RESONANCE IN THE RESTRICTED PROBLEM OF THREE BODIES WITH SHORT-PERIOD PERTURBATIONS IN THE ELLIPTIC CASE

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ABSTRACT. This is a generalization of the paper by Bhatnagar and Beena Gupta 'Resonance in the restricted problem of three bodies with short-periodic perturbations'. The motion of an asteroid moving in the gravitational field of Jupiter is considered. In the original paper it was assumed that Jupiter is moving in a circular orbit around the Sun. In the present paper we consider the orbit to be elliptic. The series occurring in the problem are expanded in powers of a small parameter ε, which represents the ratio of the mass of Jupiter to that of the Sun. The perturbations in the osculating elements are obtained up to $O(\varepsilon)$.

1. EQUATIONS OF MOTION

Let us suppose that Jupiter moves in an unperturbed elliptic orbit with the Sun at one of its foci. Take the orbital plane of Jupiter as the (x,y) plane. Let e' be the eccentricity of Jupiter's orbit, a' its semi-major axis, λ' its mean longitude, ℓ' its mean anomaly and n' its mean motion. The corresponding elements of the asteroid are denoted by e, a, λ, ℓ, and n.

The equations of motion of the asteroid with negligible mass are:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \dot{x}}, \qquad \frac{d\dot{x}}{dt} = -\frac{\partial H}{\partial x},$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial \dot{y}}, \qquad \frac{d\dot{y}}{dt} = -\frac{\partial H}{\partial y},$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial \dot{z}}, \qquad \frac{d\dot{z}}{dt} = -\frac{\partial H}{\partial z},$$

where the Hamiltonian, $H = H_0 + H_1$, is given by

$$H_0 = 1/2\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\mu}{r},$$

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 339-353.

$$H_1 = \varepsilon\ \mu\left(\frac{1}{\Delta} - \frac{xx' + yy'}{r'^3}\right).$$

In these equations (x,y,z) are the coordinates of the asteroid, (x',y',0) are the coordinates of Jupiter, Δ is the Jupiter-asteroid distance, r' the Sun-Jupiter distance, r the Sun-asteroid distance and $\mu = k^2\varepsilon$.

Let us introduce the change of variables

$$(x,y,z,\dot{x},\dot{y},\dot{z}) \to (L,G,H,\ell,g,\tilde{h})$$

defined by the canonical transformations

$$\ell = -\frac{\partial W'}{\partial L}, \quad g = -\frac{\partial W'}{\partial G}, \quad \tilde{h} = -\frac{\partial W'}{\partial H},$$

$$\dot{x} = \frac{\partial W'}{\partial x}, \quad \dot{y} = \frac{\partial W'}{\partial y}, \quad \dot{z} = \frac{\partial W'}{\partial z}.$$

Here W' is a generating function; and L,G,H,ℓ,g and h are the Delauny variables given by

$$L = \sqrt{\mu a}, \quad G = \sqrt{1-e^2}, \quad H = G \text{ Cos } i,$$

$$\ell = \ell, \quad g = \omega \quad , \quad \tilde{h} = \Omega,$$

where 1 is the inclination of the orbital plane of the asteroid with respect to the reference plane, ω the argument of the perihelion and Ω the longitude of the ascending node.

The equations of motion become

$$\frac{dL}{dt} = \frac{\partial\tilde{F}}{\partial\ell}, \qquad \frac{d\ell}{dt} = -\frac{\partial F}{\partial L},$$

$$\frac{dG}{dt} = \frac{\partial\tilde{F}}{\partial g}, \qquad \frac{dg}{dt} = -\frac{\partial\tilde{F}}{\partial G},$$

$$\frac{dH}{dt} = \frac{\partial\tilde{F}}{\partial\tilde{h}}, \qquad \frac{d\tilde{h}}{dt} = -\frac{\partial\tilde{F}}{\partial H}$$

with $\tilde{F} = \mu^2/2L^2 + F_1, \quad F_1 = \varepsilon k^2\left[\frac{1}{\Delta} - \frac{xx' + yy'}{\gamma'^3}\right].$

k is the Gaussian constant.

