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SOME REMARKS ON REGULAR AND STRONGLY REGULAR RINGS

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Introduction. This article presents some new algebraic and module theoretic characterizations of strongly regular rings. The latter uses Lambek's notion of symmetry. Strongly regular rings are shown to admit an involution and form an equational category. An example due to Paré shows that the category of regular rings and ring homomorphisms between them is not equational. Remarks on quasi-inverses and the generalized inverse of a matrix are included. The author acknowledges support from the NRC (A7752) and improvements from W. Blair received after announcement of the results.

A ring R is regular if given $x \in R$ there exists a $y \in R$ such that $x=xyx \cdot y$ is called a quasi-inverse for x. A ring R is called strongly regular if given $x \in R$ there exists $y \in R$ such that $x=x^2y$.

LEMMA 1. *The following conditions are equivalent to strong regularity for a ring R*

(i) *R is regular and 0 is the only nilpotent element of R,*

(ii) *R is regular and all idempotents in R are central,*

(iii) *R is regular and if $x, y \in R$ with $x=xyx$ then $xy=yx$.*

Proof. (i) and (ii) are in [2]. (iii) implies strong regularity and follows from (ii) since $xy=(xyx)y=(yx)(xy)=y(xy)x=yx$ because xy and yx are central.

PROPOSITION 2. *The following are equivalent for a ring R*

(i) *R is strongly regular,*

(ii) *given $x \in R$, there exists $y \in R$ and a natural number n such that $x=xyx$ and $(xy-yx)^n=0$.*

These imply the following condition and are equivalent to it if 2 is a nonzero-divisor,

(iii) *given $x \in R$, there exists a $y \in R$ such that $x=xyx$ and $x(xy-yx)=(xy-yx)x$.*

Proof. Clearly (i) implies (ii). Suppose (ii) holds. By induction there exists for each natural number n an $r_n \in R$ such that $x(yx-xy)^n=x-x^2r_n$. Hence if $(xy-yx)^n=0$ then $x(yx-xy)^n=x(-1)^n(xy-yx)^n=0$ so $x=r_nx^2$ and R is strongly regular.

Suppose that $x=xyx$ and that $x(xy-yx)=(xy-yx)x$. Then $x^2y+yx^2=2x$. Multiply this equation on both sides on the left by x and simplify to obtain $x^3y=x^2$. Similarly $yx^3=x^2$. Now $x^2y=(x^3y)y=(yx^3)y=y(x^3y)=yx^2$. Thus $2x^2y=2x$

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and therefore $x=x^2y$. The ring of 2×2 matrices over the two-element field is an example of a ring which is not strongly regular, but which does satisfy the condition in (iii) above. Thus one cannot drop the demand that 2 be a unit.

A ring R is regular if and only if every R-module is flat. One can replace right by left in this characterization and restrict attention to cyclic R-modules. Lambek has recently introduced the notion of symmetry—a right R-module M is symmetric if $mrs=0$ implies $msr=0$ for all $m \in M$ and $r, s \in R$(5). The ring R is symmetric if it is symmetric when viewed as a right R-module but this definition is not one-sided.

PROPOSITION 3. *The following are equivalent for the ring R*

(i) *R is strongly regular*

(ii) *every right R-module is flat and symmetric,*

(iii) *every cyclic right R-module is flat and symmetric,*

(iv) *R is regular and symmetric.*

Proof. (i)$\Rightarrow$(ii). Let M be a right R-module. By (i) M is flat. Let $mrs=0$ for some $m \in M$, and some $r, s \in R$ and let $I=\{x \in R \mid mx=0\}$. I is a right ideal in R. Since R is strongly regular I is a two-sided ideal. $\bar{R}=R/I$ is strongly regular and therefore it has no nilpotent elements. But $rs \in I$ so $(\bar{s}\bar{r})^2=\bar{0}$ and $\bar{s}\bar{r}=\bar{0}$ showing $msr=0$. Thus M is symmetric. Clearly (ii)$\Rightarrow$(iii). (iii)$\Rightarrow$(iv). Since every cyclic right R-module is flat, R is a regular ring [4, Prop. 4, p. 134]. R is a cyclic right R-module so R is a symmetric ring. (iv)$\Rightarrow$(i). Let $x \in R$ with $x^2=0$. By regularity $x=xyx$ for some $y \in R$. $y(x)(xy)=0$ so by symmetric $yx=y(xy)x=0$ and $x=x(yx)=0$. Since R has no non-zero nilpotent elements it is strongly regular. Clearly one is free to change right to left in the above enunciation.

