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AVERAGES OF SHIFTED CONVOLUTIONS OF $d_3(n)$

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Abstract We investigate the first and second moments of shifted convolutions of the generalized divisor function $d_3(n)$.

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1. Introduction

For any positive integer k let $d_k(n)$ denote the generalized divisor function, defined to be the Dirichlet coefficients of $\zeta(s)^k$ in the half-plane $\mathrm{Re}(s) > 1$. The study of shifted convolution sums

$$D_k(N,h) := \sum_{N<n\leqslant 2N} d_k(n)d_k(n+h)$$

is of central importance in the analytic theory of numbers. The case $k = 1$ is trivial and for $k = 2$ we have known since the work of Ingham [**6**] that

$$D_2(N,h) \sim \frac{6}{\pi^2}\sigma_{-1}(h)N\log^2 N$$

as $N \to \infty$, for given $h \in \mathbb{N}$, where $\sigma_{-1}(h) := \sum_{j|h} j^{-1}$. Several authors have since revisited this problem, achieving asymptotic formulae with h in an increasingly large range compared with N. A detailed analysis of $D_2(N,h)$ via spectral methods can be

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found in work of Motohashi [**15**]. The best results in the literature are due to Duke *et al*. [**3**] and to Meurman [**13**].

In general it is expected that $D_k(N,h)$ should be asymptotic to $c_{k,h}N\log^{2k-2}N$, for a suitable constant $c_{k,h}>0$, uniformly for h in some range. However, such a description has not yet been forthcoming for any $k\geqslant 3$, even when h is fixed. One motivation for studying the sums $D_k(N,h)$ is the deep connection that they enjoy with the asymptotic behaviour of moments

$$I_k(T) := \int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t$$

as $T\to\infty$. It is commonly believed that $I_k(T)\sim c_kT(\log T)^{k^2}$ as $T\to\infty$ for a suitable constant $c_k>0$. Keating and Snaith [**11**] have produced a conjectural interpretation of c_k using random matrix theory for Gaussian unitary ensembles. Just as for the sums $D_k(N,h)$, we have only succeeded in producing an asymptotic formula for $I_k(T)$ when $k=1$ [**4**] or $k=2$ [**5**]. The relationship between moments of the Riemann zeta function and the shifted convolution sums $D_k(N,h)$ has been explored extensively by Ivić [**8**, **9**] and, more recently, by Conrey and Gonek [**2**].

Focusing on the case $k=3$, in which setting we write $D(N,h)=D_3(N,h)$, our aim in this paper is to lend some theoretical support in favour of its expected asymptotic behaviour. If $\varphi(n)$ denotes the Euler totient function, then we set

$$H(s,q) := \sum_{d\mid q}\frac{\mu(d)}{\varphi(d)}d^sG_{q/d,d}(s),$$

with

$$G_{k,d}(s) := \sum_{e\mid d}\frac{\mu(e)}{e^s}g(s,ek) \tag{1.1}$$

and

$$g(s,q) := \prod_{p\mid q}\biggl((1-p^{-s})^3\sum_{j=0}^{\infty}\frac{d_3(p^{j+v_p(q)})}{p^{js}}\biggr).$$

Henceforth, $v_p(q)$ denotes the p-adic valuation of q. Next we define

$$\begin{aligned} P(x,q) &:= \frac{1}{2\pi\mathrm{i}}\int_{|s|=1/8}\zeta^3(s+1)H(s+1,q)\biggl(\frac{x}{q}\biggr)^s\mathrm{d}s \\ &= \mathrm{Res}_{s=0}\,\zeta^3(s+1)H(s+1,q)\biggl(\frac{x}{q}\biggr)^s \end{aligned} \tag{1.2}$$

by the Residue Theorem. Let $c_q(h)=\sum_{d\mid h,q}d\mu(q/d)$ be the Ramanujan sum and let $\varepsilon>0$. Then the work of Conrey and Gonek [**2**, Equation (30) and Conjecture 3] predicts that

$$D(N,h) = \int_N^{2N}\mathfrak{S}(x,h)\,\mathrm{d}x + O(N^{1/2+\varepsilon}), \tag{1.3}$$

uniformly for $1 \leqslant h \leqslant N^{1/2}$, where

$$\mathfrak{S}(x,h) := \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P(x,q)^2. \tag{1.4}$$

Let

$$\Delta(N,h) := D(N,h) - \int_N^{2N} \mathfrak{S}(x,h)\,\mathrm{d}x.$$

We shall lend support to (1.3) by considering both the first and second moments of $\Delta(N,h)$ as h varies over some range that is small compared with N. Beginning with the former, we shall establish the following result.

Theorem 1.1. *Assume that $1 \leqslant H \leqslant N$. Then*

$$\sum_{h \leqslant H} \Delta(N,h) \ll (H^2 + H^{1/2}N^{13/12})N^{\varepsilon}.$$

The exponents appearing in this estimate can be improved slightly for certain ranges of H. We shall not pursue this here, however. For N in the range $N^{1/6+\varepsilon} \leqslant H \leqslant N^{1-\varepsilon}$, Theorem 1.1 gives an asymptotic formula for the average

$$G(N,H) := \sum_{h \leqslant H} D(N,h). \tag{1.5}$$

It is interesting to relate Theorem 1.1 to the work of Ivić [**8**, Lemma 6], who deduces the upper bound

$$I_3(T) \ll T^{1+\varepsilon} + T^{(\alpha+3\beta-1)/2+\varepsilon}$$

for the sixth moment of the Riemann zeta function on the critical line, where $\alpha, \beta \in [0,1]$ are constants such that $\alpha + \beta \geqslant 1$ and an asymptotic formula of the shape

$$\sum_{h \leqslant H} \Delta(N,h) \ll H^{\alpha} N^{\beta+\varepsilon}$$

is valid for $1 \leqslant H \leqslant N^{1/3}$. Theorem 1.1 affords the choices $\alpha = \frac{1}{2}$ and $\beta = \frac{13}{12}$, which yield $I_3(T) \ll T^{11/8+\varepsilon}$. Unfortunately, this does not give any improvement over the well-known bound for $I_3(T)$ with exponent $\frac{5}{4} + \varepsilon$.

Turning to second moments, we shall establish the following result.

Theorem 1.2. *Assume that $N^{1/3+\varepsilon} \leqslant H \leqslant N^{1-\varepsilon}$. Then there exists $\delta > 0$ such that*

$$\sum_{h \leqslant H} |\Delta(N,h)|^2 \ll HN^{2-\delta}.$$

It follows from Theorem 1.2 that the expected asymptotic formula

$$D(N,h) \sim \int_N^{2N} \mathfrak{S}(x,h)\,\mathrm{d}x$$

holds for almost all $h \leqslant H$ if $N^{1/3+\varepsilon} \leqslant H \leqslant N^{1-\varepsilon}$. Our proof of Theorem 1.2 is based on Mikawa's investigation [**14**] of twin primes. Here the Hardy–Littlewood circle method is adapted to study the second moment of the analogous shifted convolution sum in which $d_3(n)$ is replaced by the von Mangoldt function $\Lambda(n)$. Our proof of Theorem 1.1 is simpler, being based on Perron's Formula and a bound for the sixth moment of the Riemann zeta function.

Notation

Our work will involve small positive parameters ε and δ, $\delta_1, \delta_2, \dots$. The value of ε will be allowed to vary from line to line, and δ, $\delta_1, \delta_2, \dots$ may depend on ε. All of the implied constants in our work are permitted to depend at most on these parameters.

2. Estimation of $G(N, H)$

The following two sections deal with the proof of Theorem 1.1. To this end, we evaluate separately the averages $G(N, H)$, defined in (1.5), and

$$F(N, H) := \sum_{h \leqslant H} \int_N^{2N} \mathfrak{S}(x, h) \, \mathrm{d}x. \tag{2.1}$$

We begin with the more complicated evaluation of $G(N, H)$. Changing the order of summation, we get

$$G(N, H) = \sum_{N < n \leqslant 2N} d_3(n) \sum_{h \leqslant H} d_3(n + h). \tag{2.2}$$

Using Perron's Formula, the inner sum in (2.2) can be expressed in the form

$$\sum_{h \leqslant H} d_3(n + h) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \zeta^3(s)((n + H)^s - n^s) \frac{\mathrm{d}s}{s} + O\biggl(\frac{N^{1+\varepsilon}}{T}\biggr), \tag{2.3}$$

where $c = 1 + (\log N)^{-1}$ and $2 \leqslant T \leqslant N$. Shifting the line of integration and using the Residue Theorem, we see that the integral is

$$\mathrm{Res}_{s=1} \, \zeta^3(s) \frac{(n + H)^s - n^s}{s} + \frac{1}{2\pi \mathrm{i}} \biggl(\int_{\mathcal{P}_1} + \int_{\mathcal{P}_2} + \int_{\sigma-\mathrm{i}T}^{\sigma+\mathrm{i}T} \biggr) \zeta^3(s)((n + H)^s - n^s) \frac{\mathrm{d}s}{s}, \tag{2.4}$$

where $\frac{1}{2} < \sigma < 1$ is a parameter to be fixed later, $\mathcal{P}_1$ is the line segment connecting $c - \mathrm{i}T$ and $\sigma - \mathrm{i}T$ and $\mathcal{P}_2$ is the line segment connecting $\sigma + \mathrm{i}T$ and $c + \mathrm{i}T$.

