Glasgow Mathematical Journal (2023), **65**, pp. 446–449
doi:10.1017/S0017089523000034

Glasgow Mathematical Journal

RESEARCH ARTICLE

Some characterizations of expanding and steady Ricci solitons

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Received: 11 October 2022; **Revised:** 14 February 2023; **Accepted:** 17 February 2023; **First published online:** 13 March 2023

Keywords: steady, expandig, cigar, Ricci soliton, integrability

2020 Mathematics Subject Classification: *Primary* - 53C42; *Secondary* - 53C21, 53C44

Abstract
In this short note, we deal with complete noncompact expanding and steady Ricci solitons of dimension $n \geq 3$. More precisely, under an integrability assumption, we obtain a characterization for the generalized cigar Ricci soliton and the Gaussian Ricci soliton.

1. Introduction

A gradient Ricci soliton is a Riemannian manifold Σ satisfying

$$Ric + \nabla^2 f = \lambda g,$$

where Ric denotes the Ricci tensor, $f : \Sigma \to \mathbb{R}$ is a smooth function, and $\lambda \in \mathbb{R}$. A Ricci soliton is called expanding, steady or shrinking if, respectively, $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Ricci flow was introduced by Hamilton in his seminal work [6] to study closed three manifolds with positive Ricci curvature. Ricci solitons generate self-similar solutions to the Ricci flow and often arise as singularity models of the flow; therefore, it is important to study and classify them in order to understand the geometry of singularities.

A standard example of expanding Ricci soliton is given by $(\mathbb{R}^n, g_0, -\frac{|x|^2}{4})$, where g_0 is the Euclidean metric. In fact, note that $Ric + \nabla^2 f = -\frac{1}{2}$. We recall that an expanding Ricci soliton is related to the limit solution of Type III singularities of the Ricci flow, see [7]. Besides, the characterization of expanding Ricci soliton has attracted the attention of many researchers, see for instance [2, 3, 8–11].

In the steady case, Hamilton [6] discovered the first example of a complete noncompact steady soliton on $\mathbb{R}^2$ called the cigar soliton, where the metric is given by $ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$ with potential function $f(x, y) = -\log(1 + x^2 + y^2)$, $(x, y) \in \mathbb{R}^2$. The cigar has positive Gaussian curvature $R = 4e^f$ and linear volume growth, and it is asymptotic to a cylinder of finite circumference at infinity. In the three-dimensional case, the known examples are given by quotients of $\mathbb{R}$, $\mathbb{R} \times \Sigma^2$, where Σ^2 is the cigar soliton, and the rotationally symmetric one constructed by Bryant [1].

We say that Σ is a *generalized cigar soliton*, if Σ is isometric to $M \times \mathbb{R}^{n-2}$, where M is the cigar soliton. Recently, Deruelle [5] obtained the following rigidity result to generalized cigar soliton

Theorem 1. *Let Σ be a complete nonflat noncompact steady gradient Ricci soliton of dimension $n \geq 3$ such that the sectional curvature is nonnegative and $R \in L^1(\Sigma)$. Then the universal covering of Σ is isometric to $M \times \mathbb{R}^{n-2}$, where M is the cigar soliton.*

Dedicated to my daughter Aurora Vitória.

In [2], Catino et al. obtained a suitable Bochner-type formula for the tensor $\left(Ric - \frac{R}{2}\right) e^{-f}$, where R is the scalar curvature, to guarantee that the condition $R \in L^1(\Sigma)$ in the above theorem can be relaxed to $\liminf_{r\to\infty} \frac{1}{r} \int_{B_r(0)} R = 0$. Besides, using a similar strategy they were able to prove the following rigidity result addressed to expanding Ricci solitons

Theorem 2. *Let Σ be a complete noncompact expanding gradient Ricci soliton of dimension $n \geq 3$ such that the sectional curvature is nonnegative. If $R \in L^1(\Sigma)$, then Σ is isometric to a quotient of the Gaussian soliton $\mathbb{R}^n$.*

In this paper, motivated by Deruelle [5] and Catino et al. [2], we obtain rigidity results for steady and expanding Ricci solitons under an assumption that the scalar curvature lies in $L^p(\Sigma)$, with respect to a suitable volume element. We point out that our rigidity results are obtained from a different approach. Now, we can state our first result.

