J. Austral. Math. Soc. (Series A) 52 (1992), 119–129

ANNIHILATOR EQUIVALENCE OF TORSION-FREE ABELIAN GROUPS

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(Received 1 March 1990)

Communicated by R. Lidl

Abstract

We define an equivalence relation on the class of torsion-free abelian groups under which two groups are equivalent if every pure subgroup of one has a non-zero image in the other, and each has a non-zero image in every torsion-free factor of the other.

We study the closure properties of the equivalence classes, and the structural properties of the class of all equivalence classes. Finally we identify a class of groups which satisfy Krull-Schmidt and Jordan-Hölder properties with respect to the equivalence.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*) 20 K 15.

1. Introduction

Torsion-free abelian groups are notoriously difficult to classify, even up to quasi-isomorphism. In this paper we define an equivalence relation coarser than quasi-isomorphism on the class of all torsion-free abelian groups. We show that the equivalence classes form a structure for which the fundamental classification theorems of algebra are meaningful and frequently true. For example, Kaplansky's Test Problems have a positive solution for all equivalence classes, and the Krull-Schmidt and Jordan-Hölder Theorems hold for the equivalence classes of finite rank torsion-free abelian groups having no proper nilpotent endomorphisms.

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https://doi.org/10.1017/S1446788700032900 Published online by Cambridge University Press

Let $\mathscr{C}$ denote the class of all torsion-free abelian groups, hereafter simply called "groups". For any $A \in \mathscr{C}$, define the **left annihilator** of A by ${}^{\perp}A = \{Y \in \mathscr{C} : \mathrm{Hom}(Y, A) = 0\}$ and the **right annihilator** of A by $A^{\perp} = \{X \in \mathscr{C} : \mathrm{Hom}(A, X) = 0\}$.

Define relations $<_l$, $<_r$ and $\ll$ on $\mathscr{C}$ by $A <_l B$ if ${}^{\perp}A \supseteq {}^{\perp}B$, $A <_r B$ if $A^{\perp} \supseteq B^{\perp}$, and $\ll \, = \, <_l \cap <_r$. These relations are clearly quasi-orders, and so induce equivalence relations $\sim_l$, $\sim_r$ and $\approx$ on $\mathscr{C}$ by $A \sim_l B$ if $A <_l B <_l A$, $A \sim_r B$ if $A <_r B <_r A$, and $\approx \, = \, \sim_l \cap \sim_r$. That is, for A, $B \in \mathscr{C}$, $A \approx B$ if and only if ${}^{\perp}A = {}^{\perp}B$ and $A^{\perp} = B^{\perp}$.

The notion of annihilators dates back to [1]; they have been used repeatedly in the study of torsion theories. The relations $<_r$ and $\sim_r$ were defined, and the classification program initiated, in [4]. The dual relations $<_l$ and $\sim_l$ were defined in [5]. In the latter, Wickless studied especially the finite rank groups and showed that each such group A has a pure subgroup A_l, called the **left core** of A, minimal with respect to $A \sim_l A_l$; and a factor group $A_r = A/K$, called the **right core** of A, such that K is maximal with respect to $A \sim_r A/K$.

The left core is defined as follows. If $\mathscr{E}(A)$ has no non-zero nilpotent endomorphisms, let $A_l = A$. Otherwise there exists $0 = f^2 \neq f \in \mathscr{E}(A)$. In this case let $A_1 = \ker f$ and note that A_1 is a proper pure subgroup of A. Now consider $\mathscr{E}(A_1)$ and repeat this process to obtain, after a finite number of steps, a pure subgroup A_l of A such that $\mathscr{E}(A_l)$ has no non-zero nilpotent endomorphisms. It is not hard to check that $A \sim_l A_1 \sim_l \cdots \sim_l A_l$.

