagram:

$$\begin{array}{ccc} H^{i-1}(X_{\overline{k}_0}, \mathcal{H}^i(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))) & \xrightarrow{f} \!\!\!\!\rightarrow & CH^i(X_{\overline{k}_0})\{\ell\} \\ \big\downarrow g & & \big\downarrow \varinjlim_{k/k_0} \mathrm{cl}_k \\ H^{2i-1}(X_{\overline{k}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) & \xrightarrow{\varinjlim_{k/k_0} \beta_k} & \varinjlim_{k/k_0} H^{2i}(X_k, \mathbb{Z}_\ell(i)). \end{array} \tag{3.1}$$

Here, $H^{i-1}(X_{\overline{k}_0}, \mathcal{H}^i(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))) = \varinjlim_{k/k_0} H^{i-1}(X_k, \mathcal{H}^i(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)))$, because the Gersten complex of $\mathcal{H}^i(\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ on $X_{\overline{k}_0}$ is the direct limit of Gersten complexes on X_k. Hence, the anticommutativity of (3.1) is reduced to showing, for every finite extension k/k_0 and every integer $\nu \geq 1$, the anticommutativity of

$$\begin{array}{ccc} H^{i-1}(X_k, \mathcal{H}^i(\mu_{l^\nu}^{\otimes i})) & \xrightarrow{f} \!\!\!\!\rightarrow & CH^i(X_k)[\ell^\nu] \\ \big\downarrow g & & \big\downarrow \mathrm{cl}_k \\ H^{2i-1}(X_k, \mu_{l^\nu}^{\otimes i}) & \xrightarrow{\beta_k} & H^{2i}(X_k, \mathbb{Z}_\ell(i)). \end{array} \tag{3.2}$$

To prove that (3.2) anticommutes, we proceed as in the proof of [12, Proposition 1]. Recall that each element $\alpha \in H^{i-1}(X_k, \mathcal{H}^i(\mu_{l^\nu}^{\otimes i}))$ is represented by a class $a \in H_Z^{2i-1}(X_k - Z', \mu_{l^\nu}^{\otimes i})$, where (Z, Z') is a pair of closed subsets of X_k of codimension $i-1$ and i, respectively, with $Z' \subset Z$, that vanishes under the connecting homomorphism $H^{2i-1}_{Z-Z'}(X_k - Z', \mu_{l^\nu}^{\otimes i}) \to H^{2i}_{Z'}(X_k, \mu_{l^\nu}^{\otimes i})$. We may now associate to the class a two classes $b, c \in H_Z^{2i}(X_k, \mathbb{Z}_\ell(i))$ whose images in $H^{2i}(X_k, \mathbb{Z}_\ell(i))$ are $\beta_k \circ g(\alpha)$, $-\mathrm{cl}_k \circ f(\alpha)$ respectively. The argument is as follows, using the diagram:

$$\begin{array}{ccccccc} & & & & H^{2i}_{Z'}(X_k, \mathbb{Z}_\ell(i)) & \xrightarrow{i} & H^{2i}_Z(X_k, \mathbb{Z}_\ell(i)) \\ & & & & \big\downarrow \ell^\nu & & \\ & & H^{2i-1}_{Z-Z'}(X_k - Z', \mathbb{Z}_\ell(i)) & \xrightarrow{\delta} & H^{2i}_{Z'}(X_k, \mathbb{Z}_\ell(i)) & & \\ & & \big\downarrow p & & & & \\ H^{2i-1}_Z(X_k, \mu_{\ell^\nu}^{\otimes i}) & \xrightarrow{j} & H^{2i-1}_{Z-Z'}(X_k - Z', \mu_{\ell^\nu}^{\otimes i}) & & & & \\ \big\downarrow \beta_k & & & & & & \\ H^{2i}_Z(X_k, \mathbb{Z}_\ell(i)). & & & & & & \end{array}$$

Here, the horizontal arrows are from the long exact sequences for cohomology with supports and the vertical arrows are from the long exact sequences for $H^*_{Z'}(X_k, -)$, $H^*_Z(X_k, -)$ and $H^*_{Z-Z'}(X_k, -)$ induced by the short exact sequence of inverse systems of abelian sheaves on $X_{\text{ét}}$:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mu_{\ell^{\nu'+1}}^{\otimes i} & \longrightarrow & \mu_{\ell^{\nu'+1+\nu}}^{\otimes i} & \xrightarrow{\ell^{\nu'+1}} & \mu_{\ell^\nu}^{\otimes i} & \longrightarrow & 0 \\ & & \big\downarrow \ell & & \big\downarrow \ell & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \mu_{\ell^{\nu'}}^{\otimes i} & \longrightarrow & \mu_{\ell^{\nu'+\nu}}^{\otimes i} & \xrightarrow{\ell^{\nu'}} & \mu_{\ell^\nu}^{\otimes i} & \longrightarrow & 0. \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{3.3}$$

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By the choice on a, there exists $a_1 \in H^{2i-1}_Z(X_k, \mu_{\ell^\nu}^{\otimes i})$, such that $j(a_1) = a$. Set $b := \beta_k(a_1)$. Meanwhile, after possibly enlarging $Z' \subset Z$, a lifts along p to a class $a_2 \in H^{2i-1}_{Z-Z'}(X_k - Z', \mathbb{Z}_\ell(i))$. Indeed, we may assume that: $Z - Z'$ is smooth, thus

$$H^{2i-1}_{Z-Z'}(X_k - Z', \mu_{\ell^\nu}^{\otimes i}) = H^1(Z - Z', \mu_{\ell^\nu}),$$
$$H^{2i-1}_{Z-Z'}(X_k - Z', \mathbb{Z}_\ell(i)) = H^1(Z - Z', \mathbb{Z}_\ell(1))$$

by [17, Theorem 3.17]; $a \in H^1(Z - Z', \mu_{\ell^\nu})$ lifts along the composition