Theorem 3. *Let Σ be a complete noncompact steady gradient Ricci soliton of dimension $n \geq 3$ such that the sectional curvature is nonnegative. If $Re^{-f} \in L^p_{-f}(\Sigma)$, $p > 1$, then Σ is either isometric to a quotient of $\mathbb{R}^n$ or $M \times \mathbb{R}^{n-2}$, where M is the cigar soliton.*

We recall that, from [4], a complete three-dimensional noncompact steady gradient Ricci soliton has nonnegative scalar curvature. Thus, we conclude that

Corollary 1. *Let Σ be a complete three-dimensional noncompact steady gradient Ricci soliton. If $Re^{-f} \in L^p_{-f}(\Sigma)$, $p > 1$, then Σ is either isometric to a quotient of $\mathbb{R}^3$ or $M \times \mathbb{R}$, where M is the cigar soliton.*

Analogously, we can apply the same ideas of Theorem 3 to guarantee a rigidity result addressed to complete noncompact expanding gradient Ricci soliton as follows.

Theorem 4. *Let Σ be a complete noncompact expanding gradient Ricci soliton of dimension $n \geq 3$ such that the sectional curvature is nonnegative. If $Re^{-f} \in L^p_{-f}(\Sigma)$, $p > 1$, then Σ is isometric to a quotient of the Gaussian soliton $\mathbb{R}^n$.*

2. Proof of the theorems

Let ψ be a smooth function on Σ, let us define the *weighted Laplacian* on Σ^n by

$$\Delta_\psi \varphi = \Delta\varphi - \langle \nabla\psi, \nabla\varphi \rangle$$

for all $\varphi \in C^\infty(\Sigma^n)$, where $\langle , \rangle$ denotes the Riemannian metric on Σ.

In what follows, we denote the space of Lebesgue integrable functions on Σ^n by

$$L^1(\Sigma^n) = \left\{ \varphi \in C^\infty(\Sigma^n) : \int_{\Sigma^n} |\varphi| d\Sigma < +\infty \right\},$$

where $d\Sigma$ stands for the volume element induced by the metric of Σ^n. Furthermore, given a smooth function $\psi : \Sigma \to \mathbb{R}$, we denote by $L^1_\psi(\Sigma^n)$ the set of Lebesgue integrable functions on Σ^n with respect to the modified volume element

$$d\mu = e^{-\psi} d\Sigma.$$

Given an oriented Riemannian manifold Σ^n and $p > 1$, we can consider the following space of integrable functions

$$L^p_\psi(\Sigma^n) = \{\varphi \in C^\infty(\Sigma^n) : |\varphi|^p \in L^1_\psi(\Sigma^n)\}.$$

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From a straightforward adaptation of [12, Theorem 3], we obtain the following criterion of integrability.

Lemma 1. *Let Σ^n be an n-dimensional complete oriented Riemannian manifold. If $\varphi \in C^\infty(\Sigma^n)$ is a nonnegative ψ-subharmonic function on Σ^n and $\varphi \in L^p_\psi(\Sigma^n)$, for some $p > 1$, then φ is constant.*

Now, we can prove our main result.

Proof of Theorem 3. Let $k \in \mathbb{R}$ be a constant. Thus, a straightforward calculation shows that

$$\Delta(Re^{kf}) = e^{kf}(\Delta R + 2k\langle \nabla f, \nabla R\rangle + kR\Delta f + k^2 R|\nabla f|^2). \tag{2.1}$$

Since Σ is a steady gradient Ricci soliton, from Lemma 2.3 of [10], we have

$$\Delta R = -2|Ric|^2 + \langle \nabla R, \nabla f\rangle. \tag{2.2}$$

Note that

$$e^{kf}\langle \nabla R, \nabla f\rangle = \langle \nabla(e^{kf}R), \nabla f\rangle - Rke^{kf}|\nabla f|^2. \tag{2.3}$$