To define the right core, we define a sequence A^i of successive factor groups by $A^0 = A$ and for $i \geq 0$, $A^{i+1} = A^i/\langle \mathrm{Im}\, f\rangle_*$, where $0 = f^2 \neq f \in \mathscr{E}(A^i)$, and $\langle \mathrm{Im}\, f\rangle_*$ denotes the pure subgroup of A^i generated by the image of f. The group A_r is the first A^j such that $\mathscr{E}(A^j)$ has no non-zero nilpotent endomorphisms.

In [5] it is shown that the left and right cores are unique up to quasi-isomorphism and determine the $\sim_l$ and $\sim_r$ classes of A, in the sense that $A \sim_l B$ if and only if A_l is quasi-isomorphic to B_l, and similarly for the right. These striking properties led the present authors to believe that a special rôle would be played by the $\approx$ classes of finite rank groups whose left and right cores are quasi-isomorphic. We call these "groups of core type"; they are investigated in Section 4.

The notation used is mostly standard. If $\mathscr{B}$ is a family of groups, $\oplus\mathscr{B}$ means the direct sum of the members of $\mathscr{B}$. We assume familiarity with the usual 'quasi'-concepts for finite rank groups. We denote quasi-equality and quasi-isomorphism by $\doteq$ and $\cong$ and the quasi-endomorphism ring of G by $\mathbb{Q}\mathscr{E}(G)$.

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2. Basic properties of the $\approx$ relation

Let $\mathscr{A}$ be the class of $\approx$-equivalence classes $[A]$ for A in $\mathscr{C}$, and let $\prec$ be the order induced on $\mathscr{A}$ by $\ll$.

PROPOSITION 2.1. *$\langle \mathscr{A}, \prec \rangle$ is a complete upper semi-lattice, where for all families $\mathscr{B}$ from $\mathscr{C}$,* $\sup\{[A] : A \in \mathscr{B}\} = [\oplus\mathscr{B}]$.

PROOF. We have a poset by definition, and it is routine to check that for all C in $\mathscr{B}$, $C \ll \oplus\mathscr{B}$ and that if there exists $D \in \mathscr{C}$ such that for all $C \in \mathscr{B}$, $C \ll D$, then $\oplus\mathscr{B} \ll D$.

PROPOSITION 2.2. *Let $A \rightarrowtail B \twoheadrightarrow C$ be a short exact sequence in $\mathscr{C}$. Then $B \ll A \oplus C$, and $B \approx A \oplus C$ if and only if $A <_r B$ and $C <_l B$.*

PROOF. The definitions of $<_l$ and $<_r$ imply that $B \ll A \oplus C$ and that $A <_l B$ and $C <_r B$. Hence $A <_r B$ if and only if $A \ll B$, while $C <_l B$ if and only if $C \ll B$. The rest follows from Proposition 2.1.

PROPOSITION 2.3. *For all $A \in \mathscr{C}$, $[A]$ is closed under isomorphism, arbitrary direct sums, extensions and subgroups of bounded index.*

PROOF. $[A]$ is obviously closed under isomorphism. Let $\mathscr{B}$ be a subset of $[A]$. Clearly $A \ll \oplus\mathscr{B}$ and by Proposition 2.1, $\oplus\mathscr{B} \ll A$. By Proposition 2.2, $[A]$ is closed under extensions. Let $A \geq B \geq nA$ for some $n > 0$. It is easy to check that ${}^{\perp}A = {}^{\perp}B$ and $A^{\perp} = B^{\perp}$.

We shall see in Section 4 that in the finite rank case this result is best possible in the sense that if $\mathscr{E}(A)$ has no zero divisors, then the $\approx$-class $[A]$ is precisely the closure of $\{A\}$ under the four operations of Proposition 2.3. The next Proposition shows that the relation $\approx$ satisfies Kaplansky's Test Problems [3, Chapter 6].

PROPOSITION 2.4. (1) *Let A, B, C and $D \in \mathscr{C}$ with $A \approx B \oplus C$ and $B \approx A \oplus D$. Then $A \approx B$.*

(2) *Let A and $B \in \mathscr{C}$ with $A \oplus A \approx B \oplus B$. Then $A \approx B$.*

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PROOF. Both parts follow immediately from Proposition 2.1.

