



# Auerbach Bases and Minimal Volume Sufficient Enlargements

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*Abstract.* Let  $B_Y$  denote the unit ball of a normed linear space  $Y$ . A symmetric, bounded, closed, convex set  $A$  in a finite dimensional normed linear space  $X$  is called a *sufficient enlargement* for  $X$  if, for an arbitrary isometric embedding of  $X$  into a Banach space  $Y$ , there exists a linear projection  $P: Y \rightarrow X$  such that  $P(B_Y) \subset A$ . Each finite dimensional normed space has a minimal-volume sufficient enlargement that is a parallelepiped; some spaces have “exotic” minimal-volume sufficient enlargements. The main result of the paper is a characterization of spaces having “exotic” minimal-volume sufficient enlargements in terms of Auerbach bases.

## 1 Introduction

All linear spaces considered in this paper will be over the reals. By a *space* we mean a normed linear space, unless explicitly stated otherwise. We denote by  $B_X$  the closed unit ball of a space  $X$ . We say that subsets  $A$  and  $B$  of finite dimensional linear spaces  $X$  and  $Y$ , respectively, are *linearly equivalent* if there exists a linear isomorphism  $T$  between the subspace spanned by  $A$  in  $X$  and the subspace spanned by  $B$  in  $Y$  such that  $T(A) = B$ . By a *symmetric* set  $K$  in a linear space we mean a set such that  $x \in K$  implies  $-x \in K$ .

Our terminology and notation of Banach space theory follows [6]. By  $B_p^n$ ,  $1 \leq p \leq \infty$ ,  $n \in \mathbb{N}$  we denote the closed unit ball of  $\ell_p^n$ . Our terminology and notation of convex geometry follows [17]. A Minkowski sum of finitely many line segments is called a *zonotope*. We use the term *ball* for a symmetric, bounded, closed, convex set with interior points in a finite dimensional linear space.

**Definition 1.1** ([9]) A ball in a finite dimensional normed space  $X$  is called a *sufficient enlargement* (SE) for  $X$  (or of  $B_X$ ) if, for an arbitrary isometric embedding of  $X$  into a Banach space  $Y$ , there exists a projection  $P: Y \rightarrow X$  such that  $P(B_Y) \subset A$ . A sufficient enlargement  $A$  for  $X$  is called a *minimal-volume sufficient enlargement* (MVSE) if  $\text{vol} A \leq \text{vol} D$  for each SE  $D$  for  $X$ .

It was proved in [13, Theorem 3] that each MVSE is a zonotope generated by a totally unimodular matrix, and the set of all MVSE's (for all spaces) coincides with the set of all space tiling zonotopes described in [4, 7]. It is known (see [10, Theorem 6]; the result is implicit in [5, pp. 95–97]) that a minimum-volume parallelepiped containing  $B_X$  is an MVSE for  $X$ . It was discovered (see [12, Theorem 4] and [13, Theorem 4]) that spaces  $X$  having a non-parallelepipedal MVSE are rather special:

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they should have a two-dimensional subspace whose unit ball is linearly equivalent to a regular hexagon. In dimension two this provides a complete characterization (see [12]). On the other hand, the unit ball of  $\ell_\infty^n$ ,  $n \geq 3$ , has a regular hexagonal section, but the only MVSE for  $\ell_\infty^n$  is its unit ball (so it is a parallelepiped). A natural problem arises of characterizing Banach spaces having non-parallelepipedal MVSE in dimensions  $d \geq 3$ . The main purpose of this paper is to characterize such spaces in terms of Auerbach bases. At the end of the paper we make some remarks on MVSE for  $\ell_1^n$  and study relations between the class of spaces having non-parallelepipedal MVSE and the class of spaces having a 1-complemented subspace whose unit ball is linearly equivalent to a regular hexagon.

## 2 Auerbach Bases

We need to recall some well-known results on bases in finite dimensional normed spaces. Let  $X$  be an  $n$ -dimensional normed linear space. For a vector  $x \in X$  by  $[-x, x]$  we denote the line segment joining  $-x$  and  $x$ . For  $x_1, \dots, x_k \in X$  by  $M(\{x_i\}_{i=1}^k)$  we denote the Minkowski sum of the corresponding line segments, that is,

$$M(\{x_i\}_{i=1}^k) = \{x : x = y_1 + \dots + y_k \text{ for some } y_i \in [-x_i, x_i], i = 1, \dots, k\}.$$

Let  $\{x_i\}_{i=1}^n$  be a basis in  $X$ , its *biorthogonal functionals* are defined by  $x_i^*(x_j) = \delta_{ij}$  (Kronecker delta). The basis  $\{x_i\}_{i=1}^n$  is called an *Auerbach basis* if  $\|x_i\| = \|x_i^*\| = 1$  for all  $i \in \{1, \dots, n\}$ . According to [2, Remarks to Chapter VII] H. Auerbach proved the existence of such bases for each finite dimensional  $X$ .

