FATIGUE SPACES IN ELECTROMAGNETIC-GRAVITATIONAL THEORY

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1. Introduction. In an n-dimensional manifold V^n, coordinates x^i for $i = 1, \ldots, n$, let each curve $x(t)$ for $t_0 \leq t \leq t_1$ of class C^1 define a corresponding λ by means of the integral equation

$$\lambda(t_1) = \lambda(t_0) + \int_{t_0}^{t_1} L(x, \dot{x}, \lambda) d\tau. \tag{1}$$

For a given $\lambda(t_0) = \lambda_0$, the problem of minimizing $\lambda = \lambda(t_1)$ given by (1) relative to curves joining given points x_0 and x_1 in V^n can be interpreted as a Bolza problem of minimizing

$$J = \lambda(t_0) + \int_{t_0}^{t_1} L(x, \dot{x}, \lambda) d\tau$$

in the $n+1$ dimensional space of (x, λ), subject to the restraint

$$\varphi(x, \lambda, \dot{x}, \dot{\lambda}) \equiv \dot{\lambda} - L(x, \dot{x}, \lambda) = 0$$

If L is of class C^3 in a region R of the $2n+1$ dimensional space $(x, \lambda, \dot{x})$ and if an extremal $(x(t), \lambda(t))$ of class C^1 (except at corners) exists which (along with the derivatives $\dot{x}$) lies within R, then the multiplier method applies ([1] pp. 189-202) since $\varphi_{\dot{\lambda}} \neq 0$ and the curve satisfies the Euler-Lagrange equations

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$$\frac{d}{dt} F_{\dot{x}^i} = F_{x^i} \qquad \frac{d}{dt} F_{\dot{\lambda}} = F_{\lambda}$$

where $F = L + \mu\varphi$ and $\mu = \mu(t)$ is the multiplier. Elimination of μ from this equation, as in [2] leads to the equations

$$\text{(2)} \qquad \frac{d}{dt} L_{\dot{x}^i} - L_{x^i} = L_{\dot{x}^i} L_{\lambda}, \qquad \dot{\lambda} = L$$

as necessary conditions for an extremum.

The system (2) consists in $n+1$ differential equations from which the $n+1$ unknowns x^i, λ may be found in general provided that the equations are independent.

2. An Application to Elementary Mechanics. In the particular case

$$\text{(3)} \qquad \lambda(t) = \int_{t_0}^{t} [T - V + \frac{\nu}{m}\lambda]\, d\tau + \lambda(t_0)$$

where T is the kinetic, V the potential energy,

$$T = \frac{m}{2} g_{ij}\dot{x}^i\dot{x}^j \qquad V = V(x^k)$$

it is easily verified that the right hand side of (2) gives rise to a resistance term in addition to the usual conservative force field, so that the extremals of (3) are given by

$$\text{(4)} \qquad \frac{\delta}{\delta t}(mv_i) = -\frac{\partial V}{\partial x^i} + \nu v_i \quad \text{where} \quad v_i = g_{ij}\dot{x}^j.$$

Proceeding as in elementary mechanics, the total energy $T + V$ along the trajectories satisfying Newton's law (4) can be found as follows:

$$\frac{d}{dt} T = \frac{\delta}{\delta t}(mv_i)\dot{x}^i$$

and substitution from (4) yields

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$$\frac{d}{dt} T = -\frac{d}{dt} V + 2\frac{\nu}{m} T$$

which may be written in the form

$$\frac{d}{dt}\left\{\frac{m}{\nu}(T + V)\right\} - \frac{\nu}{m}\left\{\frac{m}{\nu}(T + V)\right\} = T - V. \tag{5}$$

However, $\lambda(t)$ as defined by (3) along the extremals satisfies

$$\frac{d}{dt} \lambda(t) = T - V + \frac{\nu}{m} \lambda(t)$$

and comparing with (5) it follows that

LEMMA 1. Along the extremals of (3), viz. the curves satisfying Newton's law in the case of a conservative force field and resistance (4), the action λ is proportional to the mass times the total energy

$$\nu\lambda = m(T + V) \quad \text{for all } t, \tag{6}$$

provided $\lambda(t_0)$ is initially so chosen.

As in the classical case, the action integral (3) can be transformed into parameter invariant form with the aid of (6). Formally, by (6) the integral (3) becomes

$$\lambda(t) = \int_{t_0}^{t} 2T\, d\tau + \lambda(t_0); \ \lambda(t) = \frac{m}{\nu}(T + V)$$

and the standard procedure yields

$$\lambda(t) = \int_{t_0}^{t} \sqrt{2(\nu\lambda - mV)g_{ij}\dot{x}^i\dot{x}^j}\, d\tau + \frac{m}{\nu}(T_0 + V_0) \tag{7}$$

where T_0 and V_0 denote the value of the kinetic and potential energies initially.