3. Intrinsic criteria for $A \approx B$

In order to apply the definition of $\approx$, it is essential to be able to recognize when the group structures of A and B imply $A \approx B$.

PROPOSITION 3.1.

(1) $A <_l B$ *if and only if for all non-zero pure subgroups* C *of* A, $Hom(C, B) \neq 0$.

(2) $A <_r B$ *if and only if for all torsion-free factor groups* D *of* A, $Hom(B, D) \neq 0$.

PROOF. The proof is a routine argument from the definitions of the relations $<_l$ and $<_r$.

DEFINITION 3.2. (1) For any A and B in $\mathscr{C}$, define a smooth decreasing chain of pure subgroups of A by $A^0[B] = A$; $A^1[B] = \cap\{\ker f : f \in \mathrm{Hom}(A, B)\}$; $A^{\nu+1}[B] = (A^\nu[B])^1[B]$; if ν is a limit ordinal, $A^\nu[B] = \cap_{\mu<\nu} A^\mu[B]$. At each ordinal ν, either $A^{\nu+1}[B] < A^\nu[B]$ or $A^{\nu+1}[B] = A^\nu[B]$. Hence the chain eventually stabilizes.

The **left** B**-length** of A is $\lambda = \lambda(A[B]) = \inf\{\nu : A^\nu[B] = A^{\nu+1}[B]\}$. The B**-radical** of A is $A[B] = A^\lambda[B]$.

(2) For any A and B in $\mathscr{C}$, define a smooth increasing chain of pure subgroups of A by $A^0(B) = 0$; $A^1(B) = \langle \mathrm{tr}_B(A)\rangle_*$ (i.e. the pure subgroup of A generated by all homomorphic images of B in A); $A^{\nu+1}(B)/A^\nu(B) = (A/A^\nu(B))^1(B)$; if ν is a limit ordinal, $A^\nu(B) = \cup_{\mu<\nu} A^\mu(B)$. As before, the chain stabilizes.

The **right** B**-length** of A is $\rho = \rho(A(B)) = \inf\{\nu : A^{\nu+1}(B) = A^\nu(B)\}$. The B**-socle** of A is $A(B) = A^\rho(B)$.

PROPOSITION 3.3.

(1) *For all* A *and* $B \in \mathscr{C}$, $A <_l B$ *if and only if* $A[B] = 0$.

(2) *For all* A *and* $B \in \mathscr{C}$, $A <_r B$ *if and only if* $A(B) = A$.

PROOF. To simplify notation, put $\lambda = \lambda(A[B])$ and $\rho = \rho(A(B))$.

(1) ($\Rightarrow$) For all ν, if $A^\nu[B] \neq 0$, then by Proposition 3.1(1), $A^{\nu+1}[B]$ is a proper subgroup of $A^\nu[B]$. Hence $A^\lambda[B] = 0$.

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($\Leftarrow$) Let $X \in \mathscr{C}$ with $\operatorname{Hom}(X, B) = 0$, and let $f \in \operatorname{Hom}(X, A)$. Then for all ordinals ν, $Xf \in A^{\nu+1}[B]$; otherwise there is a $g \in \operatorname{Hom}(A^{\nu}[B], B)$ with $0 \neq f \circ g \in \operatorname{Hom}(X, B)$. Hence $Xf \in A[B] = 0$, so $X \in {}^{\perp}A$.

(2) ($\Rightarrow$) Since $\operatorname{Hom}(B, A/A^{\rho}(B)) = 0$, $\operatorname{Hom}(A, A/A^{\rho}(B)) = 0$, so $A^{\rho}(B) = A$.