**Historical comment** Banach [2] does not contain any proofs of the existence of Auerbach bases. The two dimensional case of Auerbach's result was proved in [1]. Unfortunately Auerbach's original proof in the general case seems to be lost. Proofs of the existence of Auerbach bases discussed below are taken from [3, 18]. Plichko [16] contains interesting results on relation between upper and lower Auerbach bases (which are defined below) and related references.

It is useful for us to recall the standard argument for proving the existence of Auerbach bases (it goes back at least to [18]). Consider the set  $N(= N(X))$  consisting of all subsets  $\{x_i\}_{i=1}^n \subset X$  satisfying  $\|x_i\| = 1, i \in \{1, \dots, n\}$ . It is a compact set in its natural topology, and the  $n$ -dimensional volume of  $M(\{x_i\}_{i=1}^n)$  is a continuous function on  $N$ . Hence it attains its maximum on  $N$ . Let  $U \subset N$  be the set of  $n$ -tuples on which the maximum is attained. It is easy to see that each  $\{x_i\}_{i=1}^n \in U$  is a basis (for linearly dependent sets the volume is zero). Another important observation is that  $M(\{x_i\}_{i=1}^n) \supset B_X$  if  $\{x_i\}_{i=1}^n \in U$ . In fact, if there is  $y \in B_X \setminus M(\{x_i\}_{i=1}^n)$  then (since the volume of a parallelepiped is the product of the length of its height and the  $(n - 1)$ -dimensional volume of its base), there is  $i \in \{1, \dots, n\}$  such that replacing  $x_i$  by  $y$  we get a parallelepiped whose volume is strictly greater than the volume of  $M(\{x_i\}_{i=1}^n)$ . Since we may assume  $\|y\| = 1$ , this is a contradiction with the definition of  $U$ .

The following lemma shows that each basis from  $U$  is an Auerbach basis.

**Lemma 2.1** *A system  $\{x_i\}_{i=1}^n \in N$  forms an Auerbach basis if and only if  $M(\{x_i\}_{i=1}^n) \supset B_X$ .*

**Proof** It is easy to see that

$$M(\{x_i\}_{i=1}^n) = \{x : |x_i^*(x)| \leq 1 \text{ for } i = 1, \dots, n\}$$

for each basis  $\{x_i\}_{i=1}^n$ . Hence  $M(\{x_i\}_{i=1}^n) \supset B_X$  if and only if  $\|x_i^*\| \leq 1$  for each  $i$ . It remains to observe that the equality  $\|x_i\| = 1$  implies  $\|x_i^*\| \geq 1$ ,  $i = 1, \dots, n$ . ■

This result justifies the following definition.

**Definition 2.2** A basis from  $U$  is called an *upper Auerbach basis*.

Another way of showing that each finite dimensional space  $X$  has an Auerbach basis was discovered in [3] (see also [15]). It was proved that each parallelepiped  $P$  containing  $B_X$  and having the minimum possible volume among all parallelepipeds containing  $B_X$  is of the form  $M(\{x_i\}_{i=1}^n)$  for some  $\{x_i\}_{i=1}^n \in N(X)$ . By Lemma 2.1 the corresponding system  $\{x_i\}_{i=1}^n$  is an Auerbach basis.

**Definition 2.3** A basis  $\{x_i\}_{i=1}^n$  for which  $M(\{x_i\}_{i=1}^n)$  is one of the minimum-volume parallelepipeds containing  $B_X$  is called a *lower Auerbach basis*.

The notions of lower and upper Auerbach bases are dual to each other.

**Proposition 2.4** A basis  $\{x_i\}_{i=1}^n$  in  $X$  is a lower Auerbach basis if and only if the biorthogonal sequence  $\{x_i^*\}_{i=1}^n$  is an upper Auerbach basis in  $X^*$ .

