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A NOTE ON THE DIOPHANTINE EQUATION $(na)^x + (nb)^y = (nc)^z$

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Abstract

Let (a, b, c) be a primitive Pythagorean triple satisfying $a^2 + b^2 = c^2$. In 1956, Jeśmanowicz conjectured that for any given positive integer n the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $x = y = z = 2$. In this paper, for the primitive Pythagorean triple $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ with $k = 2^s$ for some positive integer $s \geq 0$, we prove the conjecture when $n > 1$ and certain divisibility conditions are satisfied.

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1. Introduction

Let (a, b, c) be a primitive Pythagorean triple satisfying $a^2 + b^2 = c^2$. Apparently, for any given positive integer n, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \tag{1.1}$$

has the solution $x = y = z = 2$. In 1956, Sierpiński [9] showed that (1.1) has no other solution when $n = 1$ and $(a, b, c) = (3, 4, 5)$. Jeśmanowicz [4] proved the same conclusion for $n = 1$ and $(a, b, c) = (5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, $(11, 60, 61)$, and he conjectured that (1.1) has no positive integer solutions for any n other than $(x, y, z) = (2, 2, 2)$. Since then many special cases of Jeśmanowicz' conjecture have been solved for $n = 1$. In 1959, Lu [6] proved that (1.1) has the only positive integer solution $(x, y, z) = (2, 2, 2)$ if $n = 1$ and $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$. In 1965, Deḿjanenko [1] extended the results of [9] and [4] by proving that if $n = 1$ and $(a, b, c) = (2k + 1, 2k(k + 1), 2k(k + 1) + 1)$, then Jeśmanowicz' conjecture is true. In 2013, Miyazaki [8] extended the results of Lu and Deḿjanenko by proving that if (a, b, c) is a primitive Pythagorean triple such that $a \equiv \pm 1 \pmod{b}$ or $c \equiv 1 \pmod{b}$, then Jeśmanowicz' conjecture is true when $n = 1$. For more results concerning Jeśmanowicz' conjecture for $n = 1$, see [7] and [8]. When $n > 1$, only a few results

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on this conjecture are known. Let $t > 1$ be a positive integer, and let $P(t)$ denote the product of distinct prime factors of t. In 1998, Cohen and the author [3] proved that if $(a, b, c) = (2k + 1, 2k(k + 1), 2k(k + 1) + 1)$, a is a prime power and either $P(b) \mid n$ or $P(n) \nmid b$, then (1.1) has no positive integer solutions for any n other than $(x, y, z) = (2, 2, 2)$. Thereby the result of Jeśmanowicz is extended to any positive integer $n > 1$. In the case where a is not a prime power, the author [2] verified the conjecture for $(a, b, c) = (2k + 1, 2k(k + 1), 2k(k + 1) + 1) = (15, 112, 113)$. In 1999, Le [5] gave certain necessary conditions for (1.1) to have positive integer solutions (x, y, z) with $(x, y, z) \neq (2, 2, 2)$. Recently, some special cases of the Pythagorean triple $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ have been considered. For instance, Yang and Tang [11] proved that if $k = 2$, then (1.1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$, and in [10] they further showed that if $c = F_m = 2^{2^m} + 1$ and $1 \leq m \leq 4$, then Jeśmanowicz' conjecture is true. In this paper we study more cases of the Pythagorean triple $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$, and the following results will be proved.

Theorem 1.1. *Let $a = 4k^2 - 1$, $b = 4k$, $c = 4k^2 + 1$, and $k = 2^s$ for some positive integer $s \geq 0$. Suppose that the positive integer n is such that either $P(a) \mid n$ or $P(n) \nmid a$. Then the only solution of (1.1) is $x = y = z = 2$.*

Corollary 1.2 [10, first case of Theorem 2]. *Let n be any positive integer. Then the Diophantine equation $(3n)^x + (4n)^y = (5n)^z$ has no positive integer solution other than $(x, y, z) = (2, 2, 2)$.*

Theorem 1.3. *Let $a = 4k^2 - 1$, $b = 4k$, $c = 4k^2 + 1$, and $k = 2^s$ for some positive integer $s \geq 0$. Then for $1 \leq s \leq 4$, the only solution of (1.1) is $x = y = z = 2$.*

2. Lemmas

Lemma 2.1 [6, Theorem]. *Let $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ and $n = 1$. Then (1.1) has the only positive integer solution $(x, y, z) = (2, 2, 2)$.*

Lemma 2.2 [5, Theorem]. *If (x, y, z) is a solution of (1.1) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied:*