The equations of motion can be written as (Brouwer and Glemence, 1961),

$$\frac{dL}{dt} = \frac{\partial F}{\partial\ell}, \qquad \frac{d\ell}{dt} = -\frac{\partial F}{\partial L},$$

$$\frac{dG}{dt} = \frac{\partial F}{\partial g}, \qquad \frac{dg}{dt} = -\frac{\partial F}{\partial G},$$
$$\frac{dH}{dt} = \frac{\partial F}{\partial h}, \qquad \frac{dh}{dt} = -\frac{\partial F}{\partial H}, \qquad (1)$$
$$\frac{dK}{dt} = \frac{\partial F}{\partial k}, \qquad \frac{dk}{dt} = -\frac{\partial F}{\partial K}.$$

with

$$F = F_0 + F_1,$$
$$F_0 = \frac{\mu^2}{2L^2} - n'K,$$
$$F_1 = \varepsilon \, \Sigma \, C^{m_2,m_3,m_4}_{p_1,p_2,p_3,p_4} (\sin \frac{i}{2})^{2m_3} e^{m_2} e'^{m_4} \cos(p_1\ell+p_2g+p_3h+p_4k).$$

The coefficients C's are functions of a and a' of degree-1. And the D'Alembert's characteristics give

$$m_2 = |j_2| + 2k_2 = |p_1-p_2| + 2k_2,$$
$$2m_3 = |j_3| + 2k_3 = |p_2-p_3| + 2k_3, \qquad (2)$$
$$m_4 = |j_4| + 2k_4 = |p_3+p_4| + 2k_4,$$

where k_2, k_3 and k_4 are positive integers of zero. Let us introduce the new variables

$$(x_1,x_2,x_3,x_4;\ y_1,y_2,y_3,y_4)$$

defined by the following canonical transformations:

$$x_1 = L + \frac{p}{q} K, \qquad y_1 = \ell,$$
$$x_2 = -\frac{1}{q} K, \qquad y_2 = p\ell + q\omega + q(\Omega - \lambda'), \qquad (3)$$
$$x_3 = G + K, \qquad y_3 = \omega,$$
$$x_4 = -H - K, \qquad y_4 = \omega'.$$

The system of Equation (1) reduces to

$$\frac{dx_i}{dt} = \frac{\partial K'}{\partial y_i}; \quad \frac{dy_i}{dt} = \frac{\partial K'}{\partial x_i}. \quad (i = 1,2,3,4) \qquad (4)$$

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with $K' = K_0 + K_1$,

$$K_0 = \frac{\mu^2}{2(x_1 + px_2)^2} + qn'x_2, \qquad (5)$$

and

$$K_1 = \varepsilon R,$$

where

$$R = \Sigma f(a,e,i,a',e') \cos(p_1\ell + p_2\omega - p_3\omega' + p_4\ell'). \qquad (6)$$

and restrictions on p_1, p_2, p_3 and p_4 are given by the Equations (2).

2. SHORT-PERIOD PERTURBATIONS

Let us eliminate the short-periodic terms, i.e. the terms which contain mean anomaly in their argument. The elimination is achieved through the well known Von Zeipel method. Here we assume canonical transformations (x,y) to (ξ,η) defined by the generating function $W(\xi,y,\varepsilon)$ such that the new Hamiltonian $\phi(\xi,\eta,\varepsilon)$ is free from the angular variable η_1. Also we assume the two series

$$W = W_0 + W_{1/2} + W_1 + W_{3/2} + \dots ,$$

$$\phi = \phi_0 + \phi_{1/2} + \phi_1 + \phi_{3/2} + \dots ,$$

where

$$W_j = O(\varepsilon^j) \text{ and } \phi_j = O(\varepsilon^j).$$

We consider the problem by assuming that

$$|pn - qn'| \le n\,\varepsilon^{1/2}, \qquad (7)$$

where p and q are mutually prime integers. Since W does not contain time explicitly, the Hamilton-Jacobi equation will be

$$\phi(\xi;\ \frac{\partial W}{\partial \xi_2}, \frac{\partial W}{\partial \xi_3}, \frac{\partial W}{\partial \xi_4}, \varepsilon) = K(\frac{\partial W}{\partial y};\ y;\ \varepsilon). \qquad (8)$$

Here ξ means ξ_1, ξ_2, ξ_3 and ξ_4 and y means y_1, y_2, y_3 and y_4.

