ANALYTIC RESULTS FOR ROOTS OF TWO IRRATIONAL FUNCTIONS IN ELASTIC WAVE PROPAGATION

L. M. BROCK[1]

Abstract

The velocities of Rayleigh surface waves and, when they exist, Stoneley interface waves can be obtained as the roots of two irrational functions. Here previous results are extended by using standard operations related to the Wiener–Hopf technique to provide expressions in quadrature for these roots.

1. Problem description

The propagation of sound in elastic solids in the form of body waves whose speeds are defined in terms of material constants is well-established [9]. The transmission/reflection of such waves at material surfaces and interfaces has also been studied [3], and such studies have suggested the possibility of another type of wave — the Rayleigh wave — that propagates on a stress-free surface in such a manner that its effects are confined to the immediate vicinity of the surface [1]. As outlined in [1], it can be shown that the phase velocity v of a Rayleigh wave along the planar surface of a homogeneous isotropic linearly-elastic solid must satisfy the equation

$$(2s^2 - b^2)^2 - 4s^2\sqrt{s^2 - a^2}\sqrt{s^2 - a^2} = 0, \tag{1}$$

where

$$s = 1/v, \qquad a = 1/v_d, \qquad b = 1/v_r, \tag{2a-c}$$

and (v_r, v_d) are, respectively, the rotational and dilatational elastic body wave speeds given by

$$v_r = \sqrt{\frac{\mu}{\rho}}, \qquad v_d = \sqrt{2\frac{1-v}{1-2v}}v_r. \tag{3}$$

[1]Department of Mechanical Engineering, University of Kentucky, Lexington, Kentucky 40506, USA

The inverse velocity s, or slowness, is often the convenient quantity to work with. Here (μ, ρ, ν) are, respectively, the shear modulus, mass density and Poisson's ratio for the elastic half-space. For an isotropic homogeneous solid, these constants are positive and real, with $0 < \nu < 0.5$ [12], so that we always have

$$v_d > v_r \qquad (b > a). \tag{4}$$

The fact that (1) exhibits the phase velocity and not also a wave-number indicates that Rayleigh waves are non-dispersive [10].

Stoneley [14] considered the analogous problem for a wave propagating along the interface of two welded (that is, continuity of both displacement and stress exists) dissimilar homogeneous isotropic linearly-elastic solids in such a manner that its effects are confined to the immediate vicinity of the interface. As described by Cagniard [6], it can be shown that the phase velocity v of such a Stoneley wave for a planar interface must satisfy the equation

$$
\begin{aligned}
s^2 & \left[\sqrt{s^2 - a_1^2}\sqrt{s^2 - b_1^2}\sqrt{s^2 - a_2^2}\sqrt{s^2 - b_2^2} + (\Omega_1 + \Omega_2 - s^2)^2 \right] \\
& - (\Omega_2 - s^2)^2 \sqrt{s^2 - a_1^2}\sqrt{s^2 - b_1^2} - (\Omega_1 - s^2)^2 \sqrt{s^2 - a_2^2}\sqrt{s^2 - b_2^2} \\
& + \Omega_1 \Omega_2 \left(\sqrt{s^2 - a_1^2}\sqrt{s^2 - b_2^2} + \sqrt{s^2 - a_2^2}\sqrt{s^2 - b_1^2} \right) = 0,
\end{aligned} \tag{5}
$$

where the subscripts $(1, 2)$ refer to the two welded solids and

$$\Omega_1 = c_1 b_1^2, \qquad \Omega_2 = c_2 b_2^2, \qquad c_1 = \frac{1}{2} - c_2 = \frac{\mu_1}{2(\mu_2 - \mu_1)}. \tag{6}$$

Equation (2a) holds again, therefore as do, with appropriate subscripts attached, (2b,c) and (3). The absence of a wave-number in (5) indicates that Stoneley waves are also non-dispersive.