For $\frac{1}{2} \leqslant \alpha \leqslant 1$ and $|t| \geqslant 1$, Weyl's subconvexity bound is $\zeta(\alpha + \mathrm{i}t) \ll |t|^{(1-\alpha)/3+\varepsilon}$. Moreover, $\zeta(\alpha \pm \mathrm{i}T) \ll \log T$ uniformly in $1 \leqslant \alpha \leqslant c$. Hence, for $i = 1, 2$, the integrals over $\mathcal{P}_i$ in (2.4) are bounded by

$$\int_{\mathcal{P}_i} \zeta^3(s)((n + H)^s - n^s) \frac{\mathrm{d}s}{s} \ll \frac{N^\varepsilon}{T} \int_\sigma^1 T^{1-\alpha} N^\alpha \, \mathrm{d}\alpha \ll \frac{N^{1+\varepsilon}}{T}, \tag{2.5}$$

where we take into account that $2 \leqslant T \leqslant N$ and $N < n \leqslant 2N$.

Combining this with (2.2)–(2.4), we therefore obtain

$$G(N,H) = M(N,H) + E(N,H) + O\biggl(\frac{N^{2+\varepsilon}}{T}\biggr), \tag{2.6}$$

where

$$M(N,H) := \sum_{N<n\leqslant 2N} d_3(n) \operatorname{Res}_{s=1} \zeta^3(s) \frac{(n+H)^s - n^s}{s}$$

and

$$E(N,H) := \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}T}^{\sigma+\mathrm{i}T} \zeta^3(s) \sum_{N<n\leqslant 2N} d_3(n)((n+H)^s - n^s) \frac{\mathrm{d}s}{s}. \tag{2.7}$$

We proceed by writing

$$M(N,H) = \sum_{N<n\leqslant 2N} d_3(n) g(n),$$

with

$$g(x) := \operatorname{Res}_{s=1} \zeta^3(s) \frac{(x+H)^s - x^s}{s}.$$

We note that $g(x) \ll Hx^{\varepsilon}$ and $g'(x) \ll Hx^{\varepsilon-1}$. Partial summation yields

$$M(N,H) = g(2N) \sum_{N<n\leqslant 2N} d_3(n) - \int_N^{2N} g'(t) \sum_{N<n\leqslant t} d_3(n)\,\mathrm{d}t.$$

The classical work of Voronoi [**16**, Theorem 12.2] yields

$$\sum_{n\leqslant t} d_3(n) = \operatorname{Res}_{s=1} \zeta^3(s) \frac{t^s}{s} + O(t^{1/2+\varepsilon}).$$

From these results we deduce that

$$\begin{aligned} M(N,H) = g(2N)\biggl(&\operatorname{Res}_{s=1} \zeta^3(s) \frac{(2N)^s - N^s}{s}\biggr) \\ &- \int_N^{2N} g'(t) \biggl(\operatorname{Res}_{s=1} \zeta^3(s) \frac{t^s - N^s}{s}\biggr)\,\mathrm{d}t + O(HN^{1/2+\varepsilon}). \end{aligned}$$

Integration by parts now reveals that

$$M(N,H) = \int_N^{2N} g(t) \biggl(\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Res}_{s=1} \zeta^3(s) \frac{t^s - N^s}{s}\biggr)\,\mathrm{d}t + O(HN^{1/2+\varepsilon}). \tag{2.8}$$

Employing the Taylor series expansion

$$\frac{(t+H)^s - t^s}{s} = Ht^{s-1} + \frac{H^2}{2}(s-1)t^{s-2} + \cdots$$

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and the Laurent series expansion for $\zeta^3(s)$ about $s=1$, we obtain

$$\operatorname{Res}_{s=1}\zeta^3(s)\frac{(t+H)^s-t^s}{s}=H\operatorname{Res}_{s=1}\zeta^3(s)t^{s-1}+O\biggl(\frac{H^2\log t}{t}\biggr),$$

where we keep in mind that $H\leqslant N$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Res}_{s=1}\zeta^3(s)\frac{t^s-N^s}{s}=\operatorname{Res}_{s=1}\zeta^3(s)t^{s-1}\ll t^{\varepsilon}.$$

Putting these facts together in (2.8), we obtain

$$M(N,H)=H\int_N^{2N}(\operatorname{Res}_{s=1}\zeta^3(s)t^{s-1})^2\,\mathrm{d}t+O(H^2N^{\varepsilon}+HN^{1/2+\varepsilon}). \tag{2.9}$$

Our next task is to estimate $E(N,H)$ in (2.7). Applying partial summation to the sum over n, we see that

$$\begin{aligned}\sum_{N<n\leqslant 2N}d_3(n)((n+H)^s-n^s)&=\sum_{N<n\leqslant 2N}\biggl(\biggl(1+\frac{H}{n}\biggr)^s-1\biggr)d_3(n)n^s\\&=\biggl(\biggl(1+\frac{H}{2N}\biggr)^s-1\biggr)\sum_{N<n\leqslant 2N}d_3(n)n^s\\&\quad+sH\int_N^{2N}\biggl(1+\frac{H}{x}\biggr)^{s-1}\biggl(\sum_{N<n\leqslant x}d_3(n)n^s\biggr)\frac{\mathrm{d}x}{x^2}.\end{aligned}$$

It follows that

$$E(N,H)=E_1(N,H)+E_2(N,H), \tag{2.10}$$

where

$$\begin{aligned}E_1(N,H)&:=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}T}^{\sigma+\mathrm{i}T}\zeta^3(s)\biggl(\biggl(1+\frac{H}{2N}\biggr)^s-1\biggr)\biggl(\sum_{N<n\leqslant 2N}d_3(n)n^s\biggr)\frac{\mathrm{d}s}{s}\\&=\frac{1}{4\pi\mathrm{i}N}\int_0^H\int_{\sigma-\mathrm{i}T}^{\sigma+\mathrm{i}T}\zeta^3(s)\biggl(1+\frac{\theta}{2N}\biggr)^{s-1}\biggl(\sum_{N<n\leqslant 2N}d_3(n)n^s\biggr)\,\mathrm{d}s\,\mathrm{d}\theta\end{aligned}$$

and

$$E_2(N,H):=\frac{H}{2\pi\mathrm{i}}\int_N^{2N}\int_{\sigma-\mathrm{i}T}^{\sigma+\mathrm{i}T}\zeta^3(s)\biggl(1+\frac{H}{x}\biggr)^{s-1}\biggl(\sum_{N<n\leqslant x}d_3(n)n^s\biggr)\frac{\mathrm{d}s\,\mathrm{d}x}{x^2}.$$

For $i=1,2$ we may deduce that

$$E_i(N,H)\ll\frac{H}{N}\sup_{N<x\leqslant 2N}\int_{-T}^{T}|\zeta(\sigma+\mathrm{i}t)|^3\biggl|\sum_{N<n\leqslant x}d_3(n)n^{\sigma+\mathrm{i}t}\biggr|\,\mathrm{d}t. \tag{2.11}$$

Next, we transform the inner sum over n in (2.11) with a further application of Perron's Formula, obtaining

$$\sum_{N<n\leqslant x} d_3(n)n^s = \frac{1}{2\pi \mathrm{i}}\int_{c_1-2\mathrm{i}T}^{c_1+2\mathrm{i}T} \zeta^3(s_1-s)(x^{s_1}-N^{s_1})\frac{\mathrm{d}s_1}{s_1} + O\bigg(\frac{N^{1+\sigma+\varepsilon}}{T}\bigg), \tag{2.12}$$

where $s=\sigma+\mathrm{i}t$ and $c_1 = 1+\sigma+(\log N)^{-1}$. We shall shift the line of integration and use the Residue Theorem, noting that we cross the pole of the zeta function at 1 since $|t|\leqslant T$. In this way we see that the integral is

$$\mathrm{Res}_{s_1=1+s}\,\zeta^3(s_1-s)\frac{x^{s_1}-N^{s_1}}{s_1} + \frac{1}{2\pi\mathrm{i}}\bigg(\int_{\mathcal{P}_3} + \int_{\mathcal{P}_4} + \int_{2\sigma-2\mathrm{i}T}^{2\sigma+2\mathrm{i}T}\bigg)\zeta^3(s_1-s)(x^{s_1}-N^{s_1})\frac{\mathrm{d}s_1}{s_1}, \tag{2.13}$$

where $\mathcal{P}_3$ is the line segment connecting $c_1-2\mathrm{i}T$ to $2\sigma-2\mathrm{i}T$ and $\mathcal{P}_4$ is the line segment connecting $2\sigma+2\mathrm{i}T$ to $c_1+2\mathrm{i}T$.

In the same way as (2.5), we see that

$$\int_{\mathcal{P}_i}\zeta^3(s_1-s)(x^{s_1}-N^{s_1})\frac{\mathrm{d}s_1}{s_1} \ll \frac{N^{1+\sigma+\varepsilon}}{T}$$

for $i=3,4$, where we take into account that $|t|\leqslant T$.