($\Leftarrow$) Let $Y \in \mathscr{C}$ with $\operatorname{Hom}(B, Y) = 0$. Clearly $\operatorname{Hom}(A^{0}(B), Y) = 0$; assume $\operatorname{Hom}(A^{\mu}(B), Y) = 0$ for all $\mu < \nu$. Let $f \in \operatorname{Hom}(A^{\nu}(B), Y)$. Then, for all $\mu < \nu$, f induces $\bar{f} \in \operatorname{Hom} A^{\nu}(B)/A^{\mu}(B), Y)$. If $a \in A^{\nu}(B)$ with $af \neq 0$, note that $a \in A^{\mu+1}(B)\setminus A^{\mu}(B)$ for some $\mu + 1 \leq \nu$. Then $0 \neq n(a + A^{\mu}(B)) = bg$ for some $n \in \mathbb{Z}$, $b \in B$ and $g \in \operatorname{Hom}(B, A^{\mu+1}(B)/A^{\mu}(B))$. It follows that $0 \neq g \circ \bar{f} \in \operatorname{Hom}(B, Y)$, a contradiction, so $\operatorname{Hom}(A^{\nu}(B), Y) = 0$.

Hence $\operatorname{Hom}(A^{\rho}(B), Y) = \operatorname{Hom}(A, Y) = 0$, so $Y \in A^{\perp}$.

4. Groups of core type

In this section we consider only groups of finite rank. Recall from the introduction the definitions of left and right core.

DEFINITION 4.1. A group A has **core type** if $A_l \doteq\cong A_r$.

A family of groups $\mathscr{B} = \{G_1, \dots G_r\}$ for some positive integer r is called a **core system** if each $\mathscr{E}(G_j)$ has no zero-divisors and each $\operatorname{Hom}(G_i, G_j) = 0$ if $i \neq j$.

Note that a finite rank group has a division ring as quasi-endomorphism ring if and only if its endomorphism ring has no zero-divisors, [2, Vol II, p. 149].

PROPOSITION 4.2. *A group A has core type if and only if $A \approx \oplus\mathscr{B}$ for some core system $\mathscr{B}$. In this case, the groups in $\mathscr{B}$ are unique up to quasi-isomorphism.*

PROOF. Suppose A is a group of core type and let $G = A_l$. Then $A \sim_l G$ and $A \sim_r \; \dot{\cong} \; A_r \; G$, so $A \sim_r G$ as well. Thus $A \approx G$. Since G is a finite rank group with no nilpotent endomorphisms, G has a quasi-decomposition $G \doteq \oplus_{i\in[1,r]} G_i$ where for each $i \in [1, r]$, $\mathbb{Q}\mathscr{E}(G_i)$ is a division ring and $\operatorname{Hom}(G_i, G_j) = 0$ for $i \neq j$. Take $\mathscr{B} = \{G_i : i \in [1, r]\}$.

Conversely, if A is a finite rank group with $A \approx G = \oplus\mathscr{B}$ for some core system $\mathscr{B}$, then $A_l \sim_l A \sim_l G$ and $A_r \sim_r A \sim_r G$. Now A_l, A_r and G have no non-zero nilpotent endomorphisms so by [5, Theorems 12 and 13, and Lemma 18], $A_l \doteq\cong G \doteq\cong A_r$, and hence A is of core type.

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For uniqueness, suppose $A \approx G' = \oplus\mathscr{B}'$ for some other core system $\mathscr{B}'$. Then, as above, $A_l \doteq G'$, so $G \doteq G'$. Since each group in a core system is strongly indecomposable, the result follows from the uniqueness of quasi-decompositions.

COROLLARY 4.3. *If A has core type and $A \approx B$, then B has core type.*

We now use these results to prove a Krull-Schmidt Theorem for groups of core type, up to equivalence. First we need notions of indecomposability and uniqueness with respect to $\approx$. A naive approach is not sufficient, since for any group A, $A \approx A \oplus A$.

DEFINITION 4.4. A group G is **$\approx$-indecomposable** if whenever A is a non-zero pure subgroup of G with $G \approx A \oplus G/A$, then $A = G$.

The $\approx$-indecomposables are easily characterized.