**Proof** We choose a basis  $\{e_i\}_{i=1}^n$  in  $X$  and let  $\{e_i^*\}_{i=1}^n$  be its biorthogonal functionals in  $X^*$ . We normalize all volumes in  $X$  in such a way that the volume of  $M(\{e_i\}_{i=1}^n)$  is equal to 1, and all volumes in  $X^*$  in such a way that the volume of  $M(\{e_i^*\}_{i=1}^n)$  is equal to 1 (one can see that normalizations do not matter for our purposes).

Let  $K = (x_{i,j})_{i,j=1}^n$  be the matrix whose columns are coordinates of an Auerbach basis  $\{x_j\}_{j=1}^n$  with respect to  $\{e_i\}_{i=1}^n$ , and let  $K^* = (x_{i,j}^*)_{i,j=1}^n$  be a matrix whose rows are coordinates of  $\{x_i^*\}_{i=1}^n$  (which is an Auerbach basis in  $X^*$ ) with respect to  $\{e_j^*\}_{j=1}^n$ . Then  $K^* \cdot K = I$  (the identity matrix). Therefore,

$$|\det K^*| \cdot |\det K| = 1.$$

Hence  $\text{vol}(M(\{x_i\}_{i=1}^n)) \cdot \text{vol}(M(\{x_i^*\}_{i=1}^n)) = 1$ , and one of these volumes attains its maximum on the set of Auerbach bases if and only if the other attains its minimum. ■

### 3 The Main Result

**Theorem 3.1** An  $n$ -dimensional normed linear space  $X$  has a non-parallelepipedal MVSE if and only if  $X$  has a lower Auerbach basis  $\{x_i\}_{i=1}^n$  such that the unit ball of the two-dimensional subspace  $\text{lin}\{x_1, x_2\}$  is linearly equivalent to a regular hexagon.

**Proof. “Only if” part** We start by considering the case when the space  $X$  is polyhedral, that is, when  $B_X$  is a polytope. In this case we may consider  $X$  as a subspace of

$\ell_\infty^m$  for some  $m \in \mathbb{N}$ . Since  $X$  has an MVSE that is not a parallelepiped, there exists a linear projection  $P: \ell_\infty^m \rightarrow X$  such that  $P(B_\infty^m)$  has the minimal possible volume, but  $P(B_\infty^m)$  is not a parallelepiped. We consider the standard Euclidean structure on  $\ell_\infty^m$ . Let  $\{q_1, \dots, q_{m-n}\}$  be an orthonormal basis in  $\ker P$ , and let  $\{\tilde{q}_1, \dots, \tilde{q}_n\}$  be an orthonormal basis in the orthogonal complement of  $\ker P$ . As was shown in [11, Lemma 2],  $P(B_\infty^m)$  is linearly equivalent to the zonotope spanned by rows of  $\tilde{Q} = [\tilde{q}_1, \dots, \tilde{q}_n]$ . By assumption, this zonotope is not a parallelepiped. It is easy to see that this assumption is equivalent to the existence of a minimal linearly dependent collection of rows of  $\tilde{Q}$  containing  $\geq 3$  rows. This condition implies that we can reorder the coordinates in  $\ell_\infty^m$  and multiply the matrix  $\tilde{Q}$  from the right by an invertible  $n \times n$  matrix  $C_1$  in such a way that  $\tilde{Q}C_1$  has a submatrix of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix},$$

where  $a_1 \neq 0$  and  $a_2 \neq 0$ . Let  $\mathcal{X}$  be an  $m \times n$  matrix whose columns form a basis of  $X$  (considered as a subspace of  $\ell_\infty^m$ ). The argument of [11] (see conditions (1)–(3) on p. 96) implies that  $\mathcal{X}$  can be multiplied from the right by an invertible  $n \times n$  matrix  $C_2$  in such a way that  $\mathcal{X}C_2$  is of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \text{sign } a_1 & \text{sign } a_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

where there is an  $n \times n$  identity matrix at the top and all minors of the matrix  $\mathcal{X}C_2$  have absolute values  $\leq 1$ .

Observe that columns on  $\mathcal{X}C_2$  also form a basis in  $X$ . Changing signs in the first two columns and the first two coordinates of  $\ell_\infty^m$ , if necessary, we get that the subspace  $X \subset \ell_\infty^m$  is spanned by columns of the matrix

$$(3.1) \quad \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & b_{n+1,3} & \dots & b_{n+1,n} \\ b_{n+2,1} & b_{n+2,2} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & * & \dots & * \end{pmatrix},$$

in which absolute values of all minors are  $\leq 1$ . This restriction on minors implies  $|b_{i,1} - b_{i,2}| \leq 1$ ,  $|b_{i,1}| \leq 1$ , and  $|b_{i,2}| \leq 1$ . A routine verification shows that these inequalities imply that the first two columns span a subspace of  $X \subset \ell_\infty^m$  whose unit ball is linearly equivalent to a regular hexagon (see [12, p. 390] for more details).