$$H^0(Z - Z', \mathbb{G}_m) \xrightarrow{\Delta} H^1(Z - Z', \mathbb{Z}_\ell(1)) \xrightarrow{p} H^1(Z - Z', \mu_{\ell^\nu}^{\otimes i}),$$

where Δ is the connecting homomorphism for the short exact sequence of inverse systems of abelian sheaves on $X_{\text{ét}}$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{\ell^{\nu'+1}} & \longrightarrow & \mathbb{G}_m & \xrightarrow{\ell^{\nu'+1}} & \mathbb{G}_m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mu_{\ell^{\nu'}} & \longrightarrow & \mathbb{G}_m & \xrightarrow{\ell^{\nu'}} & \mathbb{G}_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(to see this, note that $p \circ \Delta$ at the direct limit over all $Z' \subset Z$ corresponds to the surjection $\oplus k(x)^\times \twoheadrightarrow \oplus k(x)^\times/\ell^\nu$, where the direct sums are over the generic points of Z). Let $a_3 = \delta(a_2)$. Then there exits $a_4 \in H^{2i}_{Z'}(X_k, \mathbb{Z}_\ell(i))$, such that $a_3 = l^\nu a_4$. Set $c := i(a_4)$. It is now direct to see that b, c satisfy the required properties.

To complete the proof, it is enough to show that $b = c$. As the category of inverse systems of abelian sheaves on $X_{\text{ét}}$ is an abelian category with enough injectives by [17, Proposition 1.1], we may take a Cartan-Eilenberg injective resolution of (3.3). Now an argument analogous to [12, p. 771] concludes the proof. □

Second Proof of Theorem 2.3. As in Section 2, let $G := H \times \mathbb{Z}/2$, where H is the Heisenberg group of order 32, let $Y \subset \mathbb{P}^N_{k_0}$ be a smooth complete intersection of dimension 15 on which G acts freely, and $X := Y/G$. By means of Proposition 3.1 and the rigidity property of λ, it is enough for us to show that $\lambda\colon CH^3(X_F)\{2\} \to H^5(X_F, \mathbb{Q}_2/\mathbb{Z}_2(3))$ is not injective for some algebraically closed field extension F of a field of definition.

The assertion in characteristic zero follows from [46, Theorem 7.2]. In positive characteristic different from 2, the assertion follows from [32, Proposition 5.3 (b)], because the group $H^5(X_{\overline{k}_0}, \mathbb{Z}_2(3))$ is torsion by construction and the composition

$$CH^2(X_{\overline{k}_0})\{2\} \xrightarrow{\lambda} H^5(X_{\overline{k}_0}, \mathbb{Q}_2/\mathbb{Z}_2(3)) \xhookrightarrow{\beta_{\overline{k}_0}} H^6(X_{\overline{k}_0}, \mathbb{Z}_2(3))$$

coincides with the cycle class map. This concludes the proof. □

We conclude this section by a remark on Schreieder's transcendental Abel-Jacobi map [38, Section 7.5].

Remark 3.3. Suppose that k_0 is of finite type over its prime field. For every finite extension k/k_0, we have the transcendental Abel-Jacobi map:

$$\lambda_{\mathrm{tr},k}\colon CH^i_0(X_k)\{\ell\} \to H^{2i-1}(X_k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X_k, \mathbb{Q}_\ell(i)),$$

where $CH^i_0(X_k)\{\ell\}$ is the kernel of $\mathrm{cl}_k\colon CH^i(X_k)\{\ell\} \to H^{2i}(X_k, \mathbb{Z}_\ell(i))$. Then one can observe that $\varinjlim_{k/k_0} \lambda_{\mathrm{tr},k} = 0$ by [38, Proposition 7.16] and weight arguments. This shows that if

$$\varinjlim_{k/k_0} CH^i_0(X_k)\{\ell\} = \mathrm{Ker}\left(CH^i(X_{\overline{k}_0})\{\ell\} \xrightarrow{\varinjlim_{k/k_0} \mathrm{cl}_k} \varinjlim_{k/k_0} H^{2i}(X_k, \mathbb{Z}_\ell(i)) \right)$$

is not zero, or equivalently by Proposition 3.1, if λ is not injective over $\overline{k}_0$, then $\lambda_{\mathrm{tr},k}$ is not injective for all sufficiently large finite extensions k/k_0 contained in $\overline{k}_0/k_0$. As described in Remark 3.2, we already have several examples with the property, and we will give further such examples in Sections 4 and 5. In Section 6, we will provide examples with $\lambda_{\mathrm{tr},k} \neq 0$.

4. Proof of Theorem 1.4

Let k_0 be a number field. Let B (respectively, E) be an abelian threefold (respectively, an elliptic curve) over k_0, and set $A := B \times E$. Suppose that A has good ordinary reduction at some prime dividing 2. For instance, one can take $k_0 = \mathbb{Q}$ and A to be the product of 4 copies of the elliptic curve

$$y^2 + xy = x^3 + 1.$$

Let ι be an involution acting on B by -1 and Y be the Kummer threefold associated to B, that is, the blow up of B/ι at the 64 singular points, so that Y is smooth and contains 64 disjoint copies of $\mathbb{P}^2$. Finally, set $X := Y \times E$. Note that the action of ι lifts to A where ι acts trivially on E, and X can also be obtained by blowing up the quotient variety A/ι along the singular locus.

In the following, we fix an embedding $k_0 \hookrightarrow \mathbb{C}$.

Lemma 4.1. $H^4_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}/2) \neq 0$.