From (2.11) and (2.12), we deduce that

$$E_i(N,H) \ll A(N,H)+B(N,H), \tag{2.14}$$

for $i=1,2$, where

$$A(N,H) := HN^{2\sigma-1}\int_{-T}^{T}\int_{-2T}^{2T}|\zeta(\sigma+\mathrm{i}t)|^3|\zeta(\sigma+\mathrm{i}(t_1-t))|^3\frac{\mathrm{d}t_1}{1+|t_1|}\,\mathrm{d}t$$

and

$$B(N,H) := HN^{\sigma+\varepsilon}\int_{-T}^{T}|\zeta(\sigma+\mathrm{i}t)|^3\frac{\mathrm{d}t}{1+|t|}.$$

Here $A(N,H)$ bounds the contribution of the third integral on the right-hand side of (2.13), and $B(N,H)$ bounds the contributions from the remaining terms.

Since $\sigma>\frac{1}{2}$, we have

$$B(N,H)\ll HN^{\sigma+\varepsilon}$$

by the familiar bound for the third moment of the Riemann zeta function. Next, using the Cauchy–Schwarz inequality, we obtain

$$A(N,H) \ll HN^{2\sigma-1}\int_{-2T}^{2T}\bigg(\int_{-T}^{T}|\zeta(\sigma+\mathrm{i}t)|^6\mathrm{d}t\bigg)^{1/2}\bigg(\int_{-T}^{T}|\zeta(\sigma+\mathrm{i}(t_1-t))|^6\,\mathrm{d}t\bigg)^{1/2}\frac{\mathrm{d}t_1}{1+|t_1|}.$$

Now we choose $\sigma=\frac{7}{12}$. By [**7**, Equation (8.80)], we have the expected bound for the sixth zeta moment on the line $\mathrm{Re}(s)=\frac{7}{12}$. Hence,

$$A(N,H)\ll HN^{2\sigma-1}T^{1+\varepsilon}\ll HN^{1/6+\varepsilon}T.$$

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It therefore follows that

$$A(N,H)+B(N,H) \ll HN^{1/6+\varepsilon}T + HN^{7/12+\varepsilon}.$$

We shall balance this bound with the estimate in (2.6) by choosing $T = H^{-1/2}N^{11/12}$. Combining this with (2.6), (2.9), (2.10) and (2.14) we now get the final asymptotic formula

$$G(N,H) = H\int_N^{2N} (\operatorname{Res}_{s=1}\zeta^3(s)t^{s-1})^2 \,\mathrm{d}t + O(H^2N^\varepsilon + H^{1/2}N^{13/12+\varepsilon}). \tag{2.15}$$

Here we have observed that $HN^{7/12} \leqslant H^{1/2}N^{13/12}$ for $H \leqslant N$.

3. Estimation of $F(N,H)$

It remains to evaluate $F(N,H)$, defined in (2.1), and to estimate the difference

$$\sum_{h\leqslant H} \Delta(N,h) = G(N,H) - F(N,H). \tag{3.1}$$

We observe that

$$\begin{aligned}\sum_{h\leqslant H} \mathfrak{S}(x,h) &= \sum_{h\leqslant H}\sum_{q=1}^{\infty} \frac{c_q(h)}{q^2}P(x,q)^2 \\ &= \sum_{q\leqslant H}\biggl(\sum_{h\leqslant H} c_q(h)\biggr)\frac{P(x,q)^2}{q^2} + \sum_{h\leqslant H}\sum_{q>H}\frac{c_q(h)}{q^2}P(x,q)^2. \end{aligned}\tag{3.2}$$

In §7, we shall show that $P(x,q) = P^*(x,q)$, where $P^*(x,q)$ is defined as in (7.2). Applying (7.3), we therefore obtain $P(x,q) \ll (qx)^\varepsilon$. Using this and the fact that $|c_q(h)| \leqslant (q,h)$, we deduce that

$$\begin{aligned}\sum_{h\leqslant H}\sum_{q>H}\frac{c_q(h)}{q^2}P(x,q)^2 &\ll x^\varepsilon \sum_{h\leqslant H}\sum_{q>H}\frac{(q,h)}{q^{2-\varepsilon}} \\ &\ll x^\varepsilon \sum_{h\leqslant H}\sum_{d\mid h}\sum_{\substack{q>H\\ d\mid q}}\frac{d}{q^{2-\varepsilon}} \\ &\ll x^\varepsilon \sum_{h\leqslant H}\sum_{d\mid h}\left(\frac{H}{d}\right)^{-1+\varepsilon}\frac{d}{d^{2-\varepsilon}} \\ &\ll (xH)^\varepsilon.\end{aligned}$$

Next, we evaluate the first sum on the right-hand side of (3.2). An old result of Carmichael [**1**] asserts that

$$\sum_{h\leqslant q} c_q(h) = 0$$

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if $q > 1$. Hence, we see that

$$\sum_{h \leqslant H} c_q(h) = \begin{cases} H + O(1) & \text{if } q = 1, \\ O(q^{1+\varepsilon}) & \text{if } q > 1. \end{cases}$$

Putting all of this together, and using the definition of $P(x, 1)$ in (1.2), we get

$$\begin{aligned}\sum_{h \leqslant H} \mathfrak{S}(x, h) &= HP(x, 1)^2 + O((xH)^\varepsilon) \\ &= H(\operatorname{Res}_{s=1} \zeta^3(s) x^{s-1})^2 + O((xH)^\varepsilon).\end{aligned}$$

This implies that

$$F(N, H) = \sum_{h \leqslant H} \int_N^{2N} \mathfrak{S}(x, h) \, \mathrm{d}x = H \int_N^{2N} (\operatorname{Res}_{s=1} \zeta^3(s) x^{s-1})^2 \, \mathrm{d}x + O(N^{1+\varepsilon}).$$

Combining this with (2.15) and (3.1), we therefore conclude the proof of Theorem 1.1.

4. Activation of the circle method

Now we turn to the proof of Theorem 1.2. We shall mimic Mikawa's [**14**] treatment of the same problem for $\Lambda(n)$ in place of $d_3(n)$. However, several of Mikawa's arguments need to be adjusted to the present situation, and additional complications will occur. In this section, we describe the general set-up of the circle method.

We begin by observing that

$$D(N, h) = \int_0^1 |S(\alpha)|^2 e(-\alpha h) \, \mathrm{d}\alpha + O(hN^\varepsilon), \tag{4.1}$$

where

$$S(\alpha) := \sum_{N < n \leqslant 2N} d_3(n) e(n\alpha).$$

Let $Q_1 := N^\delta$ and $Q := N^{1/4}$ for a small parameter $0 < \delta < \frac{1}{4}$. We divide the integration into major and minor arcs as follows. The major arcs are defined as

$$\mathfrak{M} := \bigcup_{q \leqslant Q_1} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} I_{q,a}, \quad I_{q,a} := \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right],$$

and the minor arcs are defined as

$$\mathfrak{m} := [Q^{-1}, 1 + Q^{-1}] \setminus \mathfrak{M}.$$

In the remainder of this paper we establish the following two results. Taken together with (4.1), they imply Theorem 1.2.

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Proposition 4.1. *Let $0 < \eta < 1$ and let $\delta > 0$ be sufficiently small. Then there exists $\delta_1 > 0$ depending on η and δ such that uniformly for $h \leqslant N^{1-\eta}$, we have*

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-\alpha h)\,\mathrm{d}\alpha = \int_N^{2N} \mathfrak{S}(x,h)\,\mathrm{d}x + O(N^{1-\delta_1}).$$

Proposition 4.2. *Let $0 < \eta < \frac{1}{3}$ and let $\delta > 0$ be sufficiently small. Then there exists $\delta_2 > 0$ depending on η and δ such that for $N^{1/3+\eta} \leqslant H \leqslant N^{1-\eta}$, we have*

$$\sum_{h \leqslant H} \left| \int_{\mathfrak{m}} |S(\alpha)|^2 e(-\alpha h)\,\mathrm{d}\alpha \right|^2 \ll HN^{2-\delta_2}.$$

Before we can state all the lemmas needed in our method, we need to introduce a certain Dirichlet series and compute a related residue. Let $k, q \in \mathbb{N}$ and let χ be a character modulo q. A function that will occur frequently in our analysis is the Dirichlet series

$$F_k(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n) d_3(nk)}{n^s}, \tag{4.2}$$

initially defined for $\mathrm{Re}(s) > 1$. In the following, we convert this series into an Euler product and show that it can be meromorphically continued to the half plane $\mathrm{Re}(s) > 0$, with a possible pole at $s = 1$, depending on whether the character χ is principal or not.