Plugging (2.3) and (2.2) into (2.1) and taking the trace of the steady soliton equation, we conclude that:

$$\Delta(Re^{kf}) - (2k+1)\langle \nabla(e^{kf}R), \nabla f\rangle = e^{kf}(-2|Ric|^2 - kR^2 + R|\nabla f|^2(-k^2 - k)).$$

Finally, from the definition of weighted Laplacian, we get that

$$\Delta_{(2k+1)f}(Re^{kf}) = e^{kf}(-2|Ric|^2 - kR^2 + R|\nabla f|^2(-k^2 - k))$$

Choosing $k = -1$, we conclude that

$$\Delta_{-f}(Re^{-f}) = e^{-f}(-2|Ric|^2 + R^2).$$

Since the sectional curvature of Σ is nonnegative, we get that $-2|Ric|^2 + R^2 \geq 0$. In fact, given λ_k, $k = 1, 2, ..., n$, the eigenvalue of the Ricci tensor, it is not hard to see that $\sum_{i\neq j} \lambda_i > \lambda_j$ and, therefore, $R \geq 2\lambda_j$. Thus,

$$2|Ric|^2 = 2\sum \lambda_i^2 \leq R\sum \lambda_i = R^2.$$

From above inequality, we conclude that

$$\Delta_{-f}(Re^{-f}) = e^{-f}(-2|Ric|^2 + R^2) \geq 0.$$

On the other hand, since Re^{-f} is a nonnegative function and $Re^{-f} \in L^p_{-f}(\Sigma)$, from Lemma 1, we conclude that Re^{-f} is a constant. If R is constant zero, from [5], Σ is isometric to a quotient of $\mathbb{R}^n$. If $Re^{-f} = c$, where c is a nonzero constant, we get that Σ has finite $-f$-volume and, therefore, $R \in L^1(\Sigma)$. From [5], we conclude the desired result. □

We recall that a complete three-dimensional steady gradient Ricci soliton has nonnegative sectional curvature. Thus, as a consequence of anterior result, we get that

Corollary 2. *Let Σ be a complete three-dimensional noncompact steady gradient Ricci soliton. If $Re^{-f} \in L^p_{-f}(\Sigma)$, $p > 1$, then Σ is either isometric to a quotient of $\mathbb{R}^3$ or $M \times \mathbb{R}$, where M is the cigar soliton.*

Now, we are able to prove our rigidity result, in the expanding case, as follows.

Proof of Theorem 4. In fact, since we are supposing that $Ric + \nabla^2 f = \lambda g$, from Lemma 2.3, [10], we conclude that

$$\Delta R = -2|Ric|^2 + 2R\lambda + \langle \nabla R, \nabla f\rangle.$$

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Thus, following the same steps of the anterior result, we conclude from (2.1) and above equation that

$$\Delta(Re^{kf}) - (2k+1)\langle\nabla(e^{kf}R), \nabla f\rangle = e^{kf}(-2|Ric|^2 + 2R\lambda + kR(n\lambda - R) + R|\nabla f|^2(-k^2 - k)).$$

Again, choosing $k = -1$, we conclude that

$$\Delta_{-f}(Re^{-f}) = e^{-f}(-2|Ric|^2 + R^2 + R(2-n)\lambda) \tag{2.4}$$

Since the sectional curvature is nonnegative, reasoning like the anterior result, we get that $-2|Ric|^2 + R^2 \geq 0$. Taking into account that $\lambda < 0$, we get that

$$\Delta_{-f}(Re^{-f}) \geq 0.$$

Finally, from Lemma 1, we get that Re^{-f} is a constant and, therefore, from (2.4) we guarantee that $R = 0$. Since Σ has nonnegative sectional curvature, we conclude that Σ has sectional curvature equals to zero. Thus, we conclude that Σ must be a quotient of the Gaussian soliton $\mathbb{R}^n$. □

Acknowledgments. The author is partially supported by Paraíba State Research Foundation (FAPESQ), Brazil, grant 3025/2021 and CNPq, Brazil, grant 306524/2022-8, respectively.

Data availability. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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