In the form (7) the differential geometry of [3] and [4] is applicable, and since the auto-parallel curves were shown obtainable from (7) by considering $\lambda = \lambda_0$ fixed under the integral before applying the variational methods, it follows that the equations of motion

$$\frac{\delta}{\delta t}(mv_i) = -\frac{\partial V}{\partial x^i} \tag{8}$$

are the auto-parallel curves in this geometry; the constant total energy is given by $T + V = \frac{\nu}{m}\lambda_0$. Hence equations (8) and (4) are contained within the geometry of (7) as auto-parallel and geodesic curves respectively.

3. Geometry of the Electromagnetic Action Integral. The action integral for Electromagnetic theory is usually taken in the form [5]

$$\lambda = \int \sqrt{\pm g_{ij}(x)\dot{x}^i\dot{x}^j} + A_i(x)\dot{x}^i \, d\tau \tag{9}$$

corresponding to a Finsler geometry; the vector potential A_i defines the electromagnetic field tensor $F_{ij} = A_{j/i} - A_{i/j}$ where "/" denotes covariant differentiation relative to the tensor g_{ij}.

The indicatrices of (9), viz. the set of $\dot{x}^i$'s satisfying

$$\sqrt{\pm g_{ij}(x)\dot{x}^i\dot{x}^j} + A_i(x)\dot{x}^i = 1,$$

may be written in the form

$$(\pm g_{ij} - A_iA_j)\dot{x}^i\dot{x}^j + 2A_i\dot{x}^i = 1$$

and hence define a "conic" (in n-space) whose center may be found by "completing the square". Set

$$\overset{*}{g}_{ij} = \pm g_{ij} - A_iA_j, \quad \overset{*}{g}{}^{ij}\overset{*}{g}_{jk} = \delta^i_k \tag{10}$$

(we have assumed $\det(\overset{*}{g}_{ij}) \neq 0$, but it will be shown that this is the case if $\det g_{ij} \neq 0$ and $g^{ij}A_iA_j \neq \pm 1$), then the indicatrix may be written

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$$\overset{*}{g}_{ij}(\dot{x}^i + \overset{*}{g}{}^{ir} A_r)(\dot{x}^j + \overset{*}{g}{}^{js} A_s) = 1 + \overset{*}{g}{}^{rs} A_r A_s$$

from which it is clear that the center of the conic is at the point

$$\dot{x}^i = -\overset{*}{g}{}^{ir} A_r \tag{11}$$

in the tangent space. Hence the Finsler space associated with (9) is similar to a Riemannian space except that the conic (the unit circle in the Minkowsky tangent space) is centered at $-\overset{*}{g}{}^{ir} A_r$ instead of at $\dot{x}^i = 0$.

Alternatively, the Monge cone at (x_0^i, λ_0) (see fig. 1) for the Hamilton-Jacobi equation

$$H(x^i, p_i) = 1 \quad \text{where } p_i = \frac{\partial S}{\partial x^i},$$

associated with (9) is centered about the ray determined by the vector with components $y^i = -\overset{*}{g}{}^{ir} A_r$ for $i = 1, \ldots, n$ and $y^{n+1} = 1$ in the tangent space at (x_0^i, λ_0). If the axes of these cones determines a direction field whose integral curves form a schlicht covering of some sufficiently large region R_{n+1}, then these curves may also be used to define a coordinate transformation of (a subset of) R_{n+1} as follows: any point p in R_{n+1} determines a unique integral curve Γ (see fig. 2) intersecting the subspace $\lambda = 0$ at say $(\bar{x}_0^i, 0)$. If (x_0^i, λ_0) are the coordinates of p, then the coordinate transformation T is defined by

$$T: (x_0^i, \lambda_0) \leftrightarrow (\bar{x}_0^i, \lambda_0).$$

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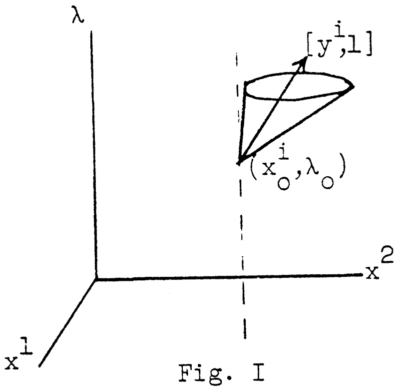