Proof. We follow the method of Diaz in [15, Section 2.1]. In this proof, we write A, B, E, X for $A_{\mathbb{C}}, B_{\mathbb{C}}, E_{\mathbb{C}}, X_{\mathbb{C}}$. Letting $A^\circ := A - (B[2] \times E)$, $U := A^\circ/\iota$ and $\pi\colon A^\circ \to U$ be the quotient map, we have the following commutative diagram:

$$\begin{array}{ccccccc}
H^4(A,\mathbb{Z}/2) & \xrightarrow{\sim} & H^4(A^\circ,\mathbb{Z}/2) & \xleftarrow{\pi^*} & H^4(U,\mathbb{Z}/2) & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & \\
H^4_{\mathrm{nr}}(A,\mathbb{Z}/2) & \hookrightarrow & H^4_{\mathrm{nr}}(A^\circ,\mathbb{Z}/2) & \xleftarrow{\pi^*} & H^4_{\mathrm{nr}}(U,\mathbb{Z}/2) & \hookleftarrow & H^4_{\mathrm{nr}}(X,\mathbb{Z}/2).
\end{array} \tag{4.1}$$

Here, the vertical arrows are the restriction maps and the horizontal arrows are the pullback maps, the injectivity of $H^4_{\mathrm{nr}}(A,\mathbb{Z}/2) \to H^4_{\mathrm{nr}}(A^\circ,\mathbb{Z}/2)$ and $H^4_{\mathrm{nr}}(X,\mathbb{Z}/2) \to H^4_{\mathrm{nr}}(U,\mathbb{Z}/2)$ is by definition of unramified cohomology, the map $H^4(A,\mathbb{Z}/2) \to H^4(A^\circ,\mathbb{Z}/2)$ is an isomorphism because $\mathrm{codim}(B[2] \times E, A) = 3$.

We need to check that (i) $\pi^*\colon H^4(U,\mathbb{Z}/2) \twoheadrightarrow H^4(A^\circ,\mathbb{Z}/2)$ and (ii) $H^4(U,\mathbb{Z}/2) \to H^4_{\mathrm{nr}}(U,\mathbb{Z}/2)$ factors through $H^4_{\mathrm{nr}}(X,\mathbb{Z}/2)$. As for (i), note that $A^\circ = (B - B[2]) \times E$ and $U = (B - B[2])/\iota \times E$. Letting $\rho\colon B - B[2] \to (B - B[2])/\iota$ be the quotient map, it is enough for us to show that $\rho^*\colon H^i((B - B[2])/\iota, \mathbb{Z}/2) \to H^i(B - B[2], \mathbb{Z}/2)$ is surjective for $i = 2, 3, 4$. Since $\mathrm{codim}(B[2], B) = 3$, the restriction map

$$\Lambda^i H^1(B,\mathbb{Z}/2) \xrightarrow{\sim} H^i(B,\mathbb{Z}/2) \to H^i(B - B[2],\mathbb{Z}/2)$$

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is an isomorphism for $i \leq 4$. So it suffices to show that $\rho^*\colon H^1((B-B[2])/\iota, \mathbb{Z}/2) \to H^1(B-B[2], \mathbb{Z}/2)$ is surjective, which follows from the fact that the short exact sequence

$$1 \to \pi_1(B-B[2]) \to \pi_1((B-B[2])/\iota) \to \{\pm 1\} \to 1$$

splits. Here, the splitting is given by the nontrivial element in the fundamental group of $\mathbb{RP}^5$ that appears as the quotient of the boundary $\mathbb{S}^5$ of an open ball neighborhood of a 2-torsion point in B, as observed in the first paragraph of the proof of [43, Theorem 1] (see also [15, p. 267]). Alternatively, (i) directly follows from [15, Corollary 2.8], because the assumptions for the statement are satisfied: $B[2] \times E$ is smooth, $\operatorname{codim}(B[2] \times E, A) = 3$, ι acts by -1 on the normal bundle $N_{B[2]\times E/A}$ and ι acts trivially on $H^1(A, \mathbb{Z}/2)$. As for (ii), the direct computation of the unramified cohomology group using the Gersten complex reduces it to the vanishing $H^3_{\mathrm{nr}}(X-U, \mathbb{Z}/2) = 0$ (see [15, Lemma 2.10]). The vanishing indeed holds because $X-U$ is 64 disjoint copies of $\mathbb{P}^2 \times E$ and

$$H^3_{\mathrm{nr}}(\mathbb{P}^2 \times E, \mathbb{Z}/2) = H^3_{\mathrm{nr}}(E, \mathbb{Z}/2) = 0.$$

Finally, a theorem of Bloch–Esnault [6, Theorem 1.2] shows that $H^4(A, \mathbb{Z}/2) \to H^4_{\mathrm{nr}}(A, \mathbb{Z}/2)$ is nonzero (here, we use the rigidity property for unramified cohomology with torsion coefficients [8, Theorem 4.4.1]). This, with (4.1), concludes the proof. □

Proposition 4.2. $\mathrm{cl}_{\mathcal{D}}\colon CH^3(X_{\mathbb{C}})\{2\} \to H^6_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(3))$ *is not injective.*

Proof. One needs to relate the fourth unramified cohomology group to the kernel of the Deligne cycle class map on torsion in codimension 3. We start with a short exact sequence given by [51, Theorem 0.2] and [26, Remark 4.2 (1)]:

$$0 \to \Lambda^5(X_{\mathbb{C}})_{\mathrm{tors}} \to H^4_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z})/H^4_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to \mathcal{T}^3(X_{\mathbb{C}}) \to 0, \tag{4.2}$$

where

$$\Lambda^5(X_{\mathbb{C}}) := H^5(X_{\mathbb{C}}, \mathbb{Z})/N^2H^5(X_{\mathbb{C}}, \mathbb{Z}),$$
$$\mathcal{T}^3(X_{\mathbb{C}}) := \operatorname{Ker}\Big(\mathrm{cl}_{\mathcal{D}}\colon CH^3(X_{\mathbb{C}})_{\mathrm{tors}} \to H^6_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(3))\Big)/\mathrm{alg}$$