PROPOSITION 4.5. *G is $\approx$-indecomposable if and only if $\mathscr{E}(G)$ has no zero-divisors.*

PROOF. ($\Rightarrow$) Let $G \doteq \oplus_{i\in[1,m]} A_i$ be a quasi-decomposition of G into strongly indecomposable quasi-summands. Then by Proposition 2.3, $G \approx \oplus_{i\in[1,m]} A_i$ so $m = 1$ and G is strongly indecomposable. Thus if $\mathscr{E}(G)$ has zero divisors, it has non-zero nil radical [2, Vol. II, p. 149] and hence contains a non-zero f with $f^2 = 0$. Let $A = \langle Gf \rangle_*$, a proper pure subgroup of G. Then $A \leq \ker f$, so f induces a homomorphism of G/A into A whose image is full in A. Hence $G/A <_l A <_l G$ and $A <_r G/A <_r G$, so by Proposition 2.2, $G \approx A \oplus G/A$, a contradiction.

($\Leftarrow$) Let A be a non-zero pure subgroup of G with $G \approx A \oplus G/A$. By Proposition 2.2, $A <_r G$ and $G/A <_l G$, so $\mathrm{Hom}(G, A) \neq 0$ and if $G/A \neq 0$, $\mathrm{Hom}(G/A, G) \neq 0$. Let $0 \neq f \in \mathscr{E}(G)$ with $Gf \leq A$ and let $0 \neq g \in \mathscr{E}(G)$ with $A \leq \ker g$. Then $f \circ g = 0$, a contradiction, so $A = G$.

COROLLARY 4.6. *Let G be a group of core type. Then there exists a core system $\mathscr{B}$ such that $G \approx \oplus\mathscr{B}$. The groups in $\mathscr{B}$ are unique up to quasi-isomorphism and are $\approx$-indecomposable.*

PROOF. This follows from Propositions 4.2 and 4.5 and the definitions.

Corollary 4.6 is analogous to the Krull-Schmidt Theorem for finite rank groups up to quasi-isomorphism. We shall now prove an analogue of the Jordan-Hölder Theorem for finite groups.

DEFINITION 4.7. Let $\mathscr{B}$ be a core system and let $G \doteq \oplus\mathscr{B}$. Let H be a group. A **left G-filtration for H** is a descending chain of pure subgroups

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$H = H_0 > H_1 > \cdots > H_n = 0$ such that for $0 \leq i < n$, H_i/H_{i+1} is quasi-isomorphic to a subgroup of some $G_{j(i)} \in \mathscr{B}$. We say H_i/H_{i+1} is **quasi-embedded** in $G_{j(i)}$.

A **G-filtration for** H is a left G-filtration such that each H_i/H_{i+1} is quasi-isomorphic to some $G_{j(i)}$.

Let $\mathscr{B}$ be a core system and let $G \doteq \oplus\mathscr{B}$. In [5] it was shown that for any group H, $H <_l G$ if and only if H has a left G-filtration. The following lemma establishes an analogous result for the relation $\ll$ and G-filtrations.

LEMMA 4.8. *Let $G \doteq \oplus\mathscr{B}$ for some core system $\mathscr{B}$. Let $0 \neq H$ be a group with $H \ll G$. Then H has a G-filtration.*

PROOF. Let $\mathscr{B} = \{G_1 \ldots G_m\}$. Since $H <_l G$, H has a left G-filtration $H = H_0 > H_1 > \cdots > H_n = 0$ with quasi-embeddings $\alpha_i : H_i/H_{i+1} \longrightarrow G_{j(i)}$ for $0 \leq i < n$ where $G_{j(i)} \in \mathscr{B}$.

The proof of the lemma is by induction on the rank r of H. If $r = 1$ then $n = 1$ and $H = H_0$ is quasi-embedded in some $G_j \in \mathscr{B}$. Moreover since $H <_r G$, $\mathrm{Hom}(G, H) \neq 0$ so some $\mathrm{Hom}(G_k, H) \neq 0$. Since $\mathrm{Hom}(G_k, G_j) = 0$ for $k \neq j$, it follows that $k = j$ and $\mathrm{Hom}(G_j, H) \neq 0$.