Fig. I

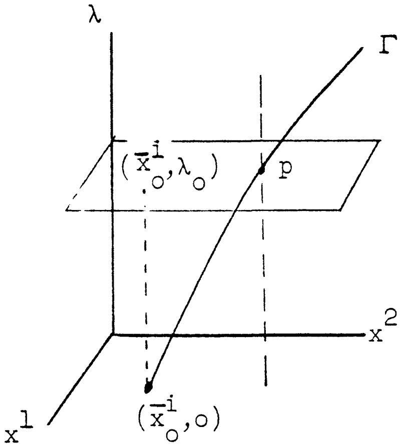

Fig. 2

Clearly λ is fixed under T. The Monge cones are now centered about the new coordinate curves $\bar{x}^i = \bar{x}^i_0$ and (since the cones were conics) the Hamilton-Jacobi equation is similar to that of a Riemannian geometry. Since the transformation depends on λ however, the partial differential equation will have the form

$$\bar{H}(\bar{x}^i, S, \bar{p}_i) = 1 \qquad \text{where } \bar{p}_i = \frac{\partial S}{\partial \bar{x}^i},$$

that is, the H-J equation for a Riemannian fatigue geometry ([3], [4]).

It is therefore to be expected that <u>any action integral of the form</u> (9) <u>may be transformed into an action integral of the form</u>

$$\lambda(t_1) = \lambda(t_0) + \int_{t_0}^{t_1} \sqrt{\bar{h}_{ij}(\bar{x}, \lambda)\dot{\bar{x}}^i \dot{\bar{x}}^j}\, dt \tag{12}$$

<u>by a transformation leaving</u> λ <u>invariant, and that this transformation is determined by the system of differential equations</u> (11). We proceed now to verify analytically the above assertion.

4. <u>The (Local) Group of Transformations.</u>[1] It follows

[1] The author gratefully acknowledges the many suggestions of the referee regarding this section.

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from (9) that the action λ satisfies $\dot{\lambda} - < A(x), \dot{x} >_g = \sqrt{\pm g(x; \dot{x}, \dot{x})}$ where $<,>_g$ denotes the inner product relative to the metric tensor g. This may be written in the alternate form

$$\dot{\lambda}^2 = \overset{*}{g}(x;\dot{x}, \dot{x}) + 2\dot{\lambda} < A(x), \dot{x} >_g \tag{13}$$

where $\overset{*}{g} = \pm g - A \otimes A$ as in (10) and $\otimes$ denotes tensor products.

LEMMA 2. <u>The tensor</u> $\overset{*}{g}$ <u>is non-singular if and only if</u> g <u>is non-singular and</u> $< A, A >_g \neq \pm 1$.

<u>Proof.</u>[1] Consider the determinant of the augmented matrix

$$\begin{vmatrix} 1 & A_j \\ 0 & \pm g_{ij} - A_i A_j \end{vmatrix} = \begin{vmatrix} 1 & A_j \\ A_i & \pm g_{ij} \end{vmatrix}$$

where the right side was obtained by elementary row operations. An expansion by the first rows yields $|\pm g_{ij}| - A_j A_i \operatorname{cof}(\pm g_{ij})$ on the right from which the lemma readily follows.

The action λ as given in (12) satisfies

$$\dot{\lambda}^2 = \underset{\lambda}{h}(\bar{x};\dot{\bar{x}}, \dot{\bar{x}}) \tag{14}$$

in terms of a metric tensor $\underset{\lambda}{h}$ which depends on λ; the previous section then indicates that (13) and (14) are equivalent under λ dependent coordinate transformations. Specifically, we consider a one parameter family of transformations $\bar{x} = \varphi_\lambda(x)$ of class C^3 in x and λ, defined on some region U of a manifold V^n; let $\varphi_\lambda^{-1}(\bar{x})$ denote the inverse transformation also of class C^3. We may consider φ_λ as defining a mapping of U onto a region

[1] The author gratefully acknowledges the many suggestions of the referee regarding this section.

U_λ; for fixed x and variable λ, $\varphi_\lambda(x)$ traces a curve whose tangent vector is $\overset{\circ}{\varphi}_\lambda(x)$ at $x_\lambda = \varphi_\lambda(x)$, where "$\circ$" denotes differentiation with respect to λ. The mapping φ_λ induces the standard mapping $\varphi'_\lambda(x)$ of vectors ζ in the tangent space at x onto the vectors $\varphi'_\lambda(x)\zeta$ in the tangent space at x_λ. The inverse mapping $\varphi'^{-1}_\lambda(x_\lambda)$ exists at x_λ if the Jacobian of φ_λ does not vanish at x.