(the notation /alg in the above equation means quotient by the algebraically trivial cycles in the kernel). It is important for us that $CH_0(X_{\mathbb{C}})$ is supported in dimension ≤ 3, because $CH_0(Y_{\mathbb{C}})$ is supported in dimension ≤ 2 by [5, Section 4 (1)]. By decomposition of the diagonal and the Bloch–Kato conjecture proved by Voevodsky, we have

$$H^4_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}) = 0 \tag{4.3}$$

(see [14, Proposition 3.3 (i)]). Moreover, [45, Theorem 1.1] yields

$$\operatorname{Coker}\Big(H^5(X_{\mathbb{C}}, \mathbb{Z})_{\mathrm{tors}} \to \Lambda^5(X_{\mathbb{C}})_{\mathrm{tors}}\Big)$$
$$\simeq \operatorname{Ker}\Big(\mathrm{cl}_{\mathcal{D}}\colon CH^3(X_{\mathbb{C}})_{\mathrm{alg,tors}} \to H^6_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(3))\Big),$$

where we write $CH^3(X_{\mathbb{C}})_{\mathrm{alg,tors}} \subset CH^3(X_{\mathbb{C}})$ for the subgroup of algebraically trivial torsion cycles. Note that $H^5(X_{\mathbb{C}}, \mathbb{Z})$ is in fact torsion free, because $Y_{\mathbb{C}}$ and $E_{\mathbb{C}}$ have torsion free cohomology (use [43, Theorem 2] for the Kummer threefold $Y_{\mathbb{C}}$), hence

$$\Lambda^5(X_{\mathbb{C}})_{\mathrm{tors}} \simeq \operatorname{Ker}\Big(\mathrm{cl}_{\mathcal{D}}\colon CH^3(X_{\mathbb{C}})_{\mathrm{alg,tors}} \to H^6_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(3))\Big). \tag{4.4}$$

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By (4.2), (4.3) and (4.4), it remains to show that $H^4_{nr}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z})\{2\} \neq 0$. This can be deduced from Lemma 4.1, because the natural map

$$H^4_{nr}(X_{\mathbb{C}}, \mathbb{Z}/2) \to H^4_{nr}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z})$$

is injective, again, by the Bloch–Kato conjecture (see [2, Theorem 1.1]). The proof is now complete. □

We prove a strengthened version of Theorem 1.4.

Theorem 4.3. *Let k_0 be a field of characteristic zero. Then there exist a fourfold product $X = Y \times E$ over k_0, where Y is a Kummer threefold and E is an elliptic curve, and a finite extension k/k_0, such that the cycle class map*

$$\mathrm{cl}\colon CH^3(X_k)[2] \to H^6(X_k, \mathbb{Z}_2(3))$$

is not injective.

Proof. Let $X = Y \times E$ be a fourfold product over a subfield $\widetilde{k}_0 \subset k_0$ that is finite over $\mathbb{Q}$, as given at the beginning of this section. Fixing an embedding $\widetilde{k}_0 \hookrightarrow \mathbb{C}$, Proposition 4.2 shows that

$$\lambda\colon CH^3(X_{\mathbb{C}})\{2\} \to H^5(X_{\mathbb{C}}, \mathbb{Q}_2/\mathbb{Z}_2(3))$$

is not injective, hence, by the rigidity property of λ, the same result holds over $\overline{\widetilde{k}}_0$, then over $\overline{k}_0$. Proposition 3.1 now shows that there exists a finite extension k/k_0, such that

$$\mathrm{cl}\colon CH^3(X_k)\{2\} \to H^6(X_k, \mathbb{Z}_2(3))$$

is not injective. This finishes the proof. □

5. Further examples in codimension three

In this section, we provide further counterexamples to Question 1.2 in codimension 3. By Proposition 3.1, this is reduced to finding examples for which Bloch's map λ is not injective over some algebraically closed field extension of a field of definition. To achieve this, we use nontorsion type counterexamples to the integral Hodge and Tate conjectures, inspired by the work of Soulé–Voisin [42].

Let k_0 be a field, ℓ be a prime number invertible in k_0, $i \geq 0$ be an integer and Y be a smooth projective variety over k_0. We define

$$\widetilde{Z}^{2i}_{\text{ét},\ell}(Y_{(k_0)_s}) \coloneqq \mathrm{Coker}\Big(H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))_{\text{tors}} \to H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))^{(1)}/H^{2i}_{\text{alg}}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))\Big),$$

where $H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))^{(1)} \subset H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))$ is the $\mathrm{Gal}((k_0)_s/k_0)$-submodule consisting of elements with open stabiliser and $H^{2i}_{\text{alg}}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))$ is the image of the cycle class map $\mathrm{cl}\colon CH^i(Y_{(k_0)_s}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))$. The group $\widetilde{Z}^{2i}_{\text{ét},\ell}(Y_{(k_0)_s})$ is well-defined because $H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))^{(1)} \subset H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))$ is saturated by [11, Lemma 4.1]. Note that $\widetilde{Z}^{2i}_{\text{ét},\ell}(Y_{(k_0)_s})_{\text{tors}} = 0$ if and only if the sublattice

$$H^{2i}_{\text{alg}}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))_{\text{tf}} \subset H^{2i}(Y_{(k_0)_s}, \mathbb{Z}_\ell(i))^{(1)}_{\text{tf}}$$

is saturated. When $k \subset \mathbb{C}$, we similarly define

$$\widetilde{Z}^{2i}(Y_{\mathbb{C}}) \coloneqq \mathrm{Coker}\Big(H^{2i}(Y_{\mathbb{C}}, \mathbb{Z})_{\text{tors}} \to \mathrm{Hdg}^{2i}(Y_{\mathbb{C}}, \mathbb{Z})/H^{2i}_{\text{alg}}(Y_{\mathbb{C}}, \mathbb{Z})\Big),$$

where $\mathrm{Hdg}^{2i}(Y_{\mathbb{C}},\mathbb{Z}) \subset H^{2i}(Y_{\mathbb{C}},\mathbb{Z})$ is the subgroup of integral Hodge classes and $H^{2i}_{\mathrm{alg}}(Y_{\mathbb{C}},\mathbb{Z}) := \mathrm{Im}\big(\mathrm{cl}\colon CH^i(Y_{\mathbb{C}}) \to H^{2i}(Y_{\mathbb{C}},\mathbb{Z})\big)$. Note that $\widetilde{Z}^{2i}(Y_{\mathbb{C}})_{\mathrm{tors}} = 0$ if and only if the sublattice

$$H^{2i}_{\mathrm{alg}}(Y_{\mathbb{C}},\mathbb{Z})_{\mathrm{tf}} \subset \mathrm{Hdg}^{2i}(Y_{\mathbb{C}},\mathbb{Z})_{\mathrm{tf}}$$

is saturated.