To start with, let $\mathrm{Re}(s) > 1$. By $\mathcal{A}_k$, we denote the set of integers whose prime divisors all divide k. Obviously, we can factor $F_k(\chi, s)$ in the form

$$F_k(\chi, s) = A_k(\chi, s) B_k(\chi, s), \tag{4.3}$$

where

$$A_k(\chi, s) := \sum_{n \in \mathcal{A}_k} \frac{\chi(n) d_3(kn)}{n^s}, \qquad B_k(\chi, s) := \sum_{(n,k)=1} \frac{\chi(n) d_3(n)}{n^s}.$$

Now we may write A_k and B_k as Euler products in the form

$$A_k(\chi, s) = \prod_{p \mid k} \sum_{j=0}^{\infty} \frac{\chi(p^j) d_3(p^{j+v_p(k)})}{p^{js}}, \tag{4.4}$$

$$B_k(\chi, s) = \prod_{p \mid k} \left(1 - \frac{\chi(p)}{p^s}\right)^3 L^3(\chi, s). \tag{4.5}$$

Obviously, $A_k(\chi, s)$ can be analytically continued to the half plane $\mathrm{Re}(s) > 0$, and $B_k(\chi, s)$ can be meromorphically continued to the whole complex plane. Moreover, $B_k(\chi, s)$ is holomorphic if χ is non-principal and has a pole at $s = 1$ if χ is principal. In the latter case, when χ is the principal character χ_0 modulo q, we have

$$B_k(\chi_0, s) = \prod_{p \mid kq} \left(1 - \frac{1}{p^s}\right)^3 \zeta^3(s). \tag{4.6}$$

Furthermore, we have the following bounds.

Lemma 4.3. *Let $k, q \in \mathbb{N}$. Let χ be a non-principal character modulo q. Then for* $\mathrm{Re}(s) > \frac{1}{2}$ *we have*

$$|F_k(\chi, s)| \ll k^{\varepsilon} |L(\chi, s)|^3. \tag{4.7}$$

Let χ_0 be the principal character modulo q. Then for $\mathrm{Re}(s) > \frac{1}{2}$ *and $s \neq 1$ we have*

$$|F_k(\chi_0, s)| \ll (kq)^{\varepsilon} |\zeta(s)|^3. \tag{4.8}$$

For $j \in \{0, 1, 2\}$ we have

$$\frac{\mathrm{d}^j}{\mathrm{d}^j x} \operatorname{Res}_{s=1} F_k(\chi_0, s) \frac{x^s}{s} \ll \frac{(qkx)^{\varepsilon} x}{x^j}. \tag{4.9}$$

Proof. We first deduce from (4.4) that

$$A_k(\chi, s) \ll \prod_{p|k} p^{\nu_p(k)\varepsilon} \sum_{j=0}^{\infty} \frac{p^{j\varepsilon}}{p^{j/2}} = \prod_{p|k} \frac{p^{\nu_p(k)\varepsilon}}{1 - p^{-1/2+\varepsilon}} \ll k^{\varepsilon},$$

provided that $\varepsilon \leqslant \frac{1}{4}$. Moreover, if χ is a Dirichlet character modulo q and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$\prod_{p|k} \left(1 - \frac{\chi(p)}{p^s}\right)^3 \ll \prod_{p|k} \left(1 + \frac{1}{\sqrt{2}}\right)^3 \ll k^{\varepsilon}.$$

Similarly, if $\mathrm{Re}(s) > \frac{1}{2}$, then

$$\prod_{p|kq} \left(1 - \frac{1}{p^s}\right)^3 \ll (kq)^{\varepsilon}.$$

Combining these estimates with (4.3), (4.5) and (4.6), we arrive at the first pair of estimates in the statement of the lemma.

Let $x > 0$. To prove (4.9), we note that

$$x^s = x \sum_{n=0}^{2} \frac{\log^n x}{n!} (s-1)^n + (s-1)^3 R_x(s),$$

where $R_x(s)$ is an entire function in s. We have

$$\begin{aligned} \operatorname{Res}_{s=1} F_k(\chi_0, s) \frac{x^s}{s} &= \frac{1}{2\pi \mathrm{i}} \int_{|s-1|=1/3} F_k(\chi_0, s) \frac{x^s}{s} \, \mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{2} \int_{|s-1|=1/3} \frac{F_k(\chi_0, s)}{s} \frac{(s-1)^n}{n!} \, \mathrm{d}s \, x \log^n x. \end{aligned} \tag{4.10}$$

The integral involving $(s-1)^3 R_x(s)$ vanishes since $F_k(\chi_0, s)$ has a triple pole at $s = 1$. We now have

$$\frac{\mathrm{d}^j}{\mathrm{d}^j x} \operatorname{Res}_{s=1} F_k(\chi_0, s) \frac{x^s}{s} = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{2} \int_{|s-1|=1/3} \frac{F_k(\chi_0, s)}{s} \frac{(s-1)^n}{n!} \, \mathrm{d}s \frac{\mathrm{d}^j}{\mathrm{d}^j x} x \log^n x$$

for $j \in \{0,1,2\}$. It is clear that

$$\frac{\mathrm{d}^j}{\mathrm{d}^j x} x \log^n x \ll x^{1-j+\varepsilon}.$$

Furthermore, using (4.8), we have, for $|s-1| = \frac{1}{3}$ and $0 \leqslant n \leqslant 2$,

$$\frac{F_k(\chi_0, s)}{s} \frac{(s-1)^n}{n!} \ll (qk)^{\varepsilon} |\zeta(s)|^3 \ll (qk)^{\varepsilon}.$$

Here we have noted that $\zeta(s)$ is bounded above by an absolute constant for s with $|s-1| = \frac{1}{3}$. Inserting these bounds into (4.10), we arrive at (4.9). □

5. Technical results

In this section we record some of the key technical facts that will be called upon in our method.

Lemma 5.1. *Let $2 < \Delta < N/2$. For arbitrary $a_n \in \mathbb{C}$ we have*

$$\int_{|\beta| \leqslant 1/\Delta} \bigg| \sum_{N<n\leqslant 2N} a_n e(\beta n) \bigg|^2 \mathrm{d}\beta \ll \Delta^{-2} \int_N^{2N} \bigg| \sum_{t<n\leqslant t+\Delta/2} a_n \bigg|^2 \mathrm{d}t + \Delta \Big(\sup_{N<n\leqslant 2N} |a_n| \Big)^2,$$

where the implied constant is absolute.

Proof. This is [**14**, Lemma 1] and is a form of the Sobolev–Gallagher inequality. □

The next two lemmas are modified versions of Lemmas 2 and 5 in [**14**], respectively, where the role of $\Lambda(n)$ is now taken by $d_3(n)$.

Lemma 5.2. *Let $k, q \in \mathbb{N}$, $\Delta, N > 1$ and let χ be a character modulo q. Set $\delta(\chi) = 1$ if χ is principal and $\delta(\chi) = 0$ otherwise. Define*

$$S(k,\chi,\Delta,N) = \int_N^{2N} \bigg| \sum_{x<n\leqslant x+\Delta} \chi(n) d_3(kn) - \delta(\chi) \operatorname{Res}_{s=1} \frac{((x+\Delta)^s - x^s) F_k(\chi_0, s)}{s} \bigg|^2 \mathrm{d}x.$$

Let $0 < \eta < \frac{5}{12}$ be given. Then there exist positive δ and δ_3 depending on η such that if $k, q \leqslant N^{\delta}$ and $N^{1/6+\eta} \leqslant \Delta \leqslant N^{1-\eta}$, we have

$$S(k,\chi,\Delta,N) \ll \Delta^2 N^{1-\delta_3}. \tag{5.1}$$

Proof. For $k = q = 1$, Ivić [**9**, Corollary 1] proved that there exists $\delta_3 > 0$ depending on η such that if $N^{1/6+\eta} \leqslant \Delta \leqslant N^{1-\eta}$, we have

$$S(1,\chi_0,\Delta,N) = \int_N^{2N} \bigg| \sum_{x<n\leqslant x+\Delta} d_3(n) - \operatorname{Res}_{s=1} \frac{((x+\Delta)^s - x^s)\zeta^3(s)}{s} \bigg|^2 \mathrm{d}x \ll \Delta^2 N^{1-\delta_3}.$$

This is based on a bound for the sixth moment of the Riemann zeta function of the expected order of magnitude on the line $\mathrm{Re}(s) = \frac{7}{12}$, which we made use of in §2.

Ivić's method can be easily generalized to yield (5.1). The only additional inputs are the following. If χ is principal, then we use the bound (4.8). If χ is non-principal, then we use the bound (4.7) and a bound for the sixth moment of $L(\chi, s)$ in place of $\zeta(s)$. Indeed, for any given $\varepsilon > 0$, we have the bound

$$\int_{-T}^{T} |L(\chi, \tfrac{7}{12} + \mathrm{i}t)|^6 \, \mathrm{d}t \ll T(NT)^{\varepsilon},$$

provided that $q \leqslant N^{\delta}$ with $\delta > 0$ small enough. The proof of this estimate is analogous to the proof of the corresponding result for the Riemann zeta function and involves a generalization of the Atkinson mean square formula for L-functions due to Meurman [**12**]. $\square$

For the remainder of this section we suppose that $\alpha \in \mathbb{R}$ is given and that there exist coprime integers a, q such that $|\alpha - a/q| \leqslant q^{-2}$ and $q < \Delta < N/2$. Our next goal in this section is a proof of the following result.