If h is a metric tensor defined on U, then φ_λ induces a metric tensor $\underset{\lambda}{h}$ defined on U_λ by

$$\underset{\lambda}{h}(x_\lambda; \zeta_\lambda, \mu_\lambda) = h\{\varphi^{-1}_\lambda(x_\lambda); \varphi'^{-1}_\lambda(x_\lambda)\zeta_\lambda, \varphi'^{-1}_\lambda(x_\lambda)\mu_\lambda\} \tag{15}$$

for arbitrary vectors ζ_λ, μ_λ in the tangent space at x_λ. Hence $\underset{\lambda}{h}$ is also bi-linear in its vector arguments. Any curve $x(t)$ in U is deformed by φ_λ into a curve in U_λ. If further $\lambda = \lambda(t)$ also depends on t the curve is further deformed into $\bar{x}(t) = \varphi_{\lambda(t)}(x(t))$ and the tangent vector at $\bar{x}$ is

$$\dot{\bar{x}} = \overset{\circ}{\varphi}_\lambda(x)\,\dot{\lambda} + \varphi'_\lambda(x)\dot{x}.$$

By the bi-linearity of $\underset{\lambda}{h}$ in (15), it follows that

$$\underset{\lambda}{h}(\bar{x}; \dot{\bar{x}}, \dot{\bar{x}}) = h(x; \dot{x}, \dot{x}) + 2\dot{\lambda}\, h(x; \dot{x}, B) + \dot{\lambda}^2 h(x; B, B) \tag{16}$$

where

$$B = \varphi'^{-1}_\lambda(\bar{x})\, \overset{\circ}{\varphi}_\lambda(x). \tag{17}$$

If (16) is substituted in (14) and the result compared to (13), it follows that

LEMMA 3. For (14) to be equivalent to (13) under the transformation φ_λ and (15), it is necessary and sufficient that

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(18) $$\overset{*}{g}(x;\dot{x},\dot{x}) = h(x;\dot{x},\dot{x})\ \{1 - <B,B>_h\}^{-1}$$

(19) $$<A(x),\dot{x}>_g = <B,\dot{x}>_h \{1 - <B,B>_h\}^{-1}$$

for all $\dot{x}$.

But (18) implies $<B,B>_h$ independent of λ and hence (19) implies $<B,\dot{x}>_h$ also independent of λ for all $\dot{x}$; therefore $B = B(x)$ is independent of λ and (17) may be written

(17') $$\overset{\circ}{\varphi}_\lambda(x) = \varphi'_\lambda(x)\, B(x)\ .$$

Since $\varphi_\lambda(\varphi_\lambda^{-1}(x)) \equiv x$, differentiation with respect to λ yields

$$\overset{\circ}{\varphi}_\lambda(\varphi_\lambda^{-1}(x)) = - \varphi'_\lambda(\varphi_\lambda^{-1}(x)) \cdot \overset{\circ}{\varphi}{}_\lambda^{-1}(x)$$

and replacing x by $\varphi_\lambda^{-1}(x)$ in (17') yields

(20) $$\overset{\circ}{\varphi}{}_\lambda^{-1}(x) = - B(\varphi_\lambda^{-1}(x)).$$

It follows [6] that φ_λ^{-1} form a local one parameter group of transformations with inverses $\varphi_{-\lambda} = \varphi_\lambda^{-1}$. Replacing λ by $-\lambda$ in (20) introduces a minus sign because of "$\circ$" and yields

LEMMA 4. <u>In order that</u> (14) <u>and</u> (13) <u>be equivalent under</u> φ_λ <u>and</u> (15), <u>it is necessary and sufficient that the local group of transformations</u> φ_λ <u>satisfy</u>

(21) $$\overset{\circ}{\varphi}_\lambda(x) = B(\varphi_\lambda(x))$$

<u>where the vector field</u> B <u>is related to</u> A <u>by</u> (19).

It remains only to relate (21) and (11). In the notation of the previous section, (18) and (19) become

(18') $$\overset{*}{g}_{ij} = \{1 - <B,B>_h\}^{-1}\, h_{ij}$$

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(19') $$g_{ij} A^j = \{1 - \langle B, B \rangle_h\}^{-1} h_{ij} B^j .$$

Hence,

$$\overset{*}{g}{}^{ij} = (1 - \langle B, B \rangle_h) h^{ij}$$

and by (19')

(22) $$\overset{*}{g}{}^{ij} A_j = B^i .$$

Hence (11) can be written $\dot{x}^i = - B^i(x)$ and comparison with (20) verifies the assertion of the previous section.

We derive now for future reference the following formulas.

LEMMA 5.