Lemma 5.1. *With the same notation as above, suppose either: $\widetilde{Z}^{2i}_{ét,\ell}(Y_{(k_0)_s})\{\ell\} \neq 0$, or $k_0 \subset \mathbb{C}$ and $\widetilde{Z}^{2i}(Y_{\mathbb{C}})\{\ell\} \neq 0$. Then there exist a finitely generated extension K_0/k_0 with $\mathrm{tr\,deg}_{k_0} K_0 = 1$ and an elliptic curve E over K_0, such that, letting $X := Y \times_{k_0} E$, the map $\lambda\colon CH^{i+1}(X_{\overline{K}_0})\{\ell\} \to H^{2i+1}(X_{\overline{K}_0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1))$ is not injective.*

Proof. We only do the first case, the second case is similar (also see [45, Proposition 3.1]). After tensor $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, the short exact sequence

$$0 \to H^{2i}_{\mathrm{alg}}(Y_{(k_0)_s},\mathbb{Z}_\ell(i)) \to H^{2i}(Y_{(k_0)_s},\mathbb{Z}_\ell(i))^{(1)} \to H^{2i}(Y_{(k_0)_s},\mathbb{Z}_\ell(i))^{(1)}/H^{2i}_{\mathrm{alg}}(Y_{(k_0)_s},\mathbb{Z}_\ell(i)) \to 0$$

yields an exact sequence

$$0 \to \widetilde{Z}^{2i}_{\text{ét},\ell}(Y_{(k_0)_s})\{\ell\} \to H^{2i}_{\mathrm{alg}}(Y_{(k_0)_s},\mathbb{Z}_\ell(i)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^{2i}(Y_{(k_0)_s},\mathbb{Z}_\ell(i)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

From the assumption, we now see that there exists a nonzero $\alpha \in CH^i(Y_{(k_0)_s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ that vanishes in $H^{2i}(Y_{(k_0)_s},\mathbb{Z}_\ell(i)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Note that, by passing to the algebraic closure, we get isomorphisms

$$CH^i(X_{(k_0)_s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\sim} CH^i(X_{\overline{k}_0}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,\ H^{2i}(X_{(k_0)_s},\mathbb{Z}_\ell(i)) \xrightarrow{\sim} H^{2i}(X_{\overline{k}_0},\mathbb{Z}_\ell(i)).$$

Let $\alpha' \in CH^i(X_{\overline{k}_0}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ be the image of α.

Let K_0/k_0 be a finitely generated field extension with $\mathrm{tr\,deg}_{k_0} K_0 = 1$ and E be an elliptic curve over K_0 with $j(E) \notin \overline{k}_0$. Fixing a component $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ of $CH^1(E_{\overline{K}_0})\{\ell\} = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^2$, we indentify α' with an element in $CH^i(Y_{\overline{k}_0}) \otimes CH^1(E_{\overline{K}_0})\{\ell\}$. Letting $X := Y \times_{k_0} E$, a theorem of Schoen [37, Theorem 0.2] shows that the image β of α' under the exterior product map

$$CH^i(Y_{\overline{k}_0}) \otimes CH^1(E_{\overline{K}_0})\{\ell\} \xrightarrow{\times} CH^{i+1}(X_{\overline{K}_0})\{\ell\}$$

is nonzero. Now it remains for us to show that $\beta \in CH^{i+1}(X_{\overline{K}_0})\{\ell\}$ is in the kernel of λ. This follows from the commutative diagram:

$$\begin{array}{ccc} CH^i(Y_{\overline{k}_0}) \otimes CH^1(E_{\overline{K}_0})\{\ell\} & \xrightarrow{\mathrm{cl}\otimes\lambda} & H^{2i}(Y_{\overline{k}_0},\mathbb{Z}_\ell(i)) \otimes H^1(E_{\overline{K}_0},\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \\ \downarrow{\scriptstyle\times} & & \downarrow{\scriptstyle\cup} \\ CH^{i+1}(X_{\overline{K}_0}) & \xrightarrow{\lambda} & H^{2i+1}(X_{\overline{K}_0},\mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)). \end{array}$$

The proof is complete. □

Lemma 5.1 can be applied to nontorsion type counterexamples to the integral Hodge conjecture [14, 15, 22, 30, 48] or the integral Tate conjecture [31, 48]. One may take $k_0 = \mathbb{Q}$ for the examples in [14, 15, 22, 30, 48] and k_0 to be a finite field for the examples in [31].

Proposition 3.1 then produces various examples of fields K of finite type over the prime fields of transcendence degree 1, prime numbers ℓ invertible in K and smooth projective K-varieties X, such that $\mathrm{cl}\colon CH^3(X)[\ell] \to H^6(X,\mathbb{Z}_\ell(3))$ is not injective. Those with the best bounds are: fourfolds in characteristic zero; eightfolds in positive characteristic.