LEMMA 4. *If R is strongly regular then given x in R there is a unique y in R with $x=xyx$ and $y=yxy$.*

Proof. Let $x \in R$ a strongly regular ring. By 1(iii) there is a z in R with $x=xzx$ and $xz=zx$. Let $y=zxz$. Then $xyx=x$ and $yxy=y$, so such a y exists. Assume that $y_1 \in R$ with $x=xy_1x$ and $y_1=y_1xy_1$. By 1(iii) x commutes with y and y_1 so $y_1=y_1xy_1=y_1xyxy_1=y_1yxy_1x=y_1yx=y_1yxyx=(xy_1x)yy=xyy=y$ which shows the uniqueness of y.

An involution on a ring R is a function $*:R\to R$ such that for all $r, s \in R$, (i) $(rs)^*=(s)^*(r)^*$, (ii) $rr^*=0\Rightarrow r=0$, and (iii) $(r^*)^*=r$.

COROLLARY 5. *A strongly regular ring admits an involution.*

Proof. Define * on R be letting x^* be the y of lemma 4. It is easy to check (i) and (ii). (iii) holds because x is the unique element for y.

Kaplansky has shown implicitly in (3, lemma 4) that if R is a regular ring with involution * then given x there is a unique y with $x=xyx$, $y=yxy$, $(xy)^*=xy$ and $(yx)^*=yx$. In the case where R is strongly regular and * is defined as above lemma 4 implies that $(e)^*=e$ for any idempotent element. Thus Kaplansky's

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result reduces to the statement of lemma 4, however in order to define the involution on R one needs the uniqueness so it does not seem to be possible to obtain lemma 4 as a corollary of Kaplansky's result. If R is the ring of $n \times n$ matrices with complex entries and if * is the conjugate transpose operator then Kaplansky's result becomes that of [1, p. 1–3] for the generalized inverse of a matrix. Thus the existence and the uniqueness of the generalized inverse is an algebraic result. Four equivalent formulations of interest are found on [1, p. 11].

It is clear from lemmas 1 and 4 that strong regularity is equivalent to the demand that given x there exists a unique y such that $x=xyx$, $y=yxy$ and $xy=yx$. Thus the full subcategory of the category of rings consisting of all strongly regular rings and all ring homomorphisms between them can be equationally defined. (See [6, p. 120] for the meaning of "equational" and "equalizer"). The following example shows that the corresponding result is not true for the full subcategory of regular rings. For if it were equational it would be closed under the formation of equalizers. If α, $\beta: R \to S$ are ring homomorphisms then the equalizer of α and β is the subring $\{r \in R \mid \alpha(r)=\beta(r)\}$ of R. Thus it suffices to exhibit R, S regular, and maps α, β such that the equalizer of α and β is not regular. Let $R=S$ be the ring of 2×2 matrices with rational entries, a regular ring. Let α be the identity mapping and let $\beta: R \to R$ be defined by $\beta\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a-b & b \\ a+c-b-d & b+d \end{bmatrix}$. The equalizer of α and β is $E=\{\begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \mid a, b \in Q\}$. Since $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, E contains no quasi-inverse for $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and therefore is not regular. Since R is a ring with involution, the same example shows that the full subcategory of regular rings with involution is not equational either. This does not contradict the earlier result of Kaplansky since the definition of an involution involves an implication as well as equations. From the above example one can construct a regular ring T containing regular subrings T_1 and T_2 such that $T_1 \cap T_2$ is not regular. Let T be the ring of 4×4 matrices over Q, let T_1 be the subring having elements of R in the upper left and lower right hand corners and zeroes elsewhere, and let T_2 be the subring having elements of R in the upper left hand corner, elements of $\beta(R)$ in the lower right hand corner, and zeroes elsewhere. It is easy to check that T, T_1, T_2, and $T_1 \cap T_2$ have the desired properties.

We close with a remark on quasi-inverses. If one calls an element x of a ring R regular if $x=xyx$ for some y then a product of regular elements if regular in the commutative case. No formula is known for a quasi-inverse of a product of two elements in terms of the elements and their quasi-inverses in the general case. No formula can exist for the quasi-inverse of a sum in view of the following example. Let p be a prime, $A=Z/(p)$, G equal the cyclic group of order p, and R the group ring AG. Every element of R is the sum of regular elements but R is not regular [4, p. 155]. It is known in the commutative case that if x_1, x_2 and x_1+x_2 are regular with unique (in the sense of lemma 4) quasi-inverses y_1, y_2 and y_3 respectively then y_1+y_2 is regular with unique quasi-inverse $x_1(1-x_2y_2)+x_2(1-x_1y_1)+x_1x_2y_3$.

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