(i) $$\overset{*}{g}{}^{ij} = g^{ij} + \frac{A^i A^j}{1 - \langle A, A \rangle_g} .$$

(ii) $$h_{ij} = \frac{\overset{*}{g}_{ij}}{1 + \overset{*}{g}{}^{ij} A_i A_j}$$

(iii) $$\langle A, A \rangle_g = \langle B, B \rangle_h = \frac{\overset{*}{g}{}^{ij} A_i A_j}{1 + \overset{*}{g}{}^{ij} A_i A_j} .$$

<u>Proof.</u> (i) may be verified directly since $\overset{*}{g}{}^{ir} \overset{*}{g}_{rj} = \delta^i_j$. Substitution of (22) in (18') yields

$$\overset{*}{g}{}^{rs} A_i A_j = \frac{\langle B, B \rangle_h}{1 - \langle B, B \rangle_h}$$

and using (i) yields (iii). Since $1 - \langle B, B \rangle_h = (1 + \overset{*}{g}{}^{ij} A_i A_j)^{-1}$ by (iii), (ii) follows from (18').

We recapitulate these results in the alternate form:

THEOREM 1. <u>Given a non-singular metric tensor</u> g_{ij},

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and define $\overset{*}{g}_{ij} = \pm g_{ij} - A_i A_j$. If $g^{ij} A_i A_j \neq \pm 1$, then $\overset{*}{g}{}^{ij}$ exists, given in lemma 5. Then the system of differential equations

$$\frac{dx^i}{d\lambda} = - \overset{*}{g}{}^{ij} A_j$$

defines a (local) group of coordinate transformations $\bar{x} = \varphi_\lambda(x)$ under which the action integral

$$(9) \qquad \lambda(t) \int_{t_0}^{t} \sqrt{\pm g(x; \dot{x}, \dot{x})} + A_i \dot{x}^i \, d\tau + \lambda_0$$

is transformed into the form

$$(12') \qquad \lambda(t) = \int_{t_0}^{t} \sqrt{\underset{\lambda}{h}(\bar{x}; \dot{\bar{x}}, \dot{\bar{x}})} \, d\tau + \lambda_0$$

where $\underset{\lambda}{h}$ is defined by (15) in terms of h, and h by lemma 5 in terms of g. Definition (15) of $\underset{\lambda}{h}$ corresponds to

$$(23) \qquad \underset{\lambda}{h}_{ij}(\bar{x}) = \bar{h}_{ij}(\bar{x}, \lambda) = \frac{\delta x^r}{\delta \bar{x}^i} \frac{\delta x^a}{\delta \bar{x}^j} h_{rs}(x).$$

Proof. Since (13) and (14) are equivalent differential equations, and $\lambda(t_0) = \lambda_0$ in both action integrals, the theorem follows provided the proper sign is chosen for the roots.

Hence any electromagnetic action integral (9) can be transformed into the form (12) by means of action dependent coordinate transformations φ_λ; the converse is not true in general in view of (15) and (18). In (12') the metric tensor may be written in the form $\bar{h}(\bar{x}, \lambda; \zeta, \mu)$ rather than $\underset{\lambda}{h}(\bar{x}; \zeta, \mu)$ and hence (15) implies the existence of a metric tensor $\underset{0}{\bar{h}}(\bar{x}; \zeta, \mu)$ such that

$$\bar{h}\{\bar{\varphi}_\lambda(\bar{x}), \lambda; \overset{\cdot}{\bar{\varphi}}_\lambda(\bar{x})\zeta, \overset{\cdot}{\bar{\varphi}}_\lambda(\bar{x})\mu\} = \underset{0}{\bar{h}}(\bar{x}; \zeta, \mu).$$

Differentiation with respect to λ then yields

$$\frac{\partial \bar{h}}{\partial \lambda} + \underset{\bar{B}}{\pounds}\bar{h} = 0$$

where $\underset{\bar{B}}{\pounds}$ denotes the Lie derivative of $\bar{h}$, for fixed λ, relative to the vector field $\bar{B}$ corresponding to the transformations $\bar{\varphi}_\lambda$. It follows that

LEMMA 6. <u>For the action integral</u> (12) <u>to be equivalent to</u> (9) <u>under transformations</u> $\bar{\varphi}_\lambda$, <u>it is necessary that a vector field</u> $\bar{B}$ <u>exist such that</u>

$$\frac{\partial \bar{h}_{ij}}{\partial \lambda} = -\bar{B}_{i/j} - \bar{B}_{j/i} \tag{24}$$

<u>where</u> "/" <u>denotes covariant differentiation of</u>

$$\bar{B}_i(\bar{x}, \lambda) = \bar{h}_{ij}(\bar{x}, \lambda)\bar{B}^j(\bar{x})$$

<u>relative to</u> $\bar{h}_{ij}$ <u>for fixed</u> λ. <u>If</u> $\bar{B}$ <u>exists, then it is unique up to a Killing vector</u> v <u>corresponding to an isometry in the Riemannian Geometry determined by</u> $\bar{h}_{ij}(\bar{x}, \lambda)$ <u>for fixed</u> λ.