Following the procedure of Giacaglia (1969) we shall have

$$\phi_0 = K_0,$$

$$\phi_{1/2} = W_{1/2} = 0,$$

$$\phi_1 = \frac{1}{2\pi q}\int_0^{2\pi q} R\, dy_1,$$

$$W_1(\xi, y, \varepsilon) = -\left(\frac{\partial K_0}{\partial \xi_1}\right)^{-1} \int (K_1 - \phi_1)\, dy_1, \tag{9}$$

$$\phi_{3/2} = 0,$$

$$W_{3/2} = -\left(\frac{\partial K_0}{\partial \xi_1}\right)^{-1} \int \left(\frac{\partial K_0}{\partial \xi_2}\right)\left(\frac{\partial W_1}{\partial y_2}\right) dy_1.$$

Thus we have established the two series of W and ϕ up to $O(\varepsilon^{3/2})$. Since we are considering terms only up to $O(\varepsilon^{3/2})$, we neglect terms of $O(\varepsilon^2)$. It may be noted that the series are of the same form as in the circular case except that the value of K differs from one case to another.

Thus in this case, i.e. up to $O(\varepsilon^{3/2})$, the Hamiltonian become

$$\phi^{(3/2)} = \phi_0 + \phi_1, \tag{10}$$

where

$$\phi_0 = \frac{\mu^2}{2(\xi_1 + p\xi_2)^2} + qn'\xi_2. \tag{11}$$

and

$$\phi_1 = \varepsilon\, \Sigma C(a^*, e^*, i^*, a'^*, e'^*) \cos\, [-p_1\eta_2 + (p_2 + p_1 q)\eta_3 - - (p_3 + p_1 q)\eta_4]. \tag{12}$$

Also up to this order we have

$$\dot{\xi}_1 = \frac{\partial \phi^{(3/2)}}{\partial\, \eta_1} = 0.$$

Hence

$$\xi_1 = \text{const.}$$

or

$$L^* + (p/q)K^* = \text{const.}$$

The short-period perturbations are given by the generating function W in an implicit form as

$$x_j = \varepsilon_j + \frac{\partial W_1}{\partial y_j} + \frac{\partial W_{3/2}}{\partial y_j} = \xi_j + \varepsilon\, \Delta\, x_j,$$

$$\eta_j = y_j + \frac{\partial W_1}{\partial \xi_j} + \frac{\partial W_{3/2}}{\partial \xi_j} = y_j + \varepsilon \, \Delta \, y_j, \tag{13}$$

where Δx_j and Δy_j are short-periodic terms.

3. ELIMINATION OF THE CRITICAL ARGUMENT

At the critical point the motion is stationary and this occurs when pn = qn'. Now we will further decrease the degrees of freedom by introducing a new transformation given by a generating function S.

Here ϕ is a function of $(\xi;\eta_2, \eta_3, \eta_4, \varepsilon)$. Let us change the Hamiltonian ϕ to $F(X;Y;\varepsilon)$ by introducing a generating function S such that the new Hamiltonian F is independent of Y_2.

Let us introduce the new variables

$$(X_1, X_2, X_3, X_4; -, Y_3, Y_4, \varepsilon)$$

with the transformation defined by the equation

$$\xi_j = \frac{\partial S}{\partial \eta_j}; \; Y_j = \frac{\partial S}{\partial X_j}. \; (j = 1,2,3,4)$$

We also assume that

$$S = S_0 + S_{1/2} + S_1 + S_{3/2} + \dots$$

$$F = F_0 + F_{1/2} + F_1 + F_{3/2} + \dots$$

and

$$S_0 = X_1\eta_1 + X_2\eta_2 + X_3\eta_3 + X_4\eta_4,$$

where

$$S_j = 0(\varepsilon^j) \text{ and } F_j = 0(\varepsilon^j).$$

In general, the stationary solution will exist for the mean motion of the orbit and it will correspond to exact mean reasonance, i.e. at the point,

$$\dot{\xi}_2 = \frac{\partial \phi}{\partial \eta_2} = 0,$$

$$\dot{\eta}_2 = -\frac{\partial \phi}{\partial \xi_2} = 0, \tag{14}$$

and

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$$pn^{**} - qn' = 0. \qquad (15)$$

Here the double asterisks denote the averaged value over η_1 and η_2.