It remains to show that the columns of (3.1) form a lower Auerbach basis in  $X$ . Let us denote the columns of (3.1) by  $\{x_i\}_{i=1}^n$  and the biorthogonal functionals of  $\{x_i\}_{i=1}^n$  (considered as vectors in  $X^*$ ) by  $\{x_i^*\}_{i=1}^n$ .

We map  $\{x_i^*\}_{i=1}^n$  onto the unit vector basis of  $\mathbb{R}^n$ . This mapping maps  $B_{X^*}$  onto the symmetric convex hull of vectors whose coordinates are rows of the matrix (3.1). In fact, using the definitions we get

$$\left\| \sum_{i=1}^n \alpha_i x_i^* \right\|_{X^*} = \max \left\{ \left| \sum_{i=1}^n \alpha_i \beta_i \right| : \max_{1 \leq j \leq m} \left| \sum_{i=1}^n \beta_i b_{ji} \right| \leq 1 \right\}.$$

Therefore, if  $\{\alpha_i\}_{i=1}^n \in \mathbb{R}^n$  is in the symmetric convex hull of  $\{b_{ji}\}_{i=1}^n \in \mathbb{R}^n$ ,  $j = 1, \dots, m$ , then

$$\left| \sum_{i=1}^n \alpha_i \beta_i \right| \leq \max_{1 \leq j \leq m} \left| \sum_{i=1}^n \beta_i b_{ji} \right| \quad \text{and} \quad \left\| \sum_{i=1}^n \alpha_i x_i^* \right\|_{X^*} \leq 1.$$

On the other hand, if  $\{\alpha_i\}_{i=1}^n$  is not in the symmetric convex hull of  $\{b_{ji}\}_{i=1}^n \in \mathbb{R}^n$ ,  $j = 1, \dots, m$ , then, by the separation theorem (see, e.g., [17, Theorem 1.3.4]), there is  $\{\beta_i\}_{i=1}^n$  such that

$$\max_{1 \leq j \leq m} \left| \sum_{i=1}^n \beta_i b_{ji} \right| \leq 1, \quad \text{but} \quad \left| \sum_{i=1}^n \alpha_i \beta_i \right| > 1,$$

and hence

$$\left\| \sum_{i=1}^n \alpha_i x_i^* \right\|_{X^*} > 1.$$

Thus the restriction on the absolute values of minors of (3.1) implies that  $\{x_i^*\}_{i=1}^n$  is an upper Auerbach basis in  $X^*$ . By Proposition 2.4,  $\{x_i\}_{i=1}^n$  is a lower Auerbach basis in  $X$ .

Now we consider the general case. Let  $Y$  be an  $n$ -dimensional space, and let  $A$  be a non-parallelepipedal MVSE for  $Y$ . By [13, Theorem 3] and [12, Lemma 1] there is a polyhedral space  $X$  such that  $B_X \supset B_Y$  and  $A$  is an SE (hence MVSE) for  $X$ . By the first part of the proof there is a lower Auerbach basis  $\{x_i\}_{i=1}^n$  in  $X$  such that the unit ball of the subspace of  $X$  spanned by  $\{x_1, x_2\}$  is linearly equivalent to a regular hexagon. The basis  $\{x_i\}_{i=1}^n$  is a lower Auerbach basis for  $Y$  too. In fact, the spaces have the same MVSE, hence a minimum-volume parallelepiped containing  $B_X$  is also a minimum-volume parallelepiped containing  $B_Y$ . It remains to show that the unit ball of the subspace spanned in  $Y$  by  $\{x_1, x_2\}$  is also a regular hexagon.

To achieve this goal we use additional information about the basis  $\{x_i\}$  that we get from the first part of the proof. Namely, we use the observation that the vertices of

the unit ball of the subspace  $\text{lin}(x_1, x_2)$  are:  $\pm x_1, \pm x_2, \pm(x_1 - x_2)$ . So it remains to show that  $(x_1 - x_2) \in B_Y$ . This has already been done in [13, pp. 617–618].