If the required vector field $\bar{B}$ exists then the corresponding transformations will be given by $x = \bar{\varphi}_\lambda(\bar{x})$ where $\overset{\circ}{\bar{\varphi}}_\lambda = \bar{B}(\bar{\varphi}_\lambda)$. Clearly $\bar{\varphi}_\lambda = \varphi_\lambda^{-1} = \varphi_{-\lambda}$, and by (20), $\bar{B} = -B$. The transition from (12) or (12') to (9) is given explicitly in the component notation by

THEOREM 2. <u>Given a Riemannian fatigue geometry</u> (12) <u>for which a vector field</u> $\bar{B}^i(\bar{x})$ <u>exists satisfying</u> (24). <u>Then the system of differential equations</u> $\frac{d\bar{x}^i}{d\lambda} = \bar{B}^i(\bar{x})$ <u>defines a (local) group of transformations</u> $x^i = \varphi^i_{-\lambda}(\bar{x})$ <u>under which</u> (12) <u>may be transformed into</u> (9) <u>if</u> $\bar{h}_{ij}\bar{B}^i\bar{B}^j \neq 1$ <u>where</u>

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(25) $\quad \overset{*}{h}_{ij} \overset{\text{DEF}}{=\!=} \dfrac{\bar{h}_{ij}}{1-\bar{h}_{rs}B^rB^s}$ (26) $\quad \bar{A}_i \overset{\text{DEF}}{=\!=} \overset{*}{h}_{ij}\bar{B}^j$

(27) $\quad \pm\bar{g}_{ij} \overset{\text{DEF}}{=\!=} \overset{*}{h}_{ij} + \bar{A}_i\bar{A}_j$ (28) $\quad g_{ij} = \dfrac{\partial\bar{x}^r}{\partial x^i}\dfrac{\partial\bar{x}^s}{\partial x^j}\bar{g}_{rs}$

and $\quad A_i = \dfrac{\partial\bar{x}^r}{\partial x^i}\bar{A}_r$.

Proof. Equations (25), (26) and (27) are simply (18'), (19') and (10) respectively where terms have been suitably relabelled since we are dealing with the converse problem. Equation (28) then corresponds to (23).

5. Physical Aspects of Fatigue Geometry. The above transformations consist essentially in showing the equivalence of the differential equations (13) and (14) under transformations leaving λ invariant. Equation (13) may be written $(i = 1, 2, 3, 4)$

$$(13') \qquad \dot{\lambda}^2 - 2A_i\dot{x}^i\dot{\lambda} + (A_iA_j \mp g_{ij})\dot{x}^i\dot{x}^j = 0$$

while (14) is

$$(14') \qquad \dot{\lambda}^2 - \bar{h}_{ij}(\bar{x}, \lambda)\dot{\bar{x}}^i\dot{\bar{x}}^j = 0.$$

One may consider the action λ (or any physical measurement) as a fifth dimension, $\lambda = x^5$. Then (14') immediately suggests an analogy with the null curves of a manifold V^5 with metric tensor $\bar{\gamma}_{55} = 1$, $\bar{\gamma}_{i5} = \bar{\gamma}_{5i} = 0$, $\bar{\gamma}_{ij} = \bar{h}_{ij}(x^\alpha)$, where $\alpha = 1, \ldots, 5$. Then V^5 admits a coordinate transformation $x^\alpha = \varphi^\alpha(\bar{x}^\beta)$ under which $x^5 = \bar{x}^5$ and further, the metric tensor becomes independent of x^5, explicitly given by $\gamma_{55} = 1$, $\gamma_{5i} = \gamma_{i5} = -A_i$, $\gamma_{ij} = -\overset{*}{g}_{ij}$. But this is precisely the Kaluza cylindrical space [7], [8] in which γ_{i5} form the components of the four vector potential. The Einstein tensor based on $\gamma_{\alpha\beta}$ then becomes a natural choice for unified field equations.