6. Proof of Theorem 1.5

Lemma 6.1. *Let k be a field and ℓ be a prime invertible in k. Then $H^2(k,\mathbb{Z}_\ell(1)) \simeq T_\ell(\mathrm{Br}(k))$. In particular, $H^2(k,\mathbb{Z}_\ell(1))$ is torsion free.*

Proof. By [29, Theorem 2.7.5], we have a short exact sequence

$$0 \to \varprojlim_m{}^1 H^1(k,\mu_{\ell^m}) \to H^2(k,\mathbb{Z}_\ell(1)) \to \varprojlim_m H^2(k,\mu_{\ell^m}) \to 0.$$

The Kummer sequence

$$1 \to \mu_{\ell^m} \to \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m} \to 1$$

gives natural identifications

$$H^1(k,\mu_{\ell^m}) = k^\times/k^{\times\ell^m}, \qquad H^2(k,\mu_{\ell^m}) = \mathrm{Br}(k)[\ell^m].$$

The induced maps $k^\times/k^{\times\ell^{m+1}} \to k^\times/k^{\times\ell^m}$ are the natural quotient maps, and, in particular, they are surjective. It follows that the sequence of the $H^1(k,\mu_{\ell^m})$ satisfies the Mittag-Leffler condition, and so $\varprojlim^1 H^1(k,\mu_{\ell^m}) = 0$. The induced maps $\mathrm{Br}(k)[\ell^{m+1}] \to \mathrm{Br}(k)[\ell^m]$ are given by multiplication by ℓ, hence, $\varprojlim H^2(k,\mu_{\ell^m}) = T_\ell(\mathrm{Br}(k))$. □

Lemma 6.2. *Let k be a global field, ℓ be a prime number invertible in k and $k(t)/k$ be a purely transcendental extension of transcendence degree 1. If $\ell = 2$, suppose that k is a totally imaginary number field or a function field. Then $H^4(k(t),\mathbb{Z}_\ell(2)) = 0$.*

Proof. By [29, Theorem 2.7.5], we have a short exact sequence

$$0 \to \varprojlim_n{}^1 H^3(k(t),\mu_{\ell^n}^{\otimes 2}) \to H^4(k(t),\mathbb{Z}_\ell(2)) \to \varprojlim_n H^4(k(t),\mu_{\ell^n}^{\otimes 2}) \to 0. \tag{6.1}$$

By [41, II.4.4, Proposition 13], we have $\mathrm{cd}_\ell(k) \leq 2$, and so [41, II.4.2, Proposition 11] implies $\mathrm{cd}_\ell(k(t)) \leq 3$. It follows that the group $H^4(k(t),\mu_{\ell^n}^{\otimes 2})$ is trivial for all $n \geq 0$, hence, $\varprojlim_n H^4(k(t),\mu_{\ell^n}^{\otimes 2}) = 0$. In view of (6.1), the proof will be complete once we show that $\varprojlim_n^1 H^3(k(t),\mu_{\ell^n}^{\otimes 2}) = 0$.

We regard $k(t)$ as the function field of $\mathbb{P}^1_k$. By [41, p. 113], we have an exact sequence

$$0 \to H^3(k,\mu_{\ell^n}^{\otimes 2}) \to H^3(k(t),\mu_{\ell^n}^{\otimes 2}) \to \oplus_{x\in(\mathbb{P}^1_k)^{(1)}} H^2(k(x),\mu_{\ell^n}) \xrightarrow{C} H^2(k,\mu_{\ell^n}) \to 0$$

which is functorial in $n \geq 0$. Since $\mathrm{cd}_\ell(k) \leq 2$, the first term $H^3(k,\mu_{\ell^n}^{\otimes 2})$ vanishes. The surjective map C is the direct sum of the corestriction maps along the field extensions $k(x)/k$, and so the point at infinity $\infty \in \mathbb{P}^1_k$ determines a section of C. We obtain a decomposition

$$H^3(k(t),\mu_{\ell^n}^{\otimes 2}) \simeq \oplus_{x\in(\mathbb{A}^1_k)^{(1)}} H^2(k(x),\mu_{\ell^n}) \simeq \oplus_{x\in(\mathbb{A}^1_k)^{(1)}} \mathrm{Br}(k(x))[\ell^n]. \tag{6.2}$$

The isomorphism on the right comes from the Kummer short exact sequence. The isomorphism (6.2) is functorial in n, where on the right, the transition maps $\mathrm{Br}(k(x))[\ell^{n+1}] \to \mathrm{Br}(k(x))[\ell^n]$ are given by multiplication by ℓ.

Suppose first that k is a totally imaginary number field or a function field. Then for every closed point x of $\mathbb{A}^1_k$, the residue field $k(x)$ is also totally imaginary. It follows from the celebrated theorem of Albert, Brauer, Hasse and Noether [29, Theorem 8.1.17] that $\mathrm{Br}(k(x))$ is divisible. Thus, the maps $\mathrm{Br}(k(x))[\ell^{n+1}] \to \mathrm{Br}(k(x))[\ell^n]$ given by multiplication by ℓ are surjective, hence, by

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(6.2) so are the transition maps $H^3(k(t),\mu_{\ell^{n+1}}^{\otimes 2}) \to H^3(k(t),\mu_{\ell^n}^{\otimes 2})$. This shows that the inverse system $\{H^3(k(t),\mu_{\ell^n}^{\otimes 2})\}_{n\geq 0}$ satisfies the Mittag-Leffler condition, and so $\varprojlim_n^1 H^3(k(t),\mu_{\ell^n}^{\otimes 2}) = 0$ by [29, Proposition 2.7.4], as desired.

Suppose now that k admits at least one real embedding. Then under our assumptions, $\ell \neq 2$. By [29, Theorem 8.1.17], the group $\mathrm{Br}(k(x))$ is the direct sum of a divisible group and a finite elementary 2-group. Then, since ℓ is odd, the maps $\mathrm{Br}(k(x))[\ell^{n+1}] \to \mathrm{Br}(k(x))[\ell^n]$ given by multiplication by ℓ are surjective and the conclusion follows as in the previous case. □

Theorem 1.5 is a special case of the following more general statement.

Theorem 6.3. *Let k be a global field, $k(t)$ be a purely transcendental extension of k of transcendence degree 1 and ℓ be a prime invertible in k. If $\ell = 2$, suppose that k is a totally imaginary number field or a function field, and if ℓ is odd, suppose that* $\mathrm{char}(k) = 0$. *Then there exists a norm variety X of dimension $\ell^2 - 1$ over $k(t)$, such that*

$$\mathrm{cl}\colon CH^2(X)[\ell] \to H^4(X,\mathbb{Z}_\ell(2))$$

is not injective.