**“If” part.** First we consider the case when  $X$  is polyhedral. Suppose that  $X$  has a lower Auerbach basis  $\{x_i\}_{i=1}^n$  and that  $x_1, x_2$  span a subspace whose unit ball is linearly equivalent to a regular hexagon. Then the biorthogonal functionals  $\{x_i^*\}_{i=1}^n$  form an upper Auerbach basis in  $X^*$ . We join to this sequence all extreme points of  $B_{X^*}$ . Since  $X$  is polyhedral, we get a finite sequence, which we denote by  $\{x_i^*\}_{i=1}^m$ . Then

$$x \mapsto \{x_i^*(x)\}_{i=1}^m$$

is an isometric embedding of  $X$  into  $\ell_\infty^m$ . Writing images of  $\{x_i\}_{i=1}^n$  as columns, we get a matrix of the form:

$$(3.2) \quad (b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ b_{n+1,1} & b_{n+1,2} & * & \dots & * \\ b_{n+2,1} & b_{n+2,2} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & * & \dots & * \end{pmatrix}.$$

Since  $\{x_i^*\}_{i=1}^n$  is an upper Auerbach basis, absolute values of all minors of this matrix do not exceed 1.

Now we use fact that the linear span of  $\{x_1, x_2\}$  is a regular hexagonal space in order to show that we may assume that at least one of the pairs  $(b_{k,1}, b_{k,2})$  in (3.2) is of the form  $(\pm 1, \pm 1)$ . (Sometimes we need to modify the matrix (3.2) to achieve this goal.)

The definition of the norm on  $\ell_\infty^m$  implies that there is a  $3 \times 2$  submatrix  $S$  of the matrix  $(b_{i,j})$  ( $i = 1, \dots, m, j = 1, 2$ ) whose columns span a regular hexagonal subspace in  $\ell_\infty^3$ , and for each  $\alpha_1, \alpha_2 \in \mathbb{R}$  the equality

$$(3.3) \quad \max_{1 \leq i \leq m} |\alpha_1 b_{i,1} + \alpha_2 b_{i,2}| = \max_{i \in A} |\alpha_1 b_{i,1} + \alpha_2 b_{i,2}|$$

holds, where  $A$  is the set of labels of rows of  $S$ .

To find such a set  $S$  we observe that for each side of the hexagon we can find  $i \in \{1, \dots, m\}$  such that the side is contained in the set of vectors of  $\ell_\infty^m$  for which the  $i$ -th coordinate is either 1 or  $-1$  (this happens because the hexagon is the intersection of the unit sphere of  $\ell_\infty^n$  with the two dimensional subspace). Picking one side from each symmetric with respect to the origin pair of sides and choosing (in the way described above) one label for each of the pairs, we get the desired set  $A$ . To see that it satisfies the stated conditions we consider the operator  $R: \ell_\infty^m \rightarrow \ell_\infty^3$  given by  $R(\{x_i\}_{i=1}^n) = \{x_i\}_{i \in A}$ . The stated condition can be described as follows: the restriction of  $R$  to the linear span of the first two columns of the matrix  $(b_{ij})$  is an

isometry. To show this it suffices to show that a vector of norm 1 is mapped to a vector of norm 1. This happens due to the construction of  $A$ .

It is clear from (3.2) that the maximum in the left-hand side of (3.3) is at least

$$\max\{|\alpha_1|, |\alpha_2|\}.$$

Hence at least one of the elements in each of the columns of  $S$  is equal to  $\pm 1$ . A simple variational argument (described below) shows that changing signs in rows of  $S$ , if necessary, we may assume that

- (1) either  $S$  contains a row of the form  $(1, 0)$  or two rows of the forms  $(1, a)$  and  $(1, -b)$ ,  $a, b > 0$ ;
- (2) either  $S$  contains a row of the form  $(0, 1)$  or two rows of the forms  $(c, 1)$  and  $(-d, 1)$ ,  $c, d > 0$ .

**Note** At this point we allow the change of sign needed for (1) and for (2) to be different.

The mentioned above variational argument consists of showing that in the cases when (1) and (2) are not satisfied there are  $\alpha_1, \alpha_2 \in \mathbb{R}$  such that

$$\max_{i \in A} |\alpha_1 b_{i,1} + \alpha_2 b_{i,2}| < \max\{|\alpha_1|, |\alpha_2|\}.$$

Let us describe the argument in one of the typical cases (all other cases can be treated similarly).