To obtain the series for S and F we will solve the Hamilton-Jacobi equation by successive approximations.

a) If we take the case of zero-order approximation then we will get

$$F_0(X_1,X_2) = \phi_0(X_1,X_2) = \frac{\mu^2}{2}(X_1 + pX_2)^{-2} + qn'X_2 \qquad (61)$$

which is constant.

b) Approximation of order $(\varepsilon^{\frac{1}{2}})$

Taking the approximation upto $O(\varepsilon^{\frac{1}{2}})$ we have

$$F_{\frac{1}{2}} = 0. \qquad (17)$$

c) Approximation of order (ε)

Taking the approximation upto $O(\varepsilon^{\frac{1}{2}})$ we have

$$\phi = \phi_0 + \phi_1$$

$$F = F_0 + F_1$$

and

$$S = S_0 + S_{\frac{1}{2}} + S_1.$$

Also from Equation (8) and taking transformations up to this order we have the Hamilton-Jacobi equation

$$\phi(X + \frac{\partial S_{\frac{1}{2}}}{\partial \eta} + \frac{\partial S_1}{\partial \eta}\,;\ \eta_2,\ \eta_3,\ \eta_4,\ \varepsilon) = F(X;\eta_3 + \frac{\partial S_{\frac{1}{2}}}{\partial X_3} + \frac{\partial S_1}{\partial X_3},$$
$$\eta_4 + \frac{\partial S_{\frac{1}{2}}}{\partial X_4} + \frac{\partial S_1}{\partial X_4},\ \varepsilon).$$

Expanding this equation in Taylor's series and considering them up to $O(\varepsilon)$ we have

$$F_1(X;\eta_3,\eta_4,\varepsilon) = \phi_1(X;\eta_2,\eta_3,\eta_4,\varepsilon) + \tfrac{1}{2}\left(\frac{\partial S_{\frac{1}{2}}}{\partial \eta_2}\right)^2 \frac{\partial^2 \phi_0}{\partial X_2^2} + \frac{\partial S_{\frac{1}{2}}}{\partial \eta_2}\frac{\partial \phi_0}{\partial X_2}.$$

In this equation both F_1 and $S_{\frac{1}{2}}$ are unknown quantities. For determining these two we consider the approximate relations:

$$\xi_2 = X_2 + \frac{\partial S_{\frac{1}{2}}}{\partial \eta_2},$$
$$Y_2 = \eta_2 + [\partial S_{\frac{1}{2}} / \partial X_2]. \qquad (18)$$

We know that X_2 is constant at any event. And by considering the Euqation (14) we see that ξ_2 is constant. Because from Equation (14) we see that up to $O(\varepsilon)$, $\partial\phi^{(1)}/\partial\eta_2 = 0$ is the necessary condition for the solution and therefore for satisfying Equation (18) we see that $S_{1/2}$ should be identically zero for the stable stationary solution.

Let $\eta_2 = \eta_2^0\ (\xi;\ \eta_3,\ \eta_4,\ \varepsilon)$ be the point of minimum of $\phi(\xi;\eta;\varepsilon)$ such that

$$\left|\frac{\partial\phi_1}{\partial\eta_2}\right|\ \eta_2 \neq \eta_2^0 = 0. \tag{19}$$

The point will exist because ϕ_1 is periodic in η_2 with period π. Now to make the condition ($S_{1/2} = 0$) sufficient for the stable stationary solution we take

$$F_1(X;\eta_3,\eta_4,\varepsilon) = \phi_1(X;\eta_2^0\ (X,\eta_3,\eta_4,\varepsilon),\ \eta_3,\eta_4,\varepsilon), \tag{20}$$

where ϕ_1 is given by Equation (12).