Lemma 5.3. *Suppose that $\Delta > N^{1/3}$ and let*

$$J = J(\alpha, \Delta) := \int_{N}^{2N} \bigg| \sum_{t < n \leqslant t+\Delta} d_3(n) e(\alpha n) \bigg|^2 \mathrm{d}t.$$

Then there exist $\delta_4 > 0$ and $F > 0$ such that

$$J \ll (\log N)^F (\Delta N(N^{1/3} + \Delta q^{-1/2} + (q\Delta)^{1/2} + q) + \Delta^2 N^{1-\delta_4} + \Delta^3).$$

The proof of this lemma requires some auxiliary results, namely slightly modified versions of Lemmas 6–8 in [**14**]. Let f and g be sequences such that $|f(n)| \leqslant \log n$ and $|g(n)| \leqslant d_5(n)$. Moreover, let $U, V, C > 0$ and define

$$J_1 := \int_{N}^{2N} \bigg| \sum_{\substack{t < mn \leqslant t+\Delta \\ U \leqslant m \leqslant 2U}} g(n) e(\alpha mn) \bigg|^2 \mathrm{d}t,$$

$$J_2 := \int_{N}^{2N} \bigg| \sum_{\substack{t < dl \leqslant t+\Delta \\ C \leqslant l \leqslant 2C}} \bigg(\sum_{\substack{mn = d \\ U \leqslant m \leqslant 2U \\ V \leqslant n \leqslant 2V}} g(n) \bigg) e(\alpha dl) \bigg|^2 \mathrm{d}t,$$

$$J_3 := \int_{N}^{2N} \bigg| \sum_{\substack{t < mn \leqslant t+\Delta \\ U \leqslant m \leqslant 2U}} f(m) g(n) e(\alpha mn) \bigg|^2 \mathrm{d}t.$$

Then we have the following bounds.

Lemma 5.4. *There exists $F > 0$ such that*

$$J_1 \ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 (N/U)^2 + \Delta^3).$$

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Proof. This is [**14**, Lemma 6] with the summation condition $m \geqslant U$ being replaced by $U \leqslant m \leqslant 2U$. The proof is similar. □

Lemma 5.5. *There exist* $\delta_4 > 0$ *and* $F > 0$ *such that*

$$J_2 \ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^3) + \Delta^2(N^{1-\delta_4} + N^{7\delta_4}U^{3/2}V^4).$$

Proof. This is [**14**, Lemma 7] with an extra summation condition $C \leqslant l \leqslant 2C$ included and the summation conditions $m \leqslant U$ and $n \leqslant V$ being replaced by $U \leqslant m \leqslant 2U$ and $V \leqslant n \leqslant 2V$. The proof is similar. □

Lemma 5.6. *If* $U < \Delta$, *then there exists* $F > 0$ *such that*

$$J_3 \ll \Delta N(\log N)^F \left(U + \frac{\Delta}{q} + \frac{\Delta}{U} + q\right).$$

Proof. This is [**14**, Lemma 8]. □

We now turn to the proof of Lemma 5.3. To prove his corresponding result [**14**, Lemma 5], with $\Lambda(n)$ in place of $d_3(n)$, Mikawa employed a Vaughan-type decomposition of Λ due to Heath-Brown. Instead, we use here the much simpler decomposition $d_3 = \mathbf{1} * \mathbf{1} * \mathbf{1}$.

Proof of Lemma 5.3. For $N \leqslant n \leqslant 3N$, we may split $d_3(n) = \sum_{abc=n} 1$ into $O((\log N)^3)$ terms of the form

$$d_{A,B,C}(n) = \sum_{\substack{A \leqslant a \leqslant 2A \\ B \leqslant b \leqslant 2B \\ C \leqslant c \leqslant 2C \\ abc=n}} 1,$$

with $\frac{1}{8}N \leqslant ABC \leqslant 3N$. Using the Cauchy–Schwarz inequality, it follows that

$$J \ll \sup_{\substack{A \leqslant B \leqslant C \\ ABC=N}} (\log N)^9 \int_N^{2N} \left| \sum_{t<n \leqslant t+\Delta} d_{A,B,C}(n) e(\alpha n) \right|^2 \mathrm{d}t.$$

Our argument can be split into the following three cases.

Case 1. Let $N^{\delta_4} \leqslant A \leqslant N^{1/3}$. We may write

$$\int_N^{2N} \left| \sum_{t<n \leqslant t+\Delta} d_{A,B,C}(n) e(\alpha n) \right|^2 \mathrm{d}t = \int_N^{2N} \left| \sum_{\substack{t<mn \leqslant t+\Delta \\ A \leqslant m \leqslant 2A}} h_{B,C}(n) e(\alpha mn) \right|^2 \mathrm{d}t,$$

with

$$h_{B,C}(n) := \sum_{\substack{B \leqslant b \leqslant 2B \\ C \leqslant c \leqslant 2C \\ bc=n}} 1.$$

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Now Lemma 5.6 with $f = 1$ and $g = h_{B,C}$ yields the existence of $F > 0$ such that

$$\int_N^{2N} \Big| \sum_{t<n\leqslant t+\Delta} d_{A,B,C}(n)e(\alpha n) \Big|^2 \,\mathrm{d}t \ll \Delta N(\log N)^F \left(A + \frac{\Delta}{q} + \frac{\Delta}{A} + q \right)$$
$$\ll \Delta N(\log N)^F (N^{1/3} + \Delta q^{-1} + \Delta N^{-\delta_4} + q).$$

Case 2. Let $A \leqslant N^{\delta_4}$ and $C \geqslant N^{1/2+\delta_4/2}$. We have

$$\int_N^{2N} \Big| \sum_{t<n\leqslant t+\Delta} d_{A,B,C}(n)e(\alpha n) \Big|^2 \,\mathrm{d}t = \int_N^{2N} \Big| \sum_{\substack{t<mn\leqslant t+\Delta \\ C\leqslant m\leqslant 2C}} h_{A,B}(n)e(\alpha mn) \Big|^2 \,\mathrm{d}t. \tag{5.2}$$

Now Lemma 5.4 with $g = h_{A,B}$ yields the existence of $F > 0$ such that

$$\int_N^{2N} \Big| \sum_{t<n\leqslant t+\Delta} d_{A,B,C}(n)e(\alpha n) \Big|^2 \,\mathrm{d}t$$
$$\ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 (N/C)^2 + \Delta^3)$$
$$\ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 N^{1-\delta_4} + \Delta^3).$$

Case 3. Let $A \leqslant N^{\delta_4}$ and $\frac{1}{8}N^{1/2-3\delta_4/2} \leqslant B \leqslant C \leqslant N^{1/2+\delta_4/2}$. By (5.2) and the definition of $h_{A,B}$, we have

$$\int_N^{2N} \Big| \sum_{t<n\leqslant t+\Delta} d_{A,B,C}(n)e(\alpha n) \Big|^2 \,\mathrm{d}t = \int_N^{2N} \Big| \sum_{\substack{t<mn\leqslant t+\Delta \\ C\leqslant m\leqslant 2C}} \Big(\sum_{\substack{A\leqslant u\leqslant 2A \\ B\leqslant v\leqslant 2B \\ uv=n}} 1 \Big) e(\alpha mn) \Big|^2 \,\mathrm{d}t.$$

Lemma 5.5 with $g = 1$ yields the existence of $F > 0$ such that

$$\int_N^{2N} \Big| \sum_{t<n\leqslant t+\Delta} d_{A,B,C}(n)e(\alpha n) \Big|^2 \,\mathrm{d}t$$
$$\ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^3) + \Delta^2 (N^{1-\delta_4} + N^{7\delta_4} A^4 B^{3/2})$$
$$\ll (\log N)^F (\Delta N(\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^3) + \Delta^2 N^{1-\delta_4},$$

provided that $\delta_4 < \frac{1}{51}$.

There are no remaining cases. Combining everything therefore leads to the statement of Lemma 5.3. □

Throughout the following, let $\chi_{0,n}$ be the principal character modulo n. In our treatment of the major arcs, we shall have to approximate the term

$$T(q, x, \Delta) := \sum_{k|q} \frac{\mu(q^*)}{\varphi(q^*)} \sum_{x/k<m\leqslant (x+\Delta)/k} \chi_{0,q^*}(m) d_3(mk),$$

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with $q^* = q/k$, by a simpler term of the form

$$\sum_{x<m\leqslant x+\Delta} p_q(m),$$

where $p_q(m)$ is a certain nicely behaved function. The remainder of this section is devoted to the computation of this function.