Let $0 \neq f \in \mathrm{Hom}(G_j, H)$ and let θ be a quasi-embedding of H into G_j. Then $0 \neq f\theta \in \mathbb{Q}\mathscr{E}(G_j)$, a division algebra. Thus $f\theta$ is a quasi-invertible endomorphism of G_j whose image is quasi-equal to G_j. Hence rank $G_j = 1$ and $H \cong G_j$, so our result holds for $r = 1$.

Now assume the result holds for all groups of rank smaller than r. Since $\mathrm{Hom}(G_1, G) \neq 0$ and $G <_l H$, $\mathrm{Hom}(G_1, H) \neq 0$. Let $0 \neq \beta : G_1 \to H$. Choose i so that $G_1\beta \leq H_i$, but $G_1\beta \not\leq H_{i+1}$. Let δ be the composite map $G_1 \xrightarrow{\beta} H_i \longrightarrow H_i/H_{i+1} \xrightarrow{\alpha_i} G_{j(i)}$.

Since $\delta \neq 0$, $j(i) = 1$ and δ is invertible in $\mathbb{Q}\mathscr{E}(G_1)$. Thus β is monic and $H_i \doteq G_1\beta \oplus H_{i+1}$. Note that $G_1\beta \cong G_1$ is quasi-pure in H since H_i is pure in H.

Let K be the purification of $G_1\beta$ in H. For $t \leq i$ let $\bar{H}_t = H_t/K$ and for $t \geq i+1$ let $\bar{H}_t = (H_t \oplus K)/K$. Then it is easy to check that $\bar{H} = \bar{H}_0 > \bar{H}_1 > \cdots > \bar{H}_i = \bar{H}_{i+1} > \cdots > \bar{H}_n = 0$ is a left G-filtration of $\bar{H} = H/K$. It follows that $\bar{H} <_l G$.

Clearly $\bar{H} <_r H <_r G$ and rank $\bar{H} <$ rank H so inductively there exists a filtration of $\bar{H}$ with each factor quasi-isomorphic to some $G_{j(i)}$. To complete the proof we recall that $K \doteq G_1$ and construct the obvious G-filtration for H.

THEOREM 4.9. *Let $G = \oplus\mathscr{B}$ for some core system $\mathscr{B}$. Let H be a group.*

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Then $H \approx G$ if and only if

(1) *there exists a quasi-embedding $\alpha : G \longrightarrow H$;*
(2) *there exists a quasi-epimorphism $\pi : H \longrightarrow G$;*
(3) *there exists a G-filtration for H.*

PROOF. ($\Leftarrow$) Condition (1) implies $G <_l H$, Condition (2) implies $G <_r H$ and Condition (3) implies that $H \ll G$.

($\Rightarrow$) Since $H \sim_l G$, $H_l \doteq G_l = G$, so there is a quasi-embedding of G in H. Since $H \sim_r G$, $H_r \doteq G_r = G$, so there is a quasi-epimorphism of H onto G.

The third condition follows from Lemma 4.8.

COROLLARY 4.10. *Let G be a group for which $\mathscr{E}(G)$ has no zero-divisors. Then $H \approx G$ if and only if H has a filtration $H = H_0 > H_1 > \cdots > H_n = 0$ with each $H_i/H_{i+1} \doteq G$.*

The necessity for the assumption in Corollary 4.10 that $\mathscr{E}(G)$ have no zero-divisors is illustrated by the following example.

EXAMPLE. Let A be a rank one group of nil type, and let G_1 and G_2 be non-quasi-isomorphic non-splitting extensions of A by A. Then $G_1 \approx G_2 \approx A$ but G_1 has no G_2-filtration.

COROLLARY 4.11. *Let G be a group for which $\mathscr{E}(G)$ has no zero-divisors. Then the set of finite rank groups equivalent to G is the closure of $\{G\}$ under the operations of quasi-isomorphism and extension.*

For the next theorem, recall Definition 3.2 for $H^j[G]$ and $H^j(G)$.