(1) $\max\{x, y\} > \min\{x, y\} > z$, *$P(n) \mid c$ and $P(n) < P(c)$;*
(2) *$x > z > y$ and $P(n) \mid b$;*
(3) *$y > z > x$ and $P(n) \mid a$.*

Lemma 2.3. *Let (a, b, c) be any primitive Pythagorean triple such that the Diophantine equation $a^x + b^y = c^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$. Then (1.1) has no positive integer solution satisfying $x > y > z$ or $y > x > z$.*

Proof. Let $(x, y, z) \neq (2, 2, 2)$ be any solution of (1.1). Since the Diophantine equation $a^x + b^y = c^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$, we must have $n > 1$. By Lemma 2.2, $P(n) \mid c$ and $P(n) < P(c)$. Suppose $n = \prod_{i=1}^{s} q_i^{\beta_i}$, $c = \prod_{i=1}^{t} q_i^{\alpha_i}$, $1 \leq s < t$. There are two cases to be considered.

Case 1. $x > y > z$. In this case, from (1.1),

$$n^{x-y}a^x + b^y = \prod_{i=1}^{s} q_i^{\alpha_i z - \beta_i (y-z)} \cdot \prod_{i=s+1}^{t} q_i^{\alpha_i z}.$$

If there is an i satisfying $\alpha_i z - \beta_i(y-z) > 0$, then we must have $q_i \mid b$, which is impossible since $\gcd(b, c) = 1$. It follows that

$$n^{x-y}a^x + b^y = \prod_{i=s+1}^{t} q_i^{\alpha_i z}. \tag{2.1}$$

Since $a^2 + b^2 = c^2$, we obtain that $c < 3a$ or $c < 3b$. Otherwise we would have $c \geq 3a$, $c \geq 3b$, and then $c^2 \geq (\frac{3}{2}(a+b))^2 > a^2 + b^2$, which is a contradiction. Therefore,

$$\prod_{i=s+1}^{t} q_i^{\alpha_i z} \leq \left(\frac{c}{q_s}\right)^z \leq \left(\frac{c}{3}\right)^z < a^z + b^z < n^{x-y}a^x + b^y,$$

which contradicts (2.1).

Case 2. $y > x > z$. As in the argument for Case 1,

$$a^x + n^{y-x}b^y = \prod_{i=s+1}^{t} q_i^{\alpha_i z}. \tag{2.2}$$

As in Case 1, from $c < 3a$ or $c < 3b$,

$$\prod_{i=s+1}^{t} q_i^{\alpha_i z} \leq \left(\frac{c}{q_s}\right)^z \leq \left(\frac{c}{3}\right)^z < a^z + b^z < a^x + n^{y-xy}b^y,$$

which contradicts (2.2). □

By Lemmas 2.2 and 2.3, we have the following corollary.

Corollary 2.4. *Let (a, b, c) be any primitive Pythagorean triple such that the Diophantine equation $a^x + b^y = c^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$. If (x, y, z) is a solution of (1.1) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied:*

(1) *$x > z > y$ and $P(n) \mid b$;*
(2) *$y > z > x$ and $P(n) \mid a$.*

For the Pythagorean triple $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$, we have the following result.

Corollary 2.5 [10, Theorem 1]. *If $4k^2 + 1$ is a Fermat prime, then (1.1) has no positive integer solution satisfying $x > y > z$ or $y > x > z$.*

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3. Proof of the main results

Proof of Theorem 1.1. We suppose that (1.1) has a solution $(x, y, z) \neq (2, 2, 2)$, and will prove that this leads to a contradiction. By Lemma 2.1, $n > 1$. There are two cases to the proof.

Case 1. If $P(n) \nmid a$, we must have $x > z > y$ and $P(n) \mid b$ by Lemma 2.2 and Corollary 2.4. From (1.1), $n^{x-y}a^x + b^y = n^{z-y}c^z$. Because $b = 4k = 2^{s+2}$, we may suppose $n = 2^\beta$ with $\beta \geq 1$. Then $2^{\beta(x-y)}a^x + 2^{(s+2)y} = 2^{\beta(z-y)}c^z$. Since $x - y > z - y$,

$$2^{\beta(x-z)}a^x + 2^{(s+2)y-\beta(z-y)} = c^z. \tag{3.1}$$

Clearly $(s + 2)y - \beta(z - y) \geq 0$. Since $x > z$, from (3.1), $(s + 2)y - \beta(z - y) = 0$. We rewrite (3.1) as

