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ON MONOTONE INCREASING REPRESENTATION FUNCTIONS

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Abstract

Let $k \geq 2$ be an integer and let A be a set of nonnegative integers. The representation function $R_{A,k}(n)$ for the set A is the number of representations of a nonnegative integer n as the sum of k terms from A. Let $A(n)$ denote the counting function of A. Bell and Shallit ['Counterexamples to a conjecture of Dombi in additive number theory', *Acta Math. Hung.*, to appear] recently gave a counterexample for a conjecture of Dombi and proved that if $A(n) = o(n^{(k-2)/k-\epsilon})$ for some $\epsilon > 0$, then $R_{\mathbb{N}\setminus A,k}(n)$ is eventually strictly increasing. We improve this result to $A(n) = O(n^{(k-2)/(k-1)})$. We also give an example to show that this bound is best possible.

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers and let A be a subset of nonnegative integers. We use A^n to denote the Cartesian product of n sets A, that is,

$$A^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in A\}.$$

Let

$$R_{A,k}(n) = |\{(a_1, a_2, \ldots, a_k) \in A^k : a_1 + a_2 + \cdots + a_k = n\}|,$$

$$R^{<}_{A,k}(n) = |\{(a_1, a_2, \ldots, a_k) \in A^k : a_1 + a_2 + \cdots + a_k = n, a_1 < a_2 < \cdots < a_k\}|,$$

$$R^{\leq}_{A,k}(n) = |\{(a_1, a_2, \ldots a_k) \in A^k : a_1 + a_2 + \cdots + a_k = n, a_1 \leq a_2 \leq \cdots \leq a_k\}|,$$

where $|\cdot|$ denotes the cardinality of a finite set. We say that $R_{A,k}(n)$ is monotonically increasing in n from a certain point on (or eventually monotone increasing) if there exists an integer n_0 such that $R_{A,k}(n+1) \geq R_{A,k}(n)$ for all integers $n \geq n_0$. We define the monotonicity of the other two representation functions $R^{<}_{A,k}(n)$ and $R^{\leq}_{A,k}(n)$ in the same way.

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We denote the counting function of the set A by

$$A(n) = \sum_{\substack{a \in A \\ a \leq n}} 1.$$

We define the lower asymptotic density of a set A of natural numbers by

$$\liminf_{n\to\infty} \frac{A(n)}{n}$$

and the asymptotic density by

$$\lim_{n\to\infty} \frac{A(n)}{n}$$

whenever the limit exists. The generating function of a set A of natural numbers is denoted by

$$G_A(x) = \sum_{a\in A} x^a.$$

Obviously, if $\mathbb{N} \setminus A$ is finite, then each of the functions $R_{A,2}(n), R^{<}_{A,2}(n)$ and $R^{\leq}_{A,2}(n)$ is eventually monotone increasing. In [4, 5], Erdős *et al.* investigated whether there exists a set A for which $\mathbb{N} \setminus A$ is infinite and the representation functions are monotone increasing from a certain point on. They proved the following theorems.

THEOREM A. *The function $R_{A,2}(n)$ is monotonically increasing from a certain point on if and only if the sequence A contains all the integers from a certain point on, that is, there exists an integer n_1 with*

$$A \cap \{n_1, n_1 + 1, n_1 + 2, \dots\} = \{n_1, n_1 + 1, n_1 + 2, \dots\}.$$

THEOREM B. *There exists an infinite set $A \subseteq \mathbb{N}$ such that $A(n) < n - cn^{1/3}$ for $n > n_0$ and $R^{<}_{A,2}(n)$ is monotone increasing from a certain point on.*

THEOREM C. *If*

$$A(n) = o\Big(\frac{n}{\log n}\Big),$$

then the functions $R^{<}_{A,2}(n)$ and $R^{\leq}_{A,2}(n)$ cannot be monotonically increasing in n from a certain point on.