Suppose that  $S$  is such that all entries in the first column are positive,  $S$  contains a row of the form  $(1, b)$  with  $b > 0$ , but not a row of the form  $(1, a)$  with  $a \leq 0$  (recall that absolute values of entries of (3.2) do not exceed 1). It is clear that we get the desired pair by letting  $\alpha_1 = 1$  and choosing  $\alpha_2 < 0$  sufficiently close to 0.

The restriction on the absolute values of the determinants implies that if the second alternative holds in (1), then  $a + b \leq 1$ , and if the second alternative holds in (2), then  $c + d \leq 1$ . This implies that the second alternative cannot hold simultaneously for (1) and (2), and thus, there is a no need for different changes of signs for (1) and (2).

Therefore it suffices to consider two cases:

- (I) The matrix  $S$  is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u & v \end{pmatrix}.$$

- (II) The matrix  $S$  is of the form

$$(3.4) \quad \begin{pmatrix} 1 & 0 \\ c & 1 \\ -d & 1 \end{pmatrix}.$$

Let us show that the fact that the columns of  $S$  span a regular hexagonal space implies that all of its  $2 \times 2$  minors have the same absolute values. It suffices to do this for any basis of the same subspace of  $\ell_\infty^3$ . The subspace should intersect two adjacent edges of the cube. Changing signs of the unit vector basis in  $\ell_\infty^n$ , if necessary, we may assume that the points of intersection are of the forms

$$(3.5) \quad \begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix} \text{ and } \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \quad |\alpha| < 1, \quad |\beta| < 1.$$

The points of intersection are vertices of the hexagon. One more vertex of the hexagon is a vector of the form

$$(3.6) \quad \begin{pmatrix} -1 \\ \gamma \\ 1 \end{pmatrix}, \quad |\gamma| < 1.$$

If the hexagon is linearly equivalent to the regular, then all parallelograms determined by pairs of vectors of the triple described in (3.5) and (3.6) should have equal areas. Therefore, the determinants of matrices formed by a unit vector and two of the vectors from the triple described in (3.5) and (3.6) should have the same absolute values. It is easy to see that the obtained equalities imply  $\alpha = \beta = 0$ . The conclusion follows.

In case **I** the equality of  $2 \times 2$  minors implies that  $|u| = |v| = 1$ , and we have found a  $(\pm 1, \pm 1)$  row.

In the case **II** we derive  $c + d = 1$ . Now we replace the element  $x_1$  in the basis consisting of columns of (3.2) by  $x_1 - cx_2$ . It is clear that the sequence we get is still a basis in the same space, and this modification does not change values of minors of size at least  $2 \times 2$ . As for minors of size  $1 \times 1$ , the only column that has to be checked is column number 1. Its  $k$ -th entry is  $b_{k,1} - cb_{k,2}$ . The condition on  $2 \times 2$  minors of the original matrix implies that  $|cb_{k,2} - b_{k,1}| \leq 1$ . The conclusion follows. On the other hand, in the row (from (3.4)) that started with  $(-d, 1)$  we get  $(-1, 1)$ , and in the row that started with  $(c, 1)$  we get  $(0, 1)$ . Reordering the coordinates of  $\ell_\infty^m$  (if necessary) we get that the space  $X$  has a basis of the form

$$(3.7) \quad (b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & b_{2,3} & \dots & b_{2,n} \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ b_{n+1,1} & b_{n+1,2} & * & \dots & * \\ b_{n+2,1} & b_{n+2,2} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & * & \dots & * \end{pmatrix}.$$

satisfying the following conditions:



- (1) The absolute values of all minors do not exceed 1;
- (2)  $|b_{n+1,1}| = |b_{n+1,2}| = 1$ .

Consider the matrix  $D$  obtained from this matrix in the following way: we keep the values of  $b_{n+1,1}$ ,  $b_{n+1,2}$ , and the entries in the first  $n$  rows with the exception of  $b_{2,3}, \dots, b_{2,n}$ , and let all other entries equal 0.

Matrix  $D$  satisfies the following condition: if some minor of  $D$  is non-zero, then the corresponding minor of (3.7) is its sign. By the results and the discussion in [11, 12], the image of  $B_\infty^m$  in  $X$  whose kernel is the orthogonal complement of  $D$  is a minimal volume projection which is not a parallelepiped. The extension property of  $\ell_\infty^m$  implies that this image is an MVSE.