$$2^{\beta(x-z)}a^x = c^z - 1. \tag{3.2}$$

Since $a = 4^{s+1} - 1 \equiv 0 \pmod 3$ and $c = 4^{s+1} + 1 \equiv -1 \pmod 3$, taking (3.2) modulo 3 gives $(-1)^z - 1 \equiv 0 \pmod 3$. It follows that $z \equiv 0 \pmod 2$. Writing $z = 2z_1$, we have $2^{\beta(x-z)}a^x = (c^{z_1} - 1)(c^{z_1} + 1)$. Let $a = a_1a_2$ with $\gcd(a_1, a_2) = 1$, $a_1^x \mid c^{z_1} + 1$ and $a_2^x \mid c^{z_1} - 1$. We observe that either $a_1 \geq 2^{s+1} + 1$ or $a_2 \geq 2^{s+1} + 1$. Suppose this is not true. Then, from $a_1 \leq 2^{s+1} - 1$ and $a_2 \leq 2^{s+1} - 1$,

$$a = a_1a_2 \leq (2^{s+1} - 1)^2 < (2^{s+1} - 1)(2^{s+1} + 1) = a,$$

which is a contradiction. If $a_1 \geq 2^{s+1} + 1$, then, from $a_1^2 \geq (2^{s+1} + 1)^2 = 4^{s+1} + 1 + 2^{s+2} > c + 1$, we get $a_1^x > a_1^z = (a_1^2)^{z_1} > (c + 1)^{z_1} > c^{z_1} + 1$, which is again a contradiction. If $a_2 \geq 2^{s+1} + 1$, we similarly get $a_2^x > c^{z_1} + 1 > c^{z_1} - 1$, which contradicts $a_2^x \mid c^{z_1} - 1$.

Case 2. If $P(a) \mid n$, we must have $x < z < y$ by Corollary 2.4. From (1.1), $a^x + n^{y-x}b^y = n^{z-x}c^z$. Since $y - x > z - x > 0$, we have $P(n) \mid a$ and $n^{z-x} \mid a^x$, which implies $P(a) = P(n)$ and $n^{z-x} = a^x$. It follows that

$$n^{y-z}b^y = c^z - 1. \tag{3.3}$$

Since $P(a) = P(n)$, $n \equiv a \equiv 0 \pmod 3$. Taking (3.3) modulo 3 gives $(-1)^z - 1 \equiv 0 \pmod 3$, which implies that z is even. Write $z = 2z_1$. Since $c \equiv 1 \pmod b$, $c^{z_1} + 1 \equiv 2 \pmod b$, so that $\gcd(c^{z_1} + 1, b) = 2$. Then, from (3.3), $(b^y/2) \mid c^{z_1} - 1$. But

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \frac{(c - a)^{z_1}(c + a)^{z_1}}{2} \geq c^{z_1} + a^{z_1} > c^{z_1} - 1,$$

which is a contradiction. □

Proof of Corollary 1.2 By Lemma 2.1, $n > 1$. Since $a = 3$, we must have $P(a) \mid n$ or $P(n) \nmid a$, which completes the proof of Corollary 1.2 by Theorem 1.1. □

Proof of Theorem 1.3. Suppose that (1.1) has a solution $(x, y, z) \neq (2, 2, 2)$. We prove that this will lead to a contradiction. By Lemma 2.1, $n > 1$. By Theorem 1.1 and Corollary 2.4, we may suppose $y > z > x$, $P(n) \mid a$ and $P(n) < P(a)$. Then, from (1.1), $a^x + n^{y-x}b^y = n^{z-x}c^z$. Since $y - x > z - x$ and $\gcd(a, c) = 1$, we must get $a^x = n^{z-x}a_1^x$

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with $\gcd(n, a_1) = 1$, so that

$$a_1^x + n^{y-z} b^y = c^z. \tag{3.4}$$

First, we observe that if $x \equiv z \equiv 0 \pmod 2$, then (3.4) cannot hold. To see this, let $x = 2x_1$ and $z = 2z_1$. From (3.4), $n^{y-z} b^y = (c^{z_1} + a_1^{x_1})(c^{z_1} - a_1^{x_1})$. As $\gcd(c^{z_1} + a_1^{x_1}, c^{z_1} - a_1^{x_1}) = 2$ implies $(b^y/2) \mid c^{z_1} + a_1^{x_1}$ or $(b^y/2) \mid c^{z_1} - a_1^{x_1}$, but on the other hand

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} \geq (8k^2)^{z_1} = (c+a)^{z_1} \geq c^{z_1} + a_1^{z_1} > c^{z_1} - a_1^{z_1},$$

we get a contradiction.

Second, we show that if $s = 1, 2, 3$ or 4, then we must have $x \equiv z \equiv 0 \pmod 2$.

We consider the cases $s = 2$ and $s = 4$ first.