THEOREM 4.12. *Let G and H be groups such that $\mathscr{E}(G)$ has no zero-divisors. Then $H \approx G$ if and only if for all $0 \leq j < \lambda(G[H])$, $H^j[G]/H^{j+1}[G] \doteq \oplus_{t(j)} G$ and for all $0 \leq j < \rho(G(H))$, $H^{j+1}(G)/H^j(G) \doteq \oplus_{s(j)} G$ for some positive integers $t(j)$ and $s(j)$.*

PROOF. The sufficiency of the condition follows immediately from Corollary 4.10.

For necessity, suppose $\mathscr{E}(G)$ has no zero-divisors and H is a group with $H \approx G$. Then by Corollary 4.10, H has a filtration $H = H_0 > H_1 > \cdots > H_n = 0$ with each $H_i/H_{i+1} \doteq G$. The proof is by induction on n, and the result is clearly true if $n = 1$. Let $n > 1$ and assume the theorem holds for all groups having a shorter such filtration. To simplify notation, denote

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$U^j[G]$ by U^j for any group U, and let $\oplus G$ denote an arbitrary finite direct sum of copies of G.

Let λ be the left G-length of H so $H^\lambda = 0$, and let $0 \le i < \lambda$ be such that $H_{n-1} \le H^i$ but $H_{n-1} \not\le H^{i+1}$. Regard H^i/H^{i+1} as a subgroup of $\oplus G$. Since

$$0 \ne W = (H_{n-1} + H^{i+1})/H_{i+1} \le H^i/H^{i+1} \le \oplus G,$$

choose a projection π onto some copy of G such that $W\pi \ne 0$. Now $H_{n-1} \doteq G$ and the composite map $H_{n-1} \hookrightarrow H^i \to H^i/H^{i+1} \xrightarrow{\pi} G$ is non-zero, so, since $\mathbb{Q}\mathscr{E}(G)$ is a division algebra, $H^i \doteq H_{n-1} \oplus X$ for some group X containing H^{i+1}.

For any group V with $H_{n-1} \le V \le H$, let $\bar{V}$ denote V/H_{n-1}. Note that $\bar{H} \approx G$ by Theorem 4.9, so by the induction assumption, $\bar{H}^j/\bar{H}^{j+1} \doteq \oplus G$ for all j. It now follows by induction on j that for $0 \le j \le i$,

$$(\bar{H})^j = (H/H_{n-1})^j[G] \cong (H^j[G])/H_{n-1} = \overline{(H^j)}.$$

We prove that $H^j/H^{j+1} \doteq \oplus G$ by considering three cases. First, let $0 \le j < i$. Then

$$H^j/H^{j+1} \cong \overline{H^j}/\overline{H^{j+1}} \cong \bar{H}^j/\bar{H}^{j+1} \doteq \oplus G.$$

The first isomorphism follows from the second isomorphism theorem, the second from the paragraph above, and the quasi-isomorphism follows from the induction assumption. Thus our theorem holds for all j with $0 \le j < i$.

Next, let $i < j < \lambda$. Then $j = i + s$ for some $s > 0$. For all $s > 0$,

$$H^{i+s} = (H^i)^s \doteq (H_{n-1} \oplus X)^s = X^s,$$

since $H_{n-1} \doteq G$. Moreover, for $s \ge 0$,

$$\bar{H}^{i+s} = (\bar{H}^i)^s = (\overline{H^i})^s \doteq X^s,$$

since $\overline{H^i} \doteq X$. Thus for $s > 0$,

$$\oplus G \doteq \bar{H}^{i+s}/\bar{H}^{i+s+1} \doteq X^s/X^{s+1} \doteq H^{i+s}/H^{i+s+1}.$$

This proves the theorem for j such that $i < j < \lambda$.