To prove the result for a general, not necessarily polyhedral, space  $X$ , consider the following polyhedral space  $Y$ : its unit ball is the intersection of the parallelepiped corresponding to a lower Auerbach basis  $\{x_i\}$  of  $X$  with whose half-spaces, which correspond to supporting hyperplanes to  $B_X$  at midpoints of sides of the regular hexagon that is the intersection of  $B_X$  with the linear span of  $x_1, x_2$ . As we have just proved the space  $Y$  has a non-parallelepipedal MVSE. Since there is a minimal-volume parallelepiped containing  $B_X$  that contains  $B_Y$ , each MVSE for  $Y$  is an MVSE for  $X$ . ■

**Remark** Theorem 3.1 solves Problem 6 posed in [14, p. 118].

## 4 Comparison of the Class of Spaces Having Non-Parallelepipedal MVSE with Different Classes of Banach Spaces

### 4.1 MVSE for $\ell_1^n$

Our first purpose is to apply Theorem 3.1 in order to analyze MVSE's of classical polyhedral spaces. For  $\ell_\infty^n$  the situation is quite simple: their unit balls are parallelepipeds and are the only MVSE for  $\ell_\infty^n$ . It turns out that the space  $\ell_1^3$  has non-parallelepipedal MVSE, and that for many other dimensions parallelepipeds are the only MVSE's for  $\ell_1^n$ . To find more on the problem of characterizing  $n$  for which the space  $\ell_1^n$  has non-parallelepipedal MVSE, one has to analyze known results on the Hadamard maximal determinant problem. See [8] for some such results and related references. In this paper we make only two simple observations.

**Proposition 4.1** *If  $n$  is such that there exists a Hadamard matrix of size  $n \times n$ , then each MVSE for  $\ell_1^n$  is a parallelepiped.*

**Proof** Each upper Auerbach basis for  $\ell_\infty^n$  in such dimensions consists of columns of Hadamard matrices. Hence their biorthogonal functionals are also (properly normalized) Hadamard matrices. It is easy to see that any two of them span a subspace in  $\ell_1^n$  isometric to  $\ell_1^2$ . ■

**Proposition 4.2** *The three-dimensional space  $\ell_1^3$  has a non-parallelepipedal MVSE.*

**Proof** The columns of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

form an upper Auerbach basis in  $\ell_\infty^3$ . The columns of the matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

form a biorthogonal system of this upper Auerbach basis. It is easy to check that the first two vectors of the biorthogonal system span a regular hexagonal subspace in  $\ell_1^3$ . ■

#### 4.2 The Shape of MVSE and Presence of a 1-Complemented Regular Hexagonal Space

It would be useful to characterize spaces having non-parallelepipedal MVSE's in terms of their complemented subspaces. The purpose of this section is to show that one of the most natural approaches to such a characterization fails. More precisely, we show that the presence of a 1-complemented subspace whose unit ball is linearly equivalent to a regular hexagon neither implies nor follows from the existence of a non-parallelepipedal MVSE.

**Proposition 4.3** *There exist spaces having 1-complemented subspaces whose unit balls are regular hexagons but such that each of their MVSE's is a parallelepiped.*

**Proof** Let  $X$  be the  $\ell_1$ -sum of a regular hexagonal space and a one-dimensional space.

- (1) The unit ball of the space does not have other sections linearly equivalent to regular hexagons. This statement can be proved using the argument presented immediately after equation (3.6).
- (2) Assume that the vertices of  $B_X$  have coordinates

$$\pm(0, 0, 1), \quad \pm(1, 0, 0), \quad \pm\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0\right).$$

Denote by  $H$  the hyperplane containing  $(1, 0, 0)$  and  $(0, 1, 0)$ . We show that a lower Auerbach basis cannot contain two vectors in  $H$ .

In fact, an easy argument shows that the volume of a parallelepiped of the form  $M(\{x_i\}_{i=1}^3)$  containing  $B_X$  and such that  $x_1, x_2 \in H$  is at least  $4\sqrt{3}$ . On the other hand, it is easy to check that the volume of a minimal-volume parallelepiped containing  $B_X$  is  $\leq 2\sqrt{3}$ . ■

**Remark** The argument of [12, pp. 393–395] implies that  $\ell_\infty$ -sums of a regular hexagonal space and any space have non-parallelepipedal MVSE's.