If $s = 2$, then $a = 7 \cdot 9$, $b = 16$, $c = 65$, so that $n = 3^\alpha, a_1 = 7$ or $n = 7^\beta, a_1 = 9$. From (3.4),

$$7^x + 3^{\alpha(y-z)} 16^y = 65^z \tag{3.5}$$

or

$$9^x + 7^{\beta(y-z)} 16^y = 65^z. \tag{3.6}$$

Considering (3.5) and (3.6) modulo 8, 16, respectively, we have $x \equiv 0 \pmod 2$. Taking modulo 3, we get $z \equiv 0 \pmod 2$.

If $s = 4$, then $a = 3 \cdot 11 \cdot 31$, $b = 64$, $c = 1025$, $n = 3^\alpha, 11^\beta, 31^\gamma, 3^\alpha 11^\beta, 3^\alpha 31^\gamma$, or $11^\beta 31^\gamma$, and, accordingly, $a_1 = 341, 93, 33, 31, 11$, or 3. From (3.4),

$$341^x + 3^{\alpha(y-z)} 64^y = 1025^z, \tag{3.7}$$
$$93^x + 11^{\beta(y-z)} 64^y = 1025^z, \tag{3.8}$$
$$33^x + 31^{\gamma(y-z)} 64^y = 1025^z, \tag{3.9}$$
$$31^x + 3^{\alpha(y-z)} 11^{\beta(y-z)} 64^y = 1025^z, \tag{3.10}$$
$$11^x + 3^{\alpha(y-z)} 31^{\gamma(y-z)} 64^y = 1025^z, \tag{3.11}$$
$$3^x + 11^{\beta(y-z)} 31^{\gamma(y-z)} 64^y = 1025^z. \tag{3.12}$$

From (3.7), (3.8), (3.10)–(3.12), taking modulo 8, we have $x \equiv 0 \pmod 2$. Taking modulo 64, (3.9) gives $x \equiv 0 \pmod 2$. Taking modulo 3, we get $z \equiv 0 \pmod 2$ from (3.7), (3.9), (3.11) and (3.12). Taking modulo 11, (3.8) and (3.10) give $93^x \equiv 31^x \equiv 2^z \pmod{11}$, thereby $1 = (\frac{2}{11})^z = (-1)^z$, where $(\frac{\cdot}{\cdot})$ is Legendre's symbol. Hence $z \equiv 0 \pmod 2$.

For the cases $s = 1$ and $s = 3$, the proofs are similar to the above proofs of cases $s = 2$ and $s = 4$. Moreover, the cases $s = 1$ and $s = 3$ have been solved in [10], so we omit the details of the proofs. □

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References

[1] V. A. Demjanenko, 'On Jeśmanowicz' problem for Pythagorean numbers', *Izv. Vyss. Ucebn. Zaved. Mat.* **48** (1965), 52–56 (in Russian).

[2] M. J. Deng, 'On the Diophantine equation $(15n)^x + (112n)^y = (113n)^z$', *J. Natur. Sci. Heilongjiang University* **24** (2007), 617–620 (in Chinese).

[3] M. Deng and G. L. Cohen, 'On the conjecture of Jeśmanowicz concerning Pythagorean triples', *Bull. Aust. Math. Soc.* **57** (1998), 515–524.

[4] L. Jeśmanowicz, 'Several remarks on Pythagorean numbers', *Wiad. Mat.* **1** (1955–1956), 196–202 (in Polish).

[5] M. H. Le, 'A note on Jeśmanowicz' conjecture concerning Pythagorean triples', *Bull. Aust. Math. Soc.* **59** (1999), 477–480.

[6] W. T. Lu, 'On the Pythagorean numbers $4n^2 - 1$; $4n$ and $4n^2 + 1$', *Acta Sci. Natur. Univ. Szechuan* **2** (1959), 39–42 (in Chinese).

[7] T. Miyazaki, 'On the conjecture of Jeśmanowicz concerning Pythagorean triples', *Bull. Aust. Math. Soc.* **80** (2009), 413–422.

[8] T. Miyazaki, 'Generalizations of classical results on Jeśmanowicz' conjecture concerning Pythagorean triples', *J. Number Theory* **133** (2013), 583–595.

[9] W. Sierpiński, 'On the equation $3^x + 4^y = 5^z$', *Wiad. Mat.* **1** (1955–1956), 194–195 (in Polish).

[10] M. Tang and Z. J. Yang, 'Jeśmanowicz' conjecture revisited', *Bull. Aust. Math. Soc.* **88** (2013), 486–491.

[11] Z. J. Yang and M. Tang, 'On the Diophantine equation $(8n)^x + (15n)^y = (17n)^z$', *Bull. Aust. Math. Soc.* **86** (2012), 348–352.

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