THEOREM D. *If $A \subseteq \mathbb{N}$ is an infinite set with*

$$\lim_{n\to\infty} \frac{n - A(n)}{\log n} = \infty,$$

then $R^{\leq}_{A,2}(n)$ cannot be monotone increasing from a certain point on.

The last theorem was proved independently by Balasubramanian [1]. Very little is known when $k > 2$. The following result was proved many years ago in [8] and independently in [6].

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THEOREM E. *If k is an integer with $k > 2$, $A \subseteq \mathbb{N}$ and $R_{A,k}(n)$ is monotonically increasing in n from a certain point on, then*

$$A(n) = o\Big(\frac{n^{2/k}}{(\log n)^{2/k}}\Big)$$

cannot hold.

Dombi [3] constructed sets A of asymptotic density $\frac{1}{2}$ such that for $k > 4$, the function $R_{A,k}(n)$ is monotone increasing from a certain point on. His constructions are based on the Rudin–Shapiro sets and Thue–Morse sequences. However, Dombi gave the following conjecture.

DOMBI'S CONJECTURE. If $\mathbb{N} \setminus A$ is infinite, then $R_{A,k}(n)$ cannot be strictly increasing.

For $k \geq 3$, Bell and Shallit [2] recently gave a counterexample of Dombi's conjecture by applying tools from automata theory and logic. They also proved the following result.

THEOREM F. *Let k be an integer with $k \geq 3$ and let $F \subseteq \mathbb{N}$ with $0 \notin F$. If $F(n) = o(n^{\alpha})$ for $\alpha < (k-2)/k$ and $A = \mathbb{N} \setminus F$, then $R_{A,k}(n)$ is eventually strictly increasing.*

In this paper, we improve this result in the following theorem.

THEOREM 1.1. *Let k be an integer with $k \geq 3$. If $A \subseteq \mathbb{N}$ satisfies*

$$A(n) \leqslant \frac{n^{(k-2)/(k-1)}}{\sqrt[k-1]{(k-2)!}} - 2$$

for all sufficiently large integers n, then $R_{\mathbb{N}\setminus A,k}(n)$ is eventually strictly increasing.

In particular, for $k = 3$, this gives the following corollary.

COROLLARY 1.2. *If $A \subseteq \mathbb{N}$ satisfies $A(n) \leqslant \sqrt{n} - 2$ for all sufficiently large integers n, then $R_{\mathbb{N}\setminus A,3}(n)$ is eventually strictly increasing.*

After we uploaded our paper to arXiv, we were informed that Mihalis Kolountzakis proved in an unpublished note that if $A \subseteq \mathbb{N}$ satisfies $A(n) \leqslant c\sqrt{n}$ for a sufficiently small positive constant c, then $R_{\mathbb{N}\setminus A,3}(n)$ is eventually strictly increasing. We improve the constant factor in the following result.

THEOREM 1.3. *If $A \subseteq \mathbb{N}$ satisfies $A(n) \leqslant (2/\sqrt{3})\sqrt{n} - 2$ for all sufficiently large integers n, then $R_{\mathbb{N}\setminus A,3}(n)$ is eventually strictly increasing.*

It turns out from the next theorem that the upper bound for the counting function of A in Theorem 1.1 is tight up to a constant factor.

THEOREM 1.4. *Suppose that $f(n)$ is a function satisfying $f(n) \to \infty$ as $n \to \infty$. Then there is a set $A \subseteq \mathbb{N}$ such that $A(n) < \sqrt[k-1]{k-1} \cdot n^{(k-2)/(k-1)} + f(n)$ for all sufficiently large integers n and $R_{\mathbb{N}\setminus A,k}(n) < R_{\mathbb{N}\setminus A,k}(n-1)$ for infinitely many positive integers n.*

Shallit [7] recently constructed a set A with positive lower asymptotic density such that the function $R_{\mathbb{N}\setminus A,3}(n)$ is strictly increasing.

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2