Proof. By (6.2) and the theorem of Albert, Brauer, Hasse and Noether [29, Theorem 8.1.17], we have $H^3(k(t),\mu_\ell^{\otimes 2}) \neq 0$. Let X be a norm variety associated to a nontrivial symbol $s \in H^3(k(t),\mu_\ell^{\otimes 2})$, as constructed by Rost [44] (see also [21, Section 5d]). The k-variety X is a smooth projective of dimension $\ell^2 - 1$. The pure Chow motive with $\mathbb{Z}_{(\ell)}$-coefficients $M(X;\mathbb{Z}_{(\ell)})$ of X contains the Rost motive $\mathcal{R}$ of s as a direct summand. By [21, Theorem RM.10], we have $CH^2(\mathcal{R}) = \mathbb{Z}/\ell$, hence, $CH^2(X)[\ell] \neq 0$ (we apply [21, Theorem RM.10] with $p = \ell$, $n = 2$, $k = 1$ and $i = 1$. By definition $b = 1 + p$, hence, $j = bk - p^i + 1 = 2$). Let $\alpha \in CH^2(\mathcal{R})[\ell]$ be a nonzero element.

If $\ell = 2$, we may construct X and α in any characteristic different from 2 as follows. Let $\mathcal{O}$ be the ring of integers of k, $\pi \in \mathcal{O}$ be a prime element and $u \in \mathcal{O}$ be, such that the class of u in the residue field $\mathcal{O}/\pi$ is not a square. The quadratic form

$$q_0 \coloneqq \langle 1,-u\rangle \otimes \langle 1,-\pi\rangle = \langle 1,-u,-\pi,u\pi\rangle$$

over k is the norm form for the quaternion algebra (u,π), hence, it is anisotropic over k. By [23, VI. Proposition 1.9], the quadratic form

$$q \coloneqq q_0 \perp t\langle 1\rangle = \langle 1,-u,-\pi,u\pi,t\rangle$$

is anisotropic over $k((t))$, hence, over $k(t)$. Let $X \subset \mathbb{P}^4_{k(t)}$ be the smooth projective quadric hypersurface over $k(t)$ defined by $q = 0$. By [20, Theorem 5.3], we have $CH^2(X)_{\mathrm{tors}} \simeq \mathbb{Z}/2$ (in the notation of [20, p. 120], $q = \langle\langle u,\pi\rangle\rangle \perp \langle t\rangle$). We let $\alpha \in CH^2(X)_{\mathrm{tors}}$ be the generator. The quadratic form q is a neighbor of the Pfister form $\langle\langle u,\pi,-t\rangle\rangle$, hence, X is a norm variety for the symbol $(u)\cup(\pi)\cup(-t) \in H^3(k(t),\mathbb{Z}/2)$.

We are going to prove that cl is not injective in codimension 2 by showing that $\mathrm{cl}(\alpha) = 0$ in $H^4(X,\mathbb{Z}_\ell(2))$. Consider the Hochschild-Serre spectral sequence in continuous ℓ-adic cohomology

$$E_2^{i,j} = H^i(k(t),H^j(X_{k(t)_s},\mathbb{Z}_\ell(2))) \Rightarrow H^{i+j}(X,\mathbb{Z}_\ell(2)). \tag{6.3}$$

It yields a filtration

$$\{0\} = F^5 \subset F^4 \subset \cdots \subset F^1 \subset F^0 = H^4(X,\mathbb{Z}_\ell(2)),$$

where F^i/F^{i+1} is a subquotient (respectively, a submodule) of $H^i(k,H^{4-i}(X_{k(t)_s},\mathbb{Z}_\ell(2)))$ for all $0 \leq i \leq 4$ (respectively, for $i = 0,1$). Let $\rho\colon M(X;\mathbb{Z}_{(\ell)}) \to M(X;\mathbb{Z}_{(\ell)})$ be the projector onto the direct summand $\mathcal{R}$, so that $\alpha \in \rho^* CH^2(X)$ (when $\ell = 2$ and X is the quadric described above, we could also take $\rho = \mathrm{id}$ in what follows). Since the Hochschild-Serre spectral sequence is natural

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with respect to correspondences, ρ and $1-\rho$ respect $F^{\cdot}$ and determine a direct sum decomposition $F^{\cdot} = \rho^*F^{\cdot} \oplus (1-\rho^*)F^{\cdot}$, where ρ^*F^i/ρ^*F^{i+1} is a subquotient (respectively, a submodule) of $H^i(k, \rho^*H^{4-i}(X_{k(t)_s}, \mathbb{Z}_\ell(2)))$ for all $0 \le i \le 4$ (respectively, for $i = 0, 1$).

The Rost motive $\mathcal{R}_{k(t)_s}$ is a finite direct sum of powers of the Tate motive. Thus, for all $j \ge 0$, we have $\rho^*H^{2j+1}(X_{k(t)_s}, \mathbb{Z}_\ell) = 0$ and $\rho^*H^{2j}(X_{k(t)_s}, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell(-j)^{\oplus r_{2j}}$ for some integers $r_{2j} \ge 0$. It follows that

$$H^1(k(t), \rho^*H^3(X_{k(t)_s}, \mathbb{Z}_\ell(2))) = H^3(k(t), \rho^*H^1(X_{k(t)_s}, \mathbb{Z}_\ell(2))) = 0.$$

