A NOTE ON THE JACOBSON AND BROWN-McCOY RADICALS

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1. Introduction. Let J(R) and G(R) respectively denote the Jacobson and Brown-McCoy radicals of the ring R and recall that $R = G(R)$ if and only if R can not be homomorphically mapped onto a simple ring with unity [1, p. 120].

In general one knows that $J(R) \subseteq G(R)$ [1, p. 118], while there do exist rings R for which $J(R) \neq G(R)$ (see [1, p. 120]). In this note we show the inequality between J and G can be sharpened in the following way: There exists a ring A with centre Z such that $J(A) \cap Z \neq G(A) \cap Z$. This is perhaps a bit surprising since $J(S) = G(S)$ whenever S is a commutative ring [1, p. 118].

Sasiada and Sulinski [3] showed by means of an example that the Jacobson radical was not the upper radical determined by the class of all simple primitive rings, thus answering a question of Kurosch (see [1, p. 113] for a discussion of this). It turns out that our ring A is a very easy example to the same effect.

2. The ring A. Let D be a commutative ring without divisors of zero which is also Jacobson radical [1, p. 103]. Kaplansky has pointed out that there is a primitive ring A whose centre is isomorphic to D [2, p. 36]. To form A, one imbeds D in its quotient field F and then takes A to be the ring of all infinite matrices of the type

$$(1) \qquad \begin{bmatrix} M & O & \cdot & \cdot & \cdot & O \\ O & d & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ O & 0 & & & & \end{bmatrix},$$

where $d \in D$ and M is an arbitrary finite square matrix with entries from F. The centre Z of A consists of the matrices diag(d, ...), hence is isomorphic to D.

*The research of the first author was supported by the National Research Council of Canada and the N.S.F., while that of the second author was supported by an H.R. MacMillan Family Fellowship.

https://doi.org/10.4153/CMB-1968-090-8 Published online by Cambridge University Press

It is easy to see that the set I of all matrices of type (1) in which d = 0 is an ideal of A which is contained in every non-zero ideal of A.

From the above observation it follows that A = G(A). Indeed, consider any homomorphic image A/K of A. If K = 0 then A/K is not simple. If $K \neq 0$ then $K \supseteq I$, whence each coset of A/K is of the form K + u, where

$$u = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & d & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & & & & \end{bmatrix}, \quad d \in D.$$

Since D is Jacobson radical, d has a quasi-inverse e and

$$v = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & & & & \end{bmatrix}$$

is a quasi-inverse of u. This implies that A/K is Jacobson radical, and therefore has no unity. Hence A = G(A).

Since A is primitive, J(A) = 0 and $J(A) \cap Z = 0$. But $G(A) \cap Z = A \cap Z \neq 0$.

At the same time we have shown that the primitive ring A can not be homomorphically mapped onto a simple primitive ring. Thus $J \neq$ upper radical determined by all simple primitive rings.

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https://doi.org/10.4153/CMB-1968-090-8 Published online by Cambridge University Press