And the general equation defining $S_{1/2}$ is given by

$$\frac{\partial S_{1/2}}{\partial\eta_2} = \frac{L^{**}}{3p^2n^{**}}\left[-qn'-pn^{**} \pm \left\{(qn'-pn^{**})^2 - \frac{6p^2n^{**}}{L^{**}}U_1\right\}^{1/2}\right], \tag{21}$$

where

$$U_1(X;\eta_2,\eta_3,\eta_4,\varepsilon) = \phi_1(X;\eta_2,\eta_3,\eta_4,\varepsilon) - \phi_q(X;\eta_2^0(X,\eta_3,\eta_4,\varepsilon)\eta_3,\eta_4,\varepsilon)$$

At the stationary solution the condition $S_{1/2} = 0$ is satisfied by Equation (21), Also from this equation we see that in general the motion will be of circulation, asymptotic or libration in η_2 if

$$\frac{6p^2n^{**}}{L^{**}}U_1 \lesseqgtr (qn' - pn^{**})^2$$

provided η_2 is taken to be maximum. U_1 is minimum at the libration centre ($\eta_2 = \eta_2^0$) where it is zero and it is maximum at the end points of the oscillation.

The amplitude of libration is given by the equation

$$U_1(X;\eta_2,\eta_3,\eta_4,\varepsilon) = \frac{L^{**}}{6p^2n^{**}}(qn' - pn^{**})^2,$$

and is obtained as

$$\eta_2 = \bar{\eta}_2\ (X;\eta_3,\eta_4,\varepsilon).$$

which is of order (ε) in this case.

Finally up to $O(\varepsilon^{3/2})$ the Hamiltonian is given by

$$F = \frac{\mu^2}{2}(X_1 + pX_2)^{-2} + qn'X_2 + F_1(X,Y_3,Y_4,\varepsilon).$$

which is a system with two degrees of freedom.

Also the parameters of the trajectory are given by the following equations:

$$a^* = a^{**} = \text{const} = \left(\frac{p^2\mu}{q^2 n'}\right)^{1/3}$$

$$K^{**} = \text{const}$$

$$\begin{aligned}
\dot{Y}_1 &= n^{**} - \frac{\partial F_1}{\partial X_1} = n^{**} - \varepsilon\ R'(X;Y_3,Y_4),\\
\dot{Y}_2 &= pn^{**} - qn' - \frac{\partial F_1}{\partial X_2} = pn^{*} - qn' - \varepsilon\ R''(X;Y_3,Y_4)\\
\dot{X}_3 &= \frac{\partial F_1}{\partial Y_3} = \varepsilon\ h'(X,Y_3,Y_4),\\
\dot{X}_4 &= \frac{\partial F_1}{\partial Y_4} = \varepsilon\ h''(X.Y_3,Y_4),\\
\dot{Y}_3 &= -\frac{\partial F_1}{\partial X_3} = \varepsilon\ F'(X,Y_3,Y_4,t),\\
\dot{Y}_4 &= -\frac{\partial F_1}{\partial X_4} = \varepsilon\ F'(X;Y_3,Y_4,t).
\end{aligned} \tag{22}$$

The period of Y_1 is $2\pi/n^{**}$ which is short, and of Y_2 is given by $2\pi/(pn^{**}-qn')$ which is long and that of Y_3, X_3, Y_4 and X_4 is very long and given by $2\pi/n^{**}\ \varepsilon$.

d) Approximation of $O(\varepsilon^{3/2})$

Taking the approximation of $O(\varepsilon^{3/2})$ we will get

$$F_{3/2} = P_{3/2}(X;\eta_2^0(X;\eta_3,\eta_4,\varepsilon),\ \eta_3,\eta_4,\varepsilon),$$

where

$$\begin{aligned}
P_{3/2}(X;\eta_3,\eta_4,\varepsilon) = {} & \frac{\partial S_{1/2}}{\partial \eta_2}\frac{\partial \phi_1}{\partial X_2} + \frac{\partial S_{1/2}}{\partial \eta_3}\frac{\partial \phi_1}{\partial X_3} + \frac{\partial S_{1/2}}{\partial \eta_4}\frac{\partial \phi_1}{\partial X_4}\\
& + \frac{1}{6}\left(\frac{\partial S_{1/2}}{\partial \eta_2}\right)^3 \frac{\partial^3 \phi_0}{\partial X_2^3} - \frac{\partial S_{1/2}}{\partial X_3}\frac{\partial F_1}{\partial \eta_3} - \frac{\partial S_{1/2}}{\partial X_4}\frac{\partial F_1}{\partial \eta_4}.
\end{aligned}$$