2. Previous results

To obtain information about possible (real) solutions of (1), one can study [1] the roots of the well-known Rayleigh function

$$R(z) = 4z^2 \alpha \beta + T^2 \tag{7}$$

in the complex z-plane. Here

$$T = b^2 - 2z^2, \qquad \alpha = \sqrt{a^2 - z^2}, \qquad \beta = \sqrt{b^2 - z^2} \tag{8a-c}$$

and $R(z)$ is rendered single-valued by introducing branch cuts for (α, β) along $\text{Im}(z) = 0$, $|\text{Re}(z)| > a$ and $\text{Im}(z) = 0$, $|\text{Re}(z)| > b$, respectively, such that $\text{Re}(\alpha, \beta) \geq 0$ everywhere in the cut z-plane. Application of the standard argument principle [7] then shows that $R(z)$ always exhibits the isolated real roots $z = \pm s_0$ of order 1, where s_0 lies in the range

$$s_0 > b. \tag{9}$$

Analogously [6], for information about (real) solutions to (5) the Stoneley function

$$S(z) = z^2(\alpha_1\beta_1\alpha_2\beta_2 + P_{12}^2) + \Omega_1\Omega_2(\alpha_1\beta_2 + \alpha_2\beta_1) - P_2^2\alpha_1\beta_1 - P_1^2\alpha_2\beta_2 \tag{10}$$

can be introduced, where (8b,c) hold again with appropriate subscripts, and

$$P_1 = \Omega_1 - z^2, \qquad P_2 = \Omega_2 - z^2, \qquad P_{12} = \Omega_1 + \Omega_2 - z^2. \tag{11}$$

In this case, the physical restrictions (4) on body wave speeds allow for a given pair of welded materials the six possible combinations

$$\begin{aligned}
b_2 > a_2 > b_1 > a_1, \\
b_2 > b_1 > a_2 > a_1, \\
b_2 > b_1 > a_1 > a_2, \\
b_1 > b_2 > a_1 > a_2, \\
b_1 > b_2 > a_2 > a_1, \\
b_1 > a_1 > b_2 > a_2,
\end{aligned} \tag{12a-f}$$

so that the function $S(z)$ can be made analytic in the z-plane cut along $\text{Im}(z) = 0$, $\min(a_1, a_2) < |\text{Re}(z)| < \max(b_1, b_2)$ by requiring that $\text{Re}(\alpha_1, \beta_1, \alpha_2, \beta_2) \geq 0$ in that cut plane. While (4) precludes the limit cases $(b_1 = a_1, b_2 = a_2)$, it is conceivable that the two dissimilar welded solids would be matched in other particulars. In this brief demonstration, however, we will treat the general situation, that is, no partial material matching occurs, so that the strict inequalities in (12) are maintained, and $\Omega_1 \neq \Omega_2$.

Then, application of the argument principle [7] shows that $S(z)$ has the isolated real roots $z = \pm s_0$ of order 1 in the range

$$s_0 > \max(b_1, b_2) \tag{13}$$

only if the welded material properties are such that

$$\sqrt{b_1^2 - b_2^2}\left[\Omega_1\Omega_2\sqrt{b_1^2 - a_2^2} - (\Omega_1 - b_1^2)^2\sqrt{b_1^2 - a_2^2}\right]$$
$$+ b_1^2(\Omega_1 + \Omega_2 - b_1^2)^2 > 0 \qquad (b_1 > b_2), \tag{14a}$$

$$\sqrt{b_2^2 - b_1^2}\left[\Omega_2\Omega_1\sqrt{b_2^2 - a_1^2} - (\Omega_2 - b_2^2)^2\sqrt{b_2^2 - a_1^2}\right]$$
$$+ b_2^2(\Omega_2 + \Omega_1 - b_2^2)^2 > 0 \qquad (b_2 > b_1). \tag{14b}$$

If (14) is not satisfied, then $S(z)$ has no roots in the cut z-plane.

Clearly, the values of s_0 for (R, S) can be found for any given set(s) of values (μ, ρ, ν), see, for example, [1, 6]. Indeed, in the case of R, (7) can be rationalized to a cubic equation in z^2, although two of the roots of that equation are extraneous. In what follows, expressions in quadrature for s_0 are obtained analytically for both (R, S) by the application of standard operations.