Finally,

$$\begin{aligned} H^i/H^{i+1} &\doteq (H_{n-1} \oplus X)/(H_{n-1} \oplus X)^1 \cong H_{n-1} \oplus X/X^1 \\ &\doteq G \oplus (\bar{H}^i/\bar{H}^{i+1}) \doteq \oplus G. \end{aligned}$$

Thus, $H^j[G]/H^{j+1}[G] \doteq \oplus G$ for $0 \le j < \lambda$. A symmetric argument establishes the same claim for $H^{j+1}(G)/H^j(G)$ and the proof is complete.

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COROLLARY 4.13. *If $\mathscr{E}(G)$ has no zero-divisors, and $H \approx G$, then $\lambda(H[G]) = \rho(H(G))$.*

PROOF. Let $\lambda = \lambda(H[G])$ and $\rho = \rho(H(G))$. Then

$$0 = H^{\lambda}[G] < H^{\lambda-1}[G] < \cdots < H^{0}[G] = H,$$

and

$$0 = H^{0}(G) < H^{1}(G) < \cdots < H^{\rho}(G) = H,$$

where each factor is quasi-isomorphic to a direct sum of copies of G. Since $H/H^{\rho-1}(G) \doteq \oplus G$, it follows that $H^{1}[G] \leq H^{\rho-1}(G)$. Suppose we have shown that $H^{k}[G] \leq H^{\rho-k}(G)$. Then

$$\begin{aligned} H^{k}[G]/\left(H^{k}[G] \cap H^{\rho-k-1}(G)\right) &\cong \left(H^{k}[G] + H^{\rho-k-1}(G)\right)/H^{\rho-k-1}(G) \\ &\leq H^{\rho-k}(G)/H^{\rho-k-1}(G). \end{aligned}$$

By Theorem 4.12, the last factor is quasi-equal to a direct sum of copies of G, so $H^{k+1}[G] \leq H^{\rho-k-1}(G)$. By induction, $H^{\rho}[G] \leq H^{0}(G) = 0$, so $\lambda \leq \rho$. A symmetric argument shows that $\rho \leq \lambda$.

We close by showing that a Jordan-Hölder type theorem holds for the filtrations obtained in Lemma 4.8 and Theorem 4.9. The result may be of some independent interest.

THEOREM 4.14. *Let $\mathscr{B}$ be a core system, let H be a group, and let $H = H_0 > H_1 > \cdots > H_{n+1} = 0$ and $H = H'_0 > H'_1 > \cdots > H'_{m+1} = 0$ be filtrations of H such that $H_i/H_{i+1} \doteq G_{j(i)}$ and $H'_i/H'_{i+1} \doteq G_{k(i)}$, where $i \mapsto G_{j(i)}$ and $i \mapsto G_{k(i)}$ are functions from $[0, n]$ and $[0, m]$ into $\mathscr{B}$. Then $n = m$ and there is a permutation σ of $[0, n]$ such that $H_i/H_{i+1} \doteq H'_{\sigma(i)}/H'_{\sigma(i)+1}$ for all $i \in [0, n]$.*

PROOF. The proof is by induction on rank H, and the result is clear if rank $H = 1$. Let rank $H = r > 1$ and suppose the result holds for all groups of rank $< r$. Choose $i \geq 0$ minimal such that $H'_m \leq H_i$ but $H'_m \not\leq H_{i+1}$. Arguing as in the proof of Lemma 4.8 we find $H_i \doteq H'_m \oplus H_{i+1}$. Then

$$H/H'_m > H'_1/H'_m > \cdots > H'_{m-1}/H'_m$$

and

$$H/H'_m > H_1/H'_m > \cdots > H_i/H'_m \doteq H_{i+1} > H_{i+2} > \cdots > H_n > 0$$

are filtrations of the group H/H'_m with factors in $\mathscr{B}$. By induction the factor groups in these new filtrations are quasi-isomorphic after rearrangement. But

https://doi.org/10.1017/S1446788700032900 Published online by Cambridge University Press

these factor groups coincide with those of the original filtrations except for the groups H'_m and H_i/H_{i+1}. Since $H_i/H_{i+1} \dot{\cong} H'_m$, the proof is complete.

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