Therefore

$$F_{3/2} = \left|\frac{\partial S_{1/2}}{\partial \eta_3}\frac{\partial \phi_1}{\partial X_3} - \frac{\partial S_{1/2}}{\partial X_3}\frac{\partial F_1}{\partial \eta_3} + \frac{\partial S_{1/2}}{\partial \eta_4}\frac{\partial \phi_1}{\partial X_4} - \frac{\partial S_{1/2}}{\partial X_4}\frac{\partial F_1}{\partial \eta_4}\right|_{\eta_2=\eta_2^0} \tag{23}$$

and η_2^0 in this case is given by the equation

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$$\left|\frac{\partial\phi^{(3/2)}}{\partial\eta_2}\right|_{\eta_2 = \eta_2^0} = 0 \quad \text{or} \quad \left|\frac{\partial\phi_1}{\partial\eta_2}\right|_{\eta_2 = \eta_2^0} = 0,$$

for $\phi_{1/2}$ and $\phi_{3/2}$ are zero. Hence, the location of the libration centre is not changed.

Also S_1 is given by the equation

$$\frac{\partial S^{(1)}}{\partial\eta_2} = \frac{L^{**}}{3p^2 n^{**}}\left[-(qn'-pn^{**}) \pm \left\{(qn'-np\quad)^2 - 6\,\frac{p^2 n^{**}}{L^{**}}\,(U_1+U_{\frac{3}{2}})\right]^{1/2}\right. \tag{24}$$

where

$$S^{(1)} = S_{1/2} + S_1,$$

and

$$U_{3/2}(X;\eta_2,\eta_3,\eta_4,\varepsilon) = P_{3/2}(X;\eta_2,\eta_3,\eta_4,\varepsilon) - F_{3/2}(X\eta_2,\eta_3,\eta_4,\varepsilon)\ldots \tag{25}$$

Since in general S_1 is real there are three possible motions in the variable η_2. The case of circulation, asymptotic motion and libration in η_2 occurs when

$$\left\{{}^{\max}_{\eta 2}\right\}\frac{6p^2 n^{**}}{L^{***}}\,(U_1 + U_{3/2}) \lesseqgtr (qn' - pn^{**})^2.$$

In the circulation and asymptotic cases S_1 is defined by choosing plus or minus sign. In the libration case the sign changes at the end points of oscillation where

$$\frac{6p^2 n^{**}}{L^{**}}\,(U_1 + U_{3/2}) = (qn' - pn^{**})^2,$$

which also gives the amplitude of libration and can be found from

$$\eta_2 = \bar{\bar{\eta}}_2(X;\eta_3,\eta_4,\varepsilon).$$

Now up to $O(\varepsilon^{3/2})$ the system is reduced to two degrees of freedom with the Hamiltonian given by

$$F = F_0 + F_1 + F_{3/2}.$$

where F_0, G_1 and $F_{3/2}$ are given by the Equations (16), (20) and (23).

Two integrals of motion can be found from the equations

$$\begin{aligned} a^{**} &= \text{const.} \\ & \qquad\ldots \\ K^{**} &= \text{const.} \end{aligned} \tag{26}$$

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and the other parameters of the trajectory can be found from the following six equations:

$$\dot{Y}_1 = n^{**} - \frac{\partial F_1}{\partial X_1} - \frac{\partial F_{3/2}}{\partial X_1} = n^{**} - \varepsilon\, U(X;Y_3,Y_4,\varepsilon),$$

$$\dot{Y}_2 = pn^{**} - qn' - \frac{\partial F_1}{\partial X_2} - \frac{\partial F_{3/2}}{\partial X_2} = pn^{**} - qn' - \varepsilon U'(X;Y_3,Y_4,\varepsilon),$$

$$\dot{X}_3 = \frac{\partial F_1}{\partial Y_4} + \frac{\partial F_{3/2}}{\partial Y_3} = \varepsilon\, V(X;Y_3,Y_4,\varepsilon), \tag{27}$$