Using Lemma 5.2 we shall aim to approximate $T(q,x,\Delta)$ in mean square by

$$T_0(q,x,\Delta) := \sum_{k|q} \frac{\mu(q^*)}{\varphi(q^*)} \operatorname{Res}_{s=1} \frac{1}{s}\left(\left(\frac{x+\Delta}{k}\right)^s - \left(\frac{x}{k}\right)^s\right) F_{k,q^*}(s), \tag{5.3}$$

where

$$F_{k,q^*}(s) = F_k(\chi_{0,q^*}, s), \tag{5.4}$$

in the notation of (4.2). Let

$$p_{k,q^*}(x) := \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{Res}_{s=1} \frac{x^s F_{k,q^*}(s)}{s}. \tag{5.5}$$

Then

$$\operatorname{Res}_{s=1} \frac{1}{s}\left(\left(\frac{x+\Delta}{k}\right)^s - \left(\frac{x}{k}\right)^s\right) F_{k,q^*}(s) = \frac{1}{k}\int_x^{x+\Delta} p_{k,q^*}\left(\frac{t}{k}\right) \mathrm{d}t.$$

Hence, we may write

$$T_0(q,x,\Delta) = \int_x^{x+\Delta} p_q(t)\,\mathrm{d}t,$$

where

$$p_q(t) := \sum_{k|q} \frac{\mu(q^*)}{\varphi(q^*)k} p_{k,q^*}\left(\frac{t}{k}\right). \tag{5.6}$$

From (4.9), it follows that

$$p_{k,q^*}(x) \ll (kq^*x)^{\varepsilon}, \qquad p'_{k,q^*}(x) \ll \frac{(kq^*x)^{\varepsilon}}{x}. \tag{5.7}$$

This, together with

$$\varphi(q^*) \gg \frac{q^*}{\log\log 10q^*}, \tag{5.8}$$

implies that

$$p_q(n) \ll \frac{(qn)^{\varepsilon}}{q}. \tag{5.9}$$

Armed with these formulae we may approximate the above integral by a sum. For x in the range $N \ll x < x+\Delta \ll N$ we see that

$$T_0(q,x,\Delta) = \sum_{x<n\leqslant x+\Delta} p_q(n) + O\left(\frac{(qN)^{\varepsilon}}{q}\right). \tag{5.10}$$

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6. Treatment of the major arcs

Now we investigate the major arcs. Let $\alpha \in I_{q,a}$ and write $\alpha = a/q + \beta$. Then we have

$$S(\alpha) = \sum_{N<n\leqslant 2N} d_3(n) e\left(\frac{an}{q}\right) e(\beta n).$$

Splitting the sum according to the value of (n, q), we obtain

$$S(\alpha) = \sum_{k|q} \sum_{\substack{N<n\leqslant 2N \\ (n,q)=k}} d_3(n) e\left(\frac{an}{q}\right) e(\beta n) = \sum_{k|q} \sum_{\substack{N/k<m\leqslant 2N/k \\ (m,q^*)=1}} d_3(mk) e\left(\frac{am}{q^*}\right) e(\beta mk),$$

where $q = q^*k$. Let $\tau(\chi)$ denote the Gauss sum associated to a Dirichlet character. Then for $(a, r) = 1$ we have the familiar identity

$$e\left(\frac{a}{r}\right) = \frac{1}{\varphi(r)} \sum_{\chi \bmod r} \chi(a)\tau(\bar{\chi}),$$

relating additive to multiplicative characters (see, for example, [**10**, Equation (3.11)]). Applying this, we may write

$$S(\alpha) = \sum_{k|q} \frac{1}{\varphi(q^*)} \sum_{\chi \bmod q^*} \tau(\bar{\chi})\chi(a) \sum_{N/k<m\leqslant 2N/k} \chi(m) d_3(mk) e(\beta mk).$$

We write $S(\alpha) = a + b + c$, where

$$a := \sum_{N<m\leqslant 2N} p_q(m) e(\beta m), \tag{6.1}$$

$$b := \sum_{k|q} \frac{1}{\varphi(q^*)} \sum_{\substack{\chi \bmod q^* \\ \chi\neq\chi_{0,q^*}}} \tau(\bar{\chi})\chi(a) \sum_{N/k<m\leqslant 2N/k} \chi(m) d_3(mk) e(\beta mk), \tag{6.2}$$

and

$$c := \sum_{k|q} \frac{\mu(q^*)}{\varphi(q^*)} \sum_{N/k<m\leqslant 2N/k} \chi_{0,q^*}(m) d_3(mk) e(\beta mk) - \sum_{N<m\leqslant 2N} p_q(m) e(\beta m). \tag{6.3}$$

Furthermore, set

$$\int_{\mathfrak{M}} |a|^2 \, \mathrm{d}\alpha = A^2, \qquad \int_{\mathfrak{M}} |b|^2 \, \mathrm{d}\alpha = B^2, \qquad \int_{\mathfrak{M}} |c|^2 \, \mathrm{d}\alpha = C^2.$$

Using the Cauchy–Schwarz inequality, we obtain

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-h\alpha) \, \mathrm{d}\alpha = \int_{\mathfrak{M}} |a|^2 e(-h\alpha) \, \mathrm{d}\alpha + O(A(B+C) + B^2 + C^2). \tag{6.4}$$

To estimate the error term in (6.4), we need bounds for A, B and C, which are provided by the following lemmas.

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Lemma 6.1. *Let $\varepsilon > 0$. Then we have $A^2 \ll N^{1+\varepsilon}$.*

Proof. Expanding $|a|^2$ and integrating, we obtain

$$A^2 \ll \sum_{q \leqslant Q_1} \frac{1}{qQ} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \sum_{N<m\leqslant 2N} p_q^2(m) + \sum_{q \leqslant Q_1} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \sum_{\substack{N<m_1\leqslant 2N \\ m_1 \neq m_2}} \sum_{N<m_2\leqslant 2N} \left| \frac{p_q(m_1)p_q(m_2)}{m_1 - m_2} \right|.$$

Now, inserting the estimate (5.9), we easily arrive at our desired result. □

Lemma 6.2. *Let $\delta > 0$ be sufficiently small. Then there exists $\delta_5 > 0$ depending on δ such that $B^2 \ll N^{1-\delta_5}$.*

Proof. By the definition of the major arcs, we have

$$B^2 = \sum_{q \leqslant Q_1} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \int_{|\beta| \leqslant 1/(qQ)} \left| \sum_{k|q} \frac{1}{\varphi(q^*)} \sum_{\substack{\chi \bmod q^* \\ \chi \neq \chi_{0,q^*}}} \tau(\bar{\chi})\chi(a) \times \sum_{N/k<m\leqslant 2N/k} \chi(m) d_3(mk) e(\beta mk) \right|^2 \mathrm{d}\beta.$$

Writing

$$\frac{1}{\phi(q^*)} = \sqrt{\frac{q^*}{\phi(q^*)}} \frac{1}{\sqrt{\phi(q^*)q^*}}$$

and applying the Cauchy–Schwarz inequality twice, we obtain

$$B^2 \ll \sum_{q \leqslant Q_1} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} g_q \sum_{k|q} \sum_{\substack{\chi \bmod q^* \\ \chi \neq \chi_{0,q^*}}} \int_{|\beta| \leqslant 1/(qQ)} \left| \sum_{N/k<m\leqslant 2N/k} \chi(m) d_3(mk) e(\beta mk) \right|^2 \mathrm{d}\beta,$$

where

$$g_q := \sum_{k|q} \frac{q^*}{\varphi(q^*)} \ll q^{\varepsilon},$$

using (5.8). Now applying Lemma 5.1 with a change of variables, we get

$$B^2 \ll \sum_{q \leqslant Q_1} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} g_q \times \sum_{k|q} \sum_{\substack{\chi \bmod q^* \\ \chi \neq \chi_{0,q^*}}} \left(\frac{k}{(qQ)^2} \int_{N/k}^{2N/k} \left| \sum_{x<m\leqslant x+qQ/(2k)} \chi(m) d_3(mk) \right|^2 \mathrm{d}x + qQN^{\varepsilon} \right).$$

Applying Lemma 5.2 and summing all relevant variables, we get the bound

$$B^2 \ll Q_1^{3+\varepsilon} N^{1-\delta_3} + Q_1^{4+\varepsilon} Q.$$

This is satisfactory if $\delta < \min\{\frac{1}{3}\delta_3, \frac{3}{16}\}$. □

Lemma 6.3. *Let $\delta > 0$ be sufficiently small. Then there exists $\delta_6 > 0$ depending on δ such that $C^2 \ll N^{1-\delta_6}$.*

Proof. First we observe that

$$\sum_{k|q} \frac{\mu(q^*)}{\varphi(q^*)} \sum_{N/k<m\leqslant 2N/k} \chi_{0,q^*}(m) d_3(mk) e(\beta mk) = \sum_{N<n\leqslant 2N} \sum_{k|(n,q)} \frac{\mu(q^*)}{\varphi(q^*)} \chi_{0,q^*}\left(\frac{n}{k}\right) d_3(n) e(\beta n).$$

Therefore, inserting the above into (6.3), we get that

$$c = \sum_{N<n\leqslant 2N} (a_n d_3(n) - p_q(n)) e(\beta n),$$

where

$$a_n = \sum_{k|(n,q)} \frac{\mu(q^*)}{\varphi(q^*)} \chi_{0,q^*}\left(\frac{n}{k}\right). \tag{6.5}$$

Hence, we have

$$C^2 = \sum_{q\leqslant Q_1} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} I(q,a), \tag{6.6}$$

with

$$I(q,a) := \int_{|\beta|<1/(qQ)} \left| \sum_{N<n\leqslant 2N} (a_n d_3(n) - p_q(n)) e(\beta n) \right|^2 \mathrm{d}\beta.$$