**Proposition 4.4** *The existence of a lower Auerbach basis with two elements spanning a regular hexagonal subspace does not imply the presence of a 1-complemented regular hexagonal subspace.*

**Proof** Consider the subspace  $X$  of  $\ell_\infty^4$  described by the equation  $x_1 + x_2 + x_3 + x_4 = 0$ . The fact that this space has a non-parallelepipedal MVSE follows immediately from the fact that the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

form a lower Auerbach basis in  $X$  (see the argument after equation (3.1)) and any two of them span a subspace whose unit ball is linearly equivalent to a regular hexagon.

So it remains to show that the space  $X$  does not have 1-complemented subspaces linearly equivalent to a regular hexagonal space. It suffices to prove the following two claims. By a *support* of a vector in  $\ell_\infty^m$  we mean the set of labels of its non-zero coordinates.

**Claim 1** *The only two-dimensional subspaces of  $X$  that have balls linearly equivalent to regular hexagons are the spaces spanned by vectors belonging to  $X$  and having intersecting two-element supports.*

**Claim 2** *Two-dimensional subspaces satisfying the conditions of Claim 1 are not 1-complemented in  $X$ .*

*Proof of Claim 1* Consider a two-dimensional subspace  $H$  of  $X$ . It is easy to check that if the unit ball of  $H$  is a hexagon, then each extreme point of the hexagon is of the form: two coordinates are 1 and  $-1$ , the remaining two are  $\alpha$  and  $-\alpha$  for some  $\alpha$  satisfying  $|\alpha| \leq 1$ . Two different forms cannot give the same extreme point unless the corresponding value of  $\alpha$  is  $\pm 1$ . Also two points of the same type cannot be present unless the corresponding values of  $\alpha$  are  $+1$  and  $-1$ . Since  $B_H$  is a hexagon, there are 3 pairs of extreme points. First we consider the case when none of  $\alpha_i, i = 1, 2, 3$ , corresponding to an extreme point is  $\pm 1$ . Then  $\pm 1$  either form a cycle or a chain in the sense shown in (4.1):

$$(4.1) \quad \begin{pmatrix} 1 & \alpha_2 & -1 \\ -1 & 1 & \alpha_3 \\ \alpha_1 & -1 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \alpha_2 & \alpha_3 \\ -1 & 1 & -\alpha_3 \\ \alpha_1 & -1 & 1 \\ -\alpha_1 & -\alpha_2 & -1 \end{pmatrix}.$$

If they form a cycle, by considering determinants (as after (3.6)) with other unit vectors, we get that all involved  $\alpha_i$  are zeros. Thus we get a subspace of the form described in the statement of the lemma.

We show that  $\pm 1$  cannot form a chain as in the second matrix in (4.1) by showing that in such a case they cannot be linearly dependent. In fact, multiplying the first column by  $\alpha_3$  and subtracting the resulting column from the third column we get

$$\begin{pmatrix} 1 & \alpha_2 & 0 \\ -1 & 1 & 0 \\ \alpha_1 & -1 & 1 - \alpha_1\alpha_3 \\ -\alpha_1 & -\alpha_2 & -1 + \alpha_1\alpha_3 \end{pmatrix}.$$

It is clear that this matrix has rank 3.

It remains to consider the case when all of the coordinated of some of the extreme points are  $\pm 1$ . Assume without loss of generality that one of the extreme points is  $(1, 1, -1, -1)$ . If there is one more  $\pm 1$  extreme point (different from  $(-1, -1, 1, 1)$ ), the section is a parallelogram.

If the other extreme point is not a  $\pm 1$  point, then it has both  $+1$  and  $-1$  either in the first two positions or in the last two positions (otherwise it is not an extreme point). In this case the section is also a parallelogram, because the norm on their linear combinations is just the  $\ell_1$ -norm.

This completes the proof of Claim 1.

*Proof of Claim 2* In fact, assume without loss of generality that we consider a two dimensional subspace spanned by the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

We need to show that there is no vector in this subspace such that projecting the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

onto it we get a projection of norm 1 on  $X$ . Assume the contrary. Let

$$\begin{pmatrix} a \\ b - a \\ -b \\ 0 \end{pmatrix}$$

be the desired vector. The condition that the images of the vectors

$$\begin{pmatrix} 1 \\ -1 \\ \pm 1 \\ \mp 1 \end{pmatrix}$$

under the projection are vectors of norm  $\leq 1$  immediately implies that  $a = (b - a) = 0$ . hence  $a = b = 0$ . Now we get a contradiction by projecting the vector

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix};$$

its image has norm 2.

This completes the proof of Claim 2 and Proposition 4.4. ■

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