In the form (12), $L(x, \dot{x}) = \sqrt{h_{ij}(x, \lambda)\dot{x}^i\dot{x}^j}$ is positive homogeneous of degree one in the $\dot{x}^i$ and, as in Finsler spaces, the $n + 1$ equations (2) are not independent. However, if the parameter t is chosen such that $\lambda' = L = 1$ and the rank of the matrix $L_{\dot{x}^i\dot{x}^j}$ is $n - 1$, then equations (2) were shown [3], [4] equivalent to the system

<u>Geodesics</u>
$$\frac{\delta \dot{x}^i}{\delta t} = -h^{ij}\partial_\lambda h_{jk}\dot{x}^k + \frac{1}{2}\partial_\lambda h_{jk}\dot{x}^k\dot{x}^i\dot{x}^j$$

where $\lambda' = L = 1$

$$\frac{\delta \dot{x}^i}{\delta t} \stackrel{\text{DEF}}{=\!=} \ddot{x}^i + \left\{ {i \atop j\ k} \right\} \dot{x}^j\dot{x}^k$$

$$\left\{ {i \atop j\ k} \right\} = \frac{1}{2} h^{ir} (\partial_i h_{rk} + \partial_k h_{ir} - \partial_r h_{ik}),$$

where $\partial_\lambda = \frac{\partial}{\partial \lambda}$ and $\partial_i = \frac{\partial}{\partial x^i}$, in a way completely analogous to the standard derivations in Riemannian or Finsler geometries. It was also shown that intrinsic differentiation leads to the equation for

<u>Auto-Parallel Curves</u>
$$\frac{\partial \dot{x}^i}{\partial t} = 0.$$

Hence the differential geometry of (12) leads naturally to two families of curves: the geodesics correspond to the geodesics of the electromagnetic action integral (9) and hence describe the motion of charged particles; the auto parallel curves are identical to the geodesics of the Riemannian geometry obtained from (12) by considering λ fixed under the integral sign before applying variational principles (except for the presence of $\lambda = t + c$ in the Christoffel symbols). It may be of interest therefore to consider the consequences of the following axioms:

I. <u>The theory of auto-parallel curves in the fatigue space should be identical to the classical theory of general relativity;</u> more precisely, the unit circles in the Minkowsky tangent spaces

$$h_{ij}(x_o^k, \lambda)\dot{x}^i\dot{x}^j = 1$$

for variable λ must be homothetic (corresponding to a uniform magnification of the metric in all directions) and the Christoffel symbols must be independent of λ. The equations of the auto parallel curves are then independent of λ, and correspond to classical gravitational theory.

II. <u>We assume the existence of an action dependent coordinate transformation</u> φ_λ <u>which permits the action to be expressed in the form</u> (9) as outlined in section 4. The purpose of this section is not to judge I and II, but to derive their consequences.

LEMMA 7. <u>Axiom I implies</u>

$$h_{ij}(x^k, \lambda) = e^{K(\lambda)}\gamma_{ij}(x^k). \tag{29}$$

<u>Proof.</u> Since the indicatrices are homothetic, it follows that

$$h_{ij}(x^k, \lambda) = \varphi(x^k, \lambda)\gamma_{ij}(x^k)$$

for some φ. If we set $\psi = \ell n\varphi$, then

$$0 = \frac{\partial\{{}_j{}^i{}_k\}}{\partial\lambda} = \frac{1}{2}\{\delta_j^i \frac{\partial^2\psi}{\partial x^k\partial\lambda} + \delta_k^i \frac{\partial^2\psi}{\partial x^j\partial\lambda} - \gamma^{ir}\gamma_{jk}\frac{\partial^2\psi}{\partial x^r\partial\lambda}\}$$

and identifying i and j, axiom I implies

$$\frac{\partial^2\psi}{\partial x^k\partial\lambda} = 0, \qquad \text{i.e. } \psi = H(x^k) + K(\lambda).$$

Absorbing $\exp H(x^k)$ into $\gamma_{ij}(x^k)$ the lemma follows.

COROLLARY. <u>The Christoffel symbols constructed from</u>

$$h_{ij}(x^k, \lambda) = e^{K(\lambda)}\gamma_{ij}(x^k)$$

are identical to those constructed from the $\gamma_{ij}(x^k)$.

LEMMA 8. Axioms I and II imply $h_{ij}(x^k, \lambda) = e^{\nu\lambda}\gamma_{ij}(x^k)$ for some constant ν.

Proof. As in section 4, there must exist a $B^i(x^k)$ satisfying (24), or alternatively, using (29)

$$K'(\lambda)e^{K(\lambda)}\gamma_{ij} = -e^{K(\lambda)} \{\gamma_{ir}B^r{}_{|j} + \gamma_{jr}B^r{}_{|i}\} \tag{30}$$

and since the Christoffel symbols are independent of λ, so also $B^r{}_{|j}$, and hence this equation implies $K'(\lambda) = \nu$, a constant as required.