Lemma 5.1 yields

$$\begin{aligned} I(q,a) &\ll \frac{1}{(qQ)^2} \int_N^{2N} \left| \sum_{t<n\leqslant t+qQ/2} (a_n d_3(n) - p_q(n)) \right|^2 \mathrm{d}t \\ &\quad + qQ\Big(\sup_{N<n\leqslant 2N} |a_n d_3(n) - p_q(n)| \Big)^2 \\ &\ll \frac{1}{(qQ)^2} \int_N^{2N} \left| \sum_{t<n\leqslant t+qQ/2} (a_n d_3(n) - p_q(n)) \right|^2 \mathrm{d}t + qQN^{\varepsilon}, \end{aligned} \tag{6.7}$$

where the last estimate comes from using (5.9) and $a_n \ll n^{\varepsilon}$. Employing (5.10), we have

$$\begin{aligned} &\left| \sum_{t<n\leqslant t+qQ/2} (a_n d_3(n) - p_q(n)) \right|^2 \\ &\qquad = \left| \sum_{t<n\leqslant t+qQ/2} a_n d_3(n) - T_0\left(q,t,\frac{qQ}{2}\right) + O\left(\frac{(qN)^{\varepsilon}}{q}\right) \right|^2 \\ &\qquad \ll \left| \sum_{t<n\leqslant t+qQ/2} a_n d_3(n) - T_0\left(q,t,\frac{qQ}{2}\right) \right|^2 + O\left(\frac{(qN)^{\varepsilon}}{q^2}\right). \end{aligned}$$

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Note that $\sum_{k|q} \mu^2(q^*)/\varphi^2(q^*) \ll 1$. Now using (5.3), (6.5) and the Cauchy–Schwarz inequality, we deduce that the first term in the last line is bounded by

$$\sum_{k|q} \left| \sum_{t/k<m\leqslant t/k+qQ/(2k)} \chi_{0,q^*}(m)d_3(mk) - \mathrm{Res}_{s=1} \frac{1}{s}\left(\left(\frac{1}{k}\left(t+\frac{qQ}{2}\right)\right)^s - \left(\frac{t}{k}\right)^s\right) F_{k,q^*}(s)\right|^2.$$

Reinserting our work back into (6.7), we see after a change of variables that

$$\begin{aligned} I(q,a) \ll{}& \frac{q^\varepsilon N^{1+\varepsilon}}{q^4Q^2} + qQN^\varepsilon \\ &+ \sum_{k|q} \frac{k}{(qQ)^2} \int_{N/k}^{2N/k} \left| \sum_{x<m\leqslant x+qQ/(2k)} \chi_{0,q^*}(m)d_3(mk)\right. \\ &\left. - \mathrm{Res}_{s=1} \frac{1}{s}\left(\left(x+\frac{qQ}{2k}\right)^s - x^s\right) F_{k,q^*}(s)\right|^2 \mathrm{d}t. \end{aligned}$$

We are now in a position to apply Lemma 5.2 to the integral on the right-hand side. This gives

$$I(q,a) \ll N^{1-\delta_3} + \frac{N^{1+\varepsilon}}{q^4Q^2} + qQN^\varepsilon \ll N^{1-\delta_3},$$

since $q \leqslant N^\delta \leqslant N^{1/4}$ and $Q = N^{1/4}$. Now, inserting the above estimate into (6.6) and summing all the relevant variables, we arrive at our desired result if $\delta < \frac{1}{2}\delta_3$. □

From (6.4) and Lemmas 6.1–6.3, we obtain the following result.

Lemma 6.4. *Let $\delta > 0$ be sufficiently small. Then there exists $\delta_7 > 0$ depending on δ such that, uniformly for h, we have*

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-h\alpha)\,\mathrm{d}\alpha = \int_{\mathfrak{M}} |a|^2 e(-h\alpha)\,\mathrm{d}\alpha + O(N^{1-\delta_7}).$$

We now turn to the computation of

$$Z(h) := \int_{\mathfrak{M}} |a|^2 e(-h\alpha)\,\mathrm{d}\alpha, \tag{6.8}$$

where a is given by (6.1). By the definition of the major arcs, we have

$$\begin{aligned} Z(h) &= \sum_{q\leqslant Q_1} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \int_{|\beta|\leqslant 1/qQ} \left| \sum_{N<m\leqslant 2N} p_q(m)e(\beta m)\right|^2 e\left(-h\left(\frac{a}{q}+\beta\right)\right)\mathrm{d}\beta \\ &= \sum_{q\leqslant Q_1} c_q(-h) \int_{|\beta|\leqslant 1/qQ} \left| \sum_{N<m\leqslant 2N} p_q(m)e(\beta m)\right|^2 e(-h\beta)\,\mathrm{d}\beta, \end{aligned}$$

where $c_q(m)$ is the Ramanujan sum.

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Expanding the square in our expression for $Z(h)$ and using (5.6), we have

$$\begin{aligned}
Z(h) &= \sum_{q\leqslant Q_1} c_q(-h) \sum_{k_1|q} \sum_{k_2|q} \frac{\mu(q/k_1)\mu(q/k_2)}{\varphi(q/k_1)k_1\varphi(q/k_2)k_2} \\
&\quad \times \int_{|\beta|\leqslant 1/qQ} \sum_{N<n_1\leqslant 2N} \sum_{N<n_2\leqslant 2N} p_{k_1,q/k_1}\Big(\frac{n_1}{k_1}\Big) p_{k_2,q/k_2}\Big(\frac{n_2}{k_2}\Big) e(\beta(n_1-n_2-h))\,\mathrm{d}\beta \\
&= \sum_{q\leqslant Q_1} c_q(-h) \sum_{k_1|q} \sum_{k_2|q} \frac{\mu(q/k_1)\mu(q/k_2)}{\varphi(q/k_1)k_1\varphi(q/k_2)k_2} \bigg\{ \int_0^1 \cdots \,\mathrm{d}\beta - \int_{1/qQ}^{1-1/qQ} \cdots \,\mathrm{d}\beta \bigg\} \\
&= \Sigma_1(h) - \Sigma_2(h), \qquad (6.9)
\end{aligned}$$

say. It easily follows that

$$\begin{aligned}
\Sigma_1(h) = \sum_{q\leqslant Q_1} c_q(-h) \sum_{k_1|q} \sum_{k_2|q} \frac{\mu(q/k_1)\mu(q/k_2)}{\varphi(q/k_1)k_1\varphi(q/k_2)k_2} \\
\times \sum_{N+h<n\leqslant N} p_{k_1,q/k_1}\Big(\frac{n}{k_1}\Big) p_{k_2,q/k_2}\Big(\frac{n-h}{k_2}\Big). \qquad (6.10)
\end{aligned}$$

Next we turn to the estimation of

$$\begin{aligned}
\Sigma_2(h) = \sum_{q\leqslant Q_1} c_q(-h) \sum_{k_1|q} \sum_{k_2|q} \frac{\mu(q/k_1)\mu(q/k_2)}{\varphi(q/k_1)k_1\varphi(q/k_2)k_2} \\
\times \int_{1/qQ}^{1-1/qQ} \bigg(\sum_{N<n_1\leqslant 2N} p_{k_1,q/k_1}\Big(\frac{n_1}{k_1}\Big) e(\beta n_1) \\
\times \sum_{N<n_2\leqslant 2N} p_{k_2,q/k_2}\Big(\frac{n_2}{k_2}\Big) e(-\beta n_2) \bigg) e(-\beta h)\,\mathrm{d}\beta.
\end{aligned}$$

Using partial summation, (5.7) and the familiar bound

$$\sum_{s<n\leqslant t} e(\beta n) \ll \|\beta\|^{-1}, \qquad (6.11)$$

where $\|\alpha\|$ is the distance of α to the nearest integer, we obtain the estimate

$$\sum_{N<n\leqslant 2N} p_{k,q/k}\Big(\frac{n}{k}\Big) e(\pm\beta n) \ll (qN)^{\varepsilon} \|\beta\|^{-1}.$$

Since $|c_q(-h)| \leqslant \varphi(q)$, it follows that $\Sigma_2(h) \ll N^{\varepsilon} Q_1 Q \ll N^{3/4}$, since $\delta < \frac{1}{4}$. Combining this with (6.9), we obtain

$$Z(h) = \Sigma_1(h) + O(N^{3/4}), \qquad (6.12)$$

uniformly for $h \in \mathbb{N}$.

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7. Computation of the singular series

We now show that our main term $\Sigma_1(h)$ in (6.10) can be approximated by the integral on the right-hand side of the estimate in Proposition 4.1. Throughout this section, we assume that $q \leqslant N^\delta$ and $k_i|q$ for $i = 1, 2$, and that $0 < \delta < \frac{1}{4}$ and $0 < \eta < 1$. In the following, we shall frequently make use of (5.7), (5.8) and the inequality $|c_q(-h)| \leqslant (q, h)$ without further mention.