$$\dot{X}_4 = \frac{\partial F_1}{\partial Y_4} + \frac{\partial F_{3/2}}{\partial Y_4} = \varepsilon\, V'(X;Y_3,Y_4,\varepsilon),$$

$$\dot{Y}_3 = -\frac{\partial F_1}{\partial X_3} - \frac{\partial F_{3/2}}{\partial X_3} = \varepsilon\, \bar{W}(X;Y_3,Y_4,\varepsilon,t),$$

$$\dot{Y}_4 = -\frac{\partial F_1}{\partial X_4} - \frac{\partial F_{3/2}}{\partial X_4} = \varepsilon\, \bar{\bar{W}}(X;Y_3,Y_4,\varepsilon,t).$$

The period of Y_1 is $2\pi/n^{**}$ which is short. The period of Y_2 is given by $2\pi/(pn^{**} - qn')$ which is long and that of X_3, X_4, Y_3 and X_4 is very long and given by $2\pi/n^{**}\, \varepsilon^{3/2}$.

4. PERTURBATIONS IN THE OSCULATING ELEMENTS UP TO $O(\varepsilon^{1/2})$

We see that up to $O(\varepsilon^0)$ there are no perturbations in the osculating elements. Up to $O(\varepsilon^{1/2})$ the variations in the osculating elements can be found out by considering the transformations:

$$\xi_j = X_j + \frac{\partial S_{1/2}}{\partial \eta_j},$$

$$\eta_j = Y_j + \frac{\partial S_{1/2}}{\partial X_j} \qquad (j = 1,2,3,4).$$

We shall first find the perturbations in Delaunay's variables and then we shall obtain the variations in the osculating elements taking terms up to $O(\varepsilon^{1/2})$. From Equations (3) we have.

$$L = x_1 + px_2, \qquad \ell = y_1,$$

$$G = x_3 + qx_2, \qquad \Omega - \lambda' = \frac{1}{q} y_2 - \frac{p}{q} y_1 - y_3,$$

$$H = qx_2 - x_4, \qquad \omega = y_3,$$

$$K = -qx_2 \quad , \qquad \omega' = y_4.$$

Also we know that

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$$L^* = \xi_1 + p\xi_2 = x_1 + \frac{\partial S_{1/2}}{\partial \eta_1} + px_2 + p\,\frac{\partial S_{1/2}}{\partial \eta_2},$$

and

$$\frac{\partial S_{1/2}}{\partial \eta_1} = 0.$$

Therefore

$$L^* = L^{**} + p\,\frac{\partial S_{1/2}}{\partial \eta_2},$$

Similarly

$$G^* = G^{**} + q\,\frac{\partial S_{1/2}}{\partial \eta_2} + \frac{\partial S_{1/2}}{\partial \eta_3},$$

$$H^* = H^{**} + q\,\frac{\partial S_{1/2}}{\partial \eta_2} - \frac{\partial S_{1/2}}{\partial \eta_4}, \qquad (28)$$

and

$$K^* = K^{**} - q\,\frac{\partial S_{1/2}}{\partial \eta_2}.$$

The variation of the mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\,[L^{**} + p\,\frac{\partial S_{1/2}}{\partial \eta_2}]^2.$$

Substituting the value of $\partial S_{1/2}/\partial \eta_1$ from Equation (21) in this equation we have

$$a^* = a_0^* \pm \Delta a^*, \qquad (29)$$

where

$$a^* = a^{**}\,(\frac{5}{3} - \frac{2}{3}\,\frac{qn'}{pn^{**}}), \qquad (30)$$

$$\Delta a^* = \frac{2}{3}\,a^{**}\,[(1 - \frac{qn'}{pn^{**}}) - \frac{1}{n^{**}L^{**}}\,U_1]^{1/2}. \qquad (31)$$

For a stationary solution, we have that

$$a^* = a_0^* = a^{**},$$

but, in general, the maximum variation from the mean value a_0^* is given by putting $\eta_2 = n_2^0$ in Equation (31), i.e.,

$$(\Delta a^*)_{max.} = \frac{2}{3}\,a^{**}(1 - \frac{qn'}{pn^{**}}).$$

Also from the second and third relation to Equation (28) we see that at exact resonance

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$$G^* = G^{**} \text{ and } H^* = H^{**}.$$

The variation in eccentricity and inclination can be found if the system of equation a* = const. and K* = const. are completely integrated.