3. Results for R

Study of (7) shows that R is holomorphic in the z-plane cut along $\text{Im}(z) = 0$, $a < |\text{Re}(z)| < b$, and behaves as

$$R \approx 2(a^2 - b^2)z^2, \qquad |z| \to \infty. \tag{15}$$

Therefore, one can introduce the function

$$G(z) = \frac{R(z)}{2(a^2 - b^2)(z^2 - s_0^2)} \tag{16}$$

which is also holomorphic in this cut plane, but exhibits no zeros there and approaches unity at infinity. It is therefore possible [1, 10, 13] to write G as the product of functions $G_\pm$ which are holomorphic in, respectively, the overlapping halves $\text{Re}(z) > -a$ and $\text{Re}(z) < a$ of the complex z-plane. Specifically, performing factorization operations formulated by Noble [10] for the well-known Wiener–Hopf technique gives

$$\ln G_\pm(z) = -\frac{1}{\pi} \int_a^b \tan^{-1} \frac{4u^2 A B}{T^2} \frac{du}{u \pm z}, \quad A = \sqrt{u^2 - a^2}, \quad B = \sqrt{u^2 - b^2}, \tag{17a-c}$$

where integration is along the positive $\text{Re}(u)$-axis. At $z = 0$ the functions R and $G_\pm$ are holomorphic, and $G_- = G_+$. Therefore, replacing the left-hand side of (16) with $G_+ G_-$ and evaluating the result at $z = 0$ gives, in view of (7), the explicit formula

$$s_0 = \frac{b^2}{\sqrt{2(b^2 - a^2)}} \frac{1}{G_+(0)} \tag{18}$$

for the root of R.

4. Results for S

The procedure for $S(z)$ when (14) holds follows by analogy. That is, a function

$$G(z) = \frac{S(z)}{4(4\Omega_1 - a_1^2 - b_1^2)(4\Omega_2 - a_2^2 - b_2^2)(z^2 - s_0^2)} \tag{19}$$

is introduced that is holomorphic and exhibits no zeros in the z-plane cut along $\text{Im}(z) = 0$, $\min(a_1, a_2) < |\text{Re}(z)| < \max(b_1, b_2)$, and approaches unity at infinity. The same factorization procedure [11] used for (16) then yields G as the product of functions $G_\pm$ that are analytic in the overlapping halves, respectively, $\text{Re}(z) > -\min(a_1, a_2)$ and $\text{Re}(z) < \min(a_1, a_2)$ of the complex z-plane. Equations (12) and (14) lead to, however, six different forms for $G_\pm$. It can be shown that when (14a) governs, then

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_1}^{b_1} C_1 + \int_{b_1}^{a_2} D_1 + \int_{a_2}^{b_2} E_1 \right) \frac{du}{u \pm z},$$

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_1}^{a_2} C_1 + \int_{a_2}^{b_1} D_2 + \int_{b_1}^{b_2} E_1 \right) \frac{du}{u \pm z}, \qquad (20a\text{-}c)$$

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_2}^{a_1} C_3 + \int_{a_1}^{b_1} D_2 + \int_{b_1}^{b_2} E_1 \right) \frac{du}{u \pm z}$$

for the cases (12a-c), respectively, while when (14b) governs, the formulas

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_2}^{a_1} C_3 + \int_{a_1}^{b_2} D_2 + \int_{b_2}^{b_1} E_4 \right) \frac{du}{u \pm z},$$

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_1}^{a_2} C_1 + \int_{a_2}^{b_2} D_2 + \int_{b_2}^{b_1} E_4 \right) \frac{du}{u \pm z}, \qquad (20d\text{-}f)$$

$$\ln G_\pm(z) = -\frac{1}{\pi} \left(\int_{a_2}^{b_2} C_3 + \int_{b_2}^{a_1} D_6 + \int_{a_1}^{b_1} E_4 \right) \frac{du}{u \pm z}$$

hold for the cases (12d-f), respectively. In (20) the relations

$$C_1 = \tan^{-1} A_1 \frac{u^2 \beta_1 \alpha_2 \beta_2 - \Omega_1 \Omega_2 \beta_2 + \beta_1 P_2^2}{\Omega_1 \Omega_1 \alpha_2 \beta_1 - \alpha_2 \beta_2 P_1^2 - u^2 P_{12}^2},$$

$$C_3 = \tan^{-1} A_2 \frac{u^2 \beta_2 \alpha_1 \beta_1 - \Omega_1 \Omega_2 \beta_1 + \beta_2 P_1^2}{\Omega_1 \Omega_1 \alpha_1 \beta_2 - \alpha_1 \beta_1 P_2^2 - u^2 P_{12}^2},$$

$$D_1 = \tan^{-1} \Omega_1 \Omega_2 \frac{\alpha_2 B_1 + A_1 \beta_2}{\alpha_2 \beta_2 P_1^2 + u^2 (P_{12}^2 - A_1 B_1 \alpha_2 \beta_2)},$$