The innermost sum on the right-hand side of (6.10) is

$$\begin{aligned}\sum_{N+h<n\leqslant 2N} & p_{k_1,q/k_1}\left(\frac{n}{k_1}\right)p_{k_2,q/k_2}\left(\frac{n-h}{k_2}\right) \\ &= \sum_{N<n\leqslant 2N} p_{k_1,q/k_1}\left(\frac{n}{k_1}\right)p_{k_2,q/k_2}\left(\frac{n-h}{k_2}\right) + O(hN^\varepsilon) \\ &= \sum_{N<n\leqslant 2N} p_{k_1,q/k_1}\left(\frac{n}{k_1}\right)p_{k_2,q/k_2}\left(\frac{n}{k_2}\right) + O(hN^\varepsilon) \\ &= \int_N^{2N} p_{k_1,q/k_1}\left(\frac{x}{k_1}\right)p_{k_2,q/k_2}\left(\frac{x}{k_2}\right)\mathrm{d}x + O(hN^\varepsilon).\end{aligned}$$

It follows that

$$\Sigma_1(h) = \sum_{q\leqslant Q_1}\frac{c_q(-h)}{q^2}\int_N^{2N}\left(\sum_{k|q}\frac{\mu(q/k)q}{\varphi(q/k)k}p_{k,q/k}\left(\frac{x}{k}\right)\right)^2\mathrm{d}x + O\left(N^\varepsilon\sum_{q=1}^\infty\frac{h(q,h)}{q^2}\right).$$

We note that uniformly for $h \leqslant N^{1-\eta}$, we have

$$N^\varepsilon\sum_{q=1}^\infty\frac{h(q,h)}{q^2} \ll N^{1-\delta_8}$$

for some $\delta_8 > 0$ depending on η, if $2\varepsilon < \eta$. Moreover, we can extend to infinity the sum over $q \leqslant Q_1$ in the main term, with acceptable error depending on δ and η. Combining everything, we obtain

$$\Sigma_1(h) = \int_N^{2N}\mathfrak{S}^*(x,h)\,\mathrm{d}x + O(N^{1-\delta_9}),$$

where δ_9 depends on η and δ and

$$\mathfrak{S}^*(x,h) := \sum_{q=1}^\infty\frac{c_q(-h)}{q^2}\left(\sum_{k|q}\frac{\mu(q/k)q}{\varphi(q/k)k}p_{k,q/k}\left(\frac{x}{k}\right)\right)^2.$$

We proceed to show that

$$\mathfrak{S}^*(x,h) = \mathfrak{S}(x,h), \tag{7.1}$$

where the right-hand side is defined as in (1.4). To begin with we write

$$\mathfrak{S}^*(x,h) = \sum_{q=1}^\infty\frac{c_q(-h)}{q^2}P^*(x,q)^2,$$

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where

$$P^*(x,q) := \sum_{d|q} \frac{\mu(d)d}{\varphi(d)} p_{q/d,d}\left(\frac{xd}{q}\right). \tag{7.2}$$

In particular, it follows from (5.7) that

$$P^*(x,q) \ll (qx)^{\varepsilon}, \tag{7.3}$$

which is not of importance in the rest of this section but was used in § 3. Recalling the definition of p_{k,q^*} from (5.5), we have

$$\begin{aligned}
p_{q/d,d}(y) &= \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Res}_{s=1} \frac{t^s F_{q/d,d}(s)}{s}\bigg|_y \\
&= \operatorname{Res}_{s=1}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^s}{s}\right)\bigg|_y F_{q/d,d}(s)\right) \\
&= \operatorname{Res}_{s=1} y^{s-1} F_{q/d,d}(s).
\end{aligned}$$

Making the change of variables $s \to s+1$, we obtain

$$\begin{aligned}
P^*(x,q) &= \sum_{d|q} \frac{\mu(d)d}{\varphi(d)} \operatorname{Res}_{s=1}\left(\frac{xd}{q}\right)^{s-1} F_{q/d,d}(s) \\
&= \operatorname{Res}_{s=0} \sum_{d|q} \frac{\mu(d)d^{s+1}x^s}{\varphi(d)q^s} F_{q/d,d}(s+1).
\end{aligned}$$

Hence,

$$P^*(x,q) = \operatorname{Res}_{s=0} \zeta^3(s+1) H^*(s+1,q)\left(\frac{x}{q}\right)^s,$$

where

$$H^*(s,q) := \sum_{d|q} \frac{\mu(d)}{\varphi(d)} d^s G^*_{q/d,d}(s)$$

and

$$G^*_{q/d,d}(s) := \frac{F_{q/d,d}(s)}{\zeta^3(s)}.$$

For the proof of (7.1), it remains to show that $G^*_{q/d,d}(s) = G_{q/d,d}(s)$, in the notation of (1.1). It suffices to check this equation for prime powers $q = p^{\alpha}$, $\alpha \in \mathbb{N}$. We recall (4.3), (4.6) and (5.4).

Case 1. If $d = 1$, then

$$G^*_{q/d,d}(s) = G^*_{q,1}(s) = \frac{F_{p^{\alpha},1}(s)}{\zeta^3(s)} = (1-p^{-s})^3 \sum_{j=0}^{\infty} \frac{d_3(p^{j+\alpha})}{p^{js}} = G_{q,1}(s) = G_{q/d,d}(s).$$

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Case 2. If $d = p^\alpha$, then

$$G^*_{q/d,d}(s) = G^*_{1,q}(s) = \frac{F_{1,p^\alpha}(s)}{\zeta^3(s)} = (1-p^{-s})^3 = G_{1,q}(s) = G_{q/d,d}(s).$$

Case 3. If $d = p^\beta$ with $1 \leqslant \beta \leqslant \alpha - 1$, then

$$G^*_{q/d,d}(s) = \frac{F_{p^{\alpha-\beta},p^\beta}(s)}{\zeta^3(s)} = (1-p^{-s})^3 d_3(p^{\alpha-\beta}) = G_{q/d,d}(s).$$

In this way we see that $G_{q/d,d}(s)$ and $G^*_{q/d,d}(s)$ match up in all cases. Combining the facts in this section, we obtain the following estimate.

Lemma 7.1. *There exists $\delta_{10} > 0$ depending on η and δ such that, uniformly for $h \leqslant N^{1-\eta}$, we have*

$$\varSigma_1(h) = \int_N^{2N} \mathfrak{S}(x,h)\,\mathrm{d}x + O(N^{1-\delta_{10}}).$$

Combining Lemma 6.4, (6.8), (6.12) and Lemma 7.1 proves Proposition 4.1.

8. Treatment of the minor arcs

This last section is concerned with the proof of Proposition 4.2, following precisely Mikawa's treatment. Expanding the square, rearranging the order of summation and integration, and using the bound (6.11), we have

$$\sum_{h\leqslant H}\left|\int_{\mathfrak{m}} |S(\alpha)|^2 e(-\alpha h)\,\mathrm{d}\alpha\right|^2 \ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S(\alpha_1)|^2|S(\alpha_2)|^2 \min\left(H, \frac{1}{\|\alpha_1-\alpha_2\|}\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_2. \tag{8.1}$$

Set $\varDelta := HN^{-\delta_{11}}$ with $0 < \delta_{11} < \eta$. We split the right-hand side of (8.1) into $I_1 + I_2$, with

$$I_1 := \int_{\mathfrak{m}}\int_{\substack{\mathfrak{m}\\|\alpha_2-\alpha_1|>1/\varDelta}} |S(\alpha_1)|^2|S(\alpha_2)|^2 \min\left(H, \frac{1}{\|\alpha_1-\alpha_2\|}\right)\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1,$$

$$I_2 := \int_{\mathfrak{m}}\int_{\substack{\mathfrak{m}\\|\alpha_2-\alpha_1|\leqslant 1/\varDelta}} |S(\alpha_1)|^2|S(\alpha_2)|^2 \min\left(H, \frac{1}{\|\alpha_1-\alpha_2\|}\right)\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1.$$

Using orthogonality and the estimate $d_3(n) \ll n^\varepsilon$, we see that

$$I_1 \ll HN^{-\delta_{11}}\left(\int_0^1 |S(\alpha)|^2\,\mathrm{d}\alpha\right)^2 \ll HN^{2-\delta_{11}/2}. \tag{8.2}$$

Furthermore, we have

$$I_2 \ll H\int_{\mathfrak{m}} |S(\alpha)|^2\left(\int_{|\beta|\leqslant 1/\varDelta} |S(\alpha+\beta)|^2\,\mathrm{d}\beta\right)\mathrm{d}\alpha. \tag{8.3}$$

In view of Lemma 5.1, the inner integral here is

$$\int_{|\beta|\leqslant 1/\Delta} |S(\alpha+\beta)|^2 \,\mathrm{d}\beta \ll \Delta^{-2} \int_N^{2N} \bigg| \sum_{t<n\leqslant t+\Delta/2} d_3(n)e(\alpha n) \bigg|^2 \mathrm{d}t + \Delta N^{\varepsilon}.$$

Now, by Dirichlet's Theorem and the definition of the minor arcs, if $\alpha \in \mathfrak{m}$, there exist a and q such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-2}, \quad (a,q)=1, \quad Q_1 < q \leqslant Q.$$

From Lemma 5.3, the definitions of Δ, Q_1, Q and the assumption $N^{1/3+\eta} \leqslant H \leqslant N^{1-\eta}$, it now follows, uniformly for $\alpha \in \mathfrak{m}$, that

$$\Delta^{-2} \int_N^{2N} \bigg| \sum_{t<n\leqslant t+\Delta/2} d_3(n)e(\alpha n) \bigg|^2 \mathrm{d}t \ll N^{1-\delta_{12}},$$

provided that $\delta_{12} < \min\{\frac{1}{2}\delta, \delta_4, \eta-\delta_{11}, \frac{1}{24}\}$. Combining this with (8.3), we therefore obtain

$$I_2 \ll HN^{1-\delta_{12}} \int_0^1 |S(\alpha)|^2 \,\mathrm{d}\alpha \ll HN^{2-\delta_{12}/2}.$$

Proposition 4.2 now follows on inserting this estimate into (8.1), together with (8.2).

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