Similarly we can find the variations in the angular variables as follows:

$$\begin{aligned} \ell^* &= \ell^{**} - \frac{\partial S_{\frac{1}{2}}}{\partial L^{**}} - \frac{\partial S_{\frac{1}{2}}}{\partial G^{**}}, \\ \omega^* &= \omega^{**} + \frac{\partial S_{\frac{1}{2}}}{\partial G^{**}}, \\ \Omega^* &= \Omega^{**} - \frac{\partial S_{\frac{1}{2}}}{\partial K^{**}} - \frac{\partial S_{\frac{1}{2}}}{\partial G^{**}}, \\ \omega'^* &= \omega'^{**} - \frac{\partial S_{\frac{1}{2}}}{\partial H^{**}}. \end{aligned} \tag{32}$$

5. PERTURBATIONS IN THE OSCULATING ELEMENTS UP TO $O(\varepsilon)$

In this case the transformations are

$$\begin{aligned} \xi_j &= X_j + \frac{\partial S_{\frac{1}{2}}}{\partial \eta_j} + \frac{\partial S_1}{\partial \eta_j}, \\ \xi_j &= Y_j + \frac{\partial S_{\frac{1}{2}}}{\partial X_j} + \frac{\partial S_1}{\partial X_j}. \end{aligned} \qquad (j=1,2,\ ,\ )$$

Again, also in this case, we shall first find the perturbations in the Delaunay variables and from that we shall obtain the variation in the osculating elements taking terms up to $0\ (\varepsilon)$.

Following the same procedure as in Section 4, the variations in the Delauny variables are given by

$$\begin{aligned} L^* &= L^{**} + p\,\frac{\partial S^{(1)}}{\partial \eta_2}, \\ G^* &= G^{**} + q\,\frac{\partial S^{(1)}}{\partial \eta_2} + \frac{\partial S^{(1)}}{\partial \eta_3}, \\ H^* &= H^{**} + q\,\frac{\partial S^{(1)}}{\partial \eta_2} - \frac{\partial S^{(1)}}{\partial \eta_4}, \\ K^* &= K^{**} - q\,\frac{\partial S^{(1)}}{\partial \eta_2}. \end{aligned} \tag{33}$$

where

$$S^{(1)} = S_{\frac{1}{2}} + S_1.$$

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The variation in the mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\,[L^{**} + p\,\frac{\partial S^{(1)}}{\partial \eta_2}]^2 .$$

Simplifying this result we get

$$a^* = a_0^* \pm \Delta a^*, \qquad (34)$$

where

$$a_0^* = a^{**}\,(\frac{5}{3} - \frac{2}{3}\,\frac{qn'}{pn^{**}}),$$

and

$$\Delta a^* = \frac{2}{3}\,a^{**}\,[(1 - \frac{qn'}{pn^{**}}) - \frac{6}{n^{**}L^{**}}\,(U_1 + U_{3/2})]^{1/2}. \qquad (35)$$

At exact resonance, we have, as before,

$$a^* = a_0^* = a^{**}.$$

In general, the maximum variation from the mean semi-major axis a_0^* is obtained by putting $\eta_2 = \eta_2^0$ in Equation (35), i.e.,

$$(\Delta a^*)_{max.} = \frac{2}{3}\,a^*\,\left|1 - \frac{qn'}{pn^{**}}\right|.$$

The variations in eccentricity and inclination can be found if integrals of Equation (26) are completely known. The variations in the angular variables are given by

$$\ell^* = \ell^{**} - \frac{\partial S^{(1)}}{\partial L^{**}} - \frac{\partial S^{(1)}}{\partial G^{**}},$$

$$\omega^* = \omega^{**} + \frac{\partial S^{(1)}}{\partial G^{**}},$$

$$\Omega^* = \Omega^{**} - \frac{\partial S^{(1)}}{\partial K^{**}} - \frac{\partial S^{(1)}}{\partial G^{**}}.$$

$$\omega'^* = \omega'^{**} - \frac{\partial S^{(1)}}{\partial H^{**}}.$$

Hence we can find perturbations in all the osculating elements.

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