$$D_2 = \tan^{-1} \frac{1}{u^2} \frac{A_1 (\Omega_1 \Omega_2 \beta_2 - \beta_1 P_2^2) + A_2 (\Omega_1 \Omega_2 \beta_1 - \beta_2 P_1^2)}{P_{12}^2 - A_1 \beta_1 A_2 \beta_2}, \qquad (21a\text{-}g)$$

$$D_6 = \tan^{-1} \Omega_1 \Omega_2 \frac{A_2 \beta_1 + \alpha_1 B_2}{\alpha_1 \beta_1 P_2^2 + u^2 (P_{12}^2 - \alpha_1 \beta_1 A_2 B_2)},$$

$$E_1 = \tan^{-1} \beta_2 \frac{u^2 A_1 B_1 A_2 + \Omega_1 \Omega_2 A_1 - A_2 P_1^2}{\Omega_1 \Omega_2 A_2 B_1 - A_1 B_1 P_2^2 + u^2 P_{12}^2},$$

$$E_4 = \tan^{-1} \beta_1 \frac{u^2 A_1 A_2 B_2 + \Omega_1 \Omega_2 A_2 - A_1 P_2^2}{\Omega_1 \Omega_2 A_1 B_2 - A_2 B_2 P_1^2 + u^2 P_{12}^2}$$

TABLE 1.

	μ(GPa)	ρ(kg/m^3)	ν	v(m/s, Rayleigh)
aluminium	25.9	2768	0.33	2856
brass	39.3	8145	0.35	2019
copper	42.7	8941	0.35	2043
steel	73.1	7778	0.3	2841
titanium	41.4	4512	0.34	2825

and (17b,c) apply, with appropriate subscripts attached. In (20) integration is over the positive Re(u)-axis. By evaluating (19) at $z = 0$, where $G_\pm$ and S are all holomorphic and $G_- = G_+$, the explicit result

$$s_0 = \sqrt{\frac{2(\Omega_2 b_1 - \Omega_1 b_2)(\Omega_2 a_1 - \Omega_1 a_2)}{(4\Omega_1 - a_1^2 - b_1^2)(4\Omega_2 - a_2^2 - b_2^2)} \frac{1}{G_+(0)}} \qquad (22)$$

can be obtained when, of course, (14) holds. In view of (3), (4), (6) and (12), the argument of the radical in (22) is always positive.

5. Numerical values

The expression (18) was checked against a table of values [1] for the Rayleigh wave phase velocity and found to agree to the three significant figures given. Then (18) and (22) were used to calculate Rayleigh and Stoneley phase velocity values for various standard [2] engineering materials, and those values were compared with the roots of, respectively, (1) and (5). The integrations arising in (18) and (22) were performed by standard Gaussian quadrature [8], while the roots of (1) and (5) were obtained by a standard [8] false position iterative scheme. The values obtained agreed to four significant figures. A list of Rayleigh values is given in Table 1 for five materials, along with the corresponding (typical) values (μ, ρ, ν). Table 2 then lists the corresponding Stoneley values for those combinations of welded materials in Table 1 for which Stoneley waves exist, that is (14) holds, along with the corresponding Rayleigh values. It is noted that the Stoneley value is in each case larger than the Rayleigh values for either of the welded materials.

6. Comments

It should be noted that the empirical formulas for the Rayleigh/Stoneley wave phase velocity can readily be derived from plots of numerically-determined values,

TABLE 2.

	v(m/s, Rayleigh-1)	v(m/s, Rayleigh-2)	v(m/s, Stoneley)
aluminium-1	2856	2841	3025
steel-2			
aluminium-1	2856	2825	3027
titanium-2			

for example [1]. Indeed, such formulas are simpler than (18) or (22). Moreover, the numerical integrations in (18) and (22) required more computational time than the numerical root-finding schemes used for (1) and (5).

However, (18) and (22) are exact, and allow a more analytic study of the dependence of Rayleigh/Stoneley velocities upon material properties to be made, especially in the Stoneley case when the partial material matching neglected here arises. Moreover, the procedure used to develop these formulas demonstrates that certain roots of two physically-relevant irrational functions can be obtained analytically. This procedure has been extended [4, 5] to cases of homogeneous isotropic linear coupled thermoelastic materials in which, due to the coupling between the mechanical and temperature fields, the Rayleigh/Stoneley signals are dispersive, that is, the phase velocity depends upon wave number.

Acknowledgment

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