Since $H^0(X_{k(t)_s}, \mathbb{Z}_\ell(2)) \simeq \mathbb{Z}_\ell(2)$, the direct summand $\rho^*H^0(X_{k(t)_s}, \mathbb{Z}_\ell(2))$ is either 0 or $\mathbb{Z}_\ell(2)$ (as $CH^0(\mathcal{R}) = \mathbb{Z}_{(\ell)}$ by [21, Theorem RM.10], we actually have $\rho^*H^0(X_{k(t)_s}, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell(2)$). Thus, by Lemma 6.2,

$$H^4(k(t), \rho^*H^0(X_{k(t)_s}, \mathbb{Z}_\ell(2))) = 0.$$

We deduce that $\rho^*F^1 = \rho^*F^2$ and $\rho^*F^3 = \rho^*F^4 = \rho^*F^5 = 0$. Therefore, $\rho^*F^1 = \rho^*F^2/\rho^*F^3$, that is, we have an exact sequence

$$0 \to \rho^*F^2/\rho^*F^3 \to \rho^*H^4(X, \mathbb{Z}_\ell(2)) \to \rho^*H^4(X_{k(t)_s}, \mathbb{Z}_\ell(2)). \tag{6.4}$$

We know that $\rho^*H^4(X_{k(t)_s}, \mathbb{Z}_\ell(2)) \simeq \mathbb{Z}_\ell^{\oplus r_4}$ is torsion free. By Lemma 6.1, the group

$$H^2(k(t), \rho^*H^2(X_{k(t)_s}, \mathbb{Z}_\ell(2))) \simeq T_\ell(\mathrm{Br}(k(t)))^{\oplus r_2}$$

is also torsion free. By [19, p. 262 and footnote 3] and [16] (see also the announcement in [17, Remark 6.15(b)]), all differentials in (6.3) are torsion, hence, ρ^*F^2/ρ^*F^3 is torsion free. Now (6.4) implies that $\rho^*H^4(X, \mathbb{Z}_\ell(2))$ is torsion free. Since $\mathrm{cl}(\alpha) \in \rho^*H^4(X, \mathbb{Z}_\ell(2))$ and $\ell\,\mathrm{cl}(\alpha) = 0$, we conclude that $\mathrm{cl}(\alpha) = 0$. □

Remark 6.4 (Colliot-Thélène). We sketch a more direct proof of the fact, used in the proof of Theorem 6.3, that the group $H^3(k(t), \mu_\ell^{\otimes 2})$ is nonzero. We first note that if a symbol $(a, b) \in \mathrm{Br}(k)[\ell] = H^2(k, \mu_\ell)$ is nonzero, then the residue of $(a, b, t) \in H^3(k(t), \mu_\ell^{\otimes 2})$ is nonzero, hence, $(a, b, t) \neq 0$. Therefore, it suffices to show that $\mathrm{Br}(k)[\ell] \neq 0$ for all global fields k.

One can show that $\mathrm{Br}(k)[2] \neq 0$ by constructing a conic $X^2 - aY^2 - bT^2 = 0$ over k without rational points. If ℓ is odd, one can construct a nonzero element of $\mathrm{Br}(k)[\ell]$ by taking a cyclic extension K/k of degree ℓ, a place v where K/k is inert (using the Chebotarev density theorem [50]), an element $c \in k_v^\times$ which is not a norm from $K_v^\times$ and approximating c by an element of $k^\times$.

Remark 6.5. One might wonder if there exist a number field k, a prime number ℓ, a nontrivial mod ℓ symbol s of degree $n+1$ and a norm variety X for s for which $\mathrm{cl}\colon CH^2(X)[\ell] \to H^4(X, \mathbb{Z}_\ell(2))$ is not injective. If ℓ is odd, this is impossible, as $\mathrm{cd}_\ell(k) = 2$. Suppose now that $\ell = 2$, so that X is the quadric hypersurface associated to a Pfister neighbor q of rank $2^n + 1$. By [20, Theorem 6.1], $CH^2(X)_{\mathrm{tors}}$ is either 0 or $\mathbb{Z}/2$. Let $\mathcal{R}$ be the Rost motive of X: it is a direct summand of $M(X; \mathbb{Z}_{(2)})$. By [21, Theorem RM.10], $CH^2(\mathcal{R})[2] \neq 0$ if and only if there exists $1 \le i \le n-1$, such that $2^n - 2^i = 2$, that is, if and only if $n = 2$. If this is the case, then $CH^2(X)_{\mathrm{tors}} \simeq \mathbb{Z}/2$ and $\dim(X) = 3$.

By definition of norm variety, for every field extension F/k, we have $X(F) \neq \emptyset$ (that is, q_F is isotropic) if and only if s_F is trivial. Therefore, by a theorem of Rost [35, Theorem 5] (see also [52, Lemma 2.1], or follow the construction of the isomorphisms in [21, Theorem RM.10]), the natural pullback map $CH^*(\mathcal{R}) \to CH^*(\mathcal{R}_F)$ is injective for all field extensions F/k, such that q_F is anisotropic.

Recall that every form of degree 5 over a p-adic field is isotropic (see [23, Chapter XI, Example 6.2(4)]). Thus, if q is isotropic at all real places of k, then q is isotropic at all places of k, and so it is isotropic by the Hasse-Minkowski principle [24, Chapter VI, Principle 3.1], hence, $CH^2(X)$ is torsion free by [20, Theorem 6.1]. Suppose now that there exists one real embedding $k \subset \mathbb{R}$, such that $q_{\mathbb{R}}$ is not isotropic. We have a commutative square

$$\begin{array}{ccc} CH^2(X)/2 & \xrightarrow{\sim} & CH^2(X_{\mathbb{R}})/2 \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^4(X, \mathbb{Z}/2) & \longrightarrow & H^4(X_{\mathbb{R}}, \mathbb{Z}/2), \end{array}$$

where the vertical maps are the cycle class maps in étale cohomology and the horizontal maps are induced by base change. The vertical map on the right is injective by [9, Proposition 2.5]. Since $CH^2(X)_{\text{tors}} \simeq \mathbb{Z}/2$, we deduce that $\mathrm{cl}\colon CH^2(X)_{\text{tors}} \to H^4(X, \mathbb{Z}/2)$ is injective, and so $\mathrm{cl}\colon CH^2(X)_{\text{tors}} \to H^4(X, \mathbb{Z}_2(2))$ is also injective.

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