The Riemann fatigue space therefore reduces to

$$\lambda = \int_{t_0} \sqrt{e^{\nu\lambda}\, \gamma_{ij}(x^k)\dot{x}^i\dot{x}^j}\; dt + \lambda(t_0)$$

and, using the formulas for the auto-parallel and geodesics, since $\dfrac{\partial h_{ij}}{\partial\lambda} = \nu h_{ij}$,

Auto-parallel

$$\frac{\delta \dot{x}^i}{\delta t} = 0 \tag{31}$$

Geodesics

$$\frac{\delta \dot{x}^i}{\delta t} = -\frac{\nu}{2}\,\dot{x}^i \tag{32}$$

where the parameter is chosen such that

$$e^{\nu\lambda}\gamma_{ij}(x^k)\dot{x}^i\dot{x}^j = 1 \tag{33}$$

and the Christoffel symbols are constructed from the $\gamma_{ij}(x^k)$.

The classical form of the action integral for electromagnetic theory is then obtained by solving (30) for B^r, that is

(34) $$\nu\gamma_{ij} = - B_{i|j} - B_{j|i}$$

and considering the action dependent coordinate transformation corresponding to the solution of the system $\dot{x}^i = +B^i$, as in section (4).

While the auto-parallel theory coincides with classical theory, the geodesic equation (32) contains a resistance term, so to speak. This resistance term gives rise to the action integral in the form (9) since, under coordinate transformations of the form φ_λ,

(35) $$\dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^j} \dot{x}^j + \frac{\partial \bar{x}^i}{d\lambda} \dot{\lambda} .$$

But if it is assumed that (31) and (32) are also applicable when the velocity of the particle or wave is c, then $\dot{\lambda} = 0$ and (35) reduces to the standard form. Since the geodesics refer to electromagnetic phenomena, it would seem that (32), rather than (31), should refer to the path of light, thereby implying a spectral shift corresponding to a cosmological constant ν.

6. An Illustrative Example. If the space is empty, but $\nu \neq 0$ so that a four potential still exists, one would expect the action integral to be of the form $(x^4 = ict)$

$$\lambda = \int_{t_0} \sqrt{e^{\nu\lambda} \delta_{ij} \dot{\bar{x}}^i \dot{\bar{x}}^j} \, dt + \lambda(t_0).$$

Then equation (34) becomes

$$\nu\delta_{ij} = - \delta_{ir} \bar{B}^r{}_{|j} - \delta_{jr} \bar{B}^r{}_{|i}$$

which has the trivial solution $\bar{B}^r = - \frac{\nu}{2} \bar{x}^r$. The transformation to the classical integral is found by solving $\dot{\bar{x}}^i = - \frac{\nu}{2} \bar{x}^i$, that is $\bar{x}^i = \alpha^i \exp(-\frac{\nu}{2}t)$, and hence φ_λ is given by

$$\bar{x}^i = x^i \exp(-\frac{\nu}{2}\lambda).$$

Substituting into $\dot{\lambda}^2 = e^{\nu\lambda}\delta_{ij}\dot{\bar{x}}^i\dot{\bar{x}}^j$ yields

$$(1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s)\dot{\lambda}^2 = \delta_{ij}\dot{x}^i\dot{x}^j - \nu\delta_{ij}x^i\dot{x}^j\dot{\lambda}$$

and comparison with (13) yields

$$\overset{*}{g}_{ij} = \frac{\delta_{ij}}{1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s}, \quad A_j = -\frac{\nu}{2}\frac{\delta_{ij}x^i}{1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s}$$

from which it follows that $g_{ij} = \overset{*}{g}_{ij} + A_i A_j$ is given by

$$g_{ij} = \frac{1}{1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s} \cdot \left\{ \delta_{ij} + \frac{\nu^2}{4}\frac{\delta_{ir}\delta_{js}x^r x^s}{1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s} \right\}$$

while $F_{ij} = 0$ since

$$A_j = \frac{2}{\nu}\frac{\partial}{\partial x_j}(\ell n(1 - \frac{\nu^2}{4}\delta_{rs}x^r x^s)).$$

Alternatively, the action integral may be written

$$\lambda = \int \Phi \cdot \{\sqrt{S_{ij}\dot{x}^i\dot{x}^j} - \frac{\nu}{2}\Phi\delta_{ij}x^i\dot{x}^j\} \, dt$$

where S_{ij} is de Sitter's solution to the Einstein equations in empty space [7], p. 182,

$$S_{ij} = \delta_{ij} + \frac{\delta_{ir}\delta_{js}x^r x^s}{\frac{4}{\nu^2} - \delta_{rs}x^r x^s}.$$

and $\Phi = \left(1 - \frac{\nu^2}{4} \delta_{rs} x^r x^s\right)^{-\frac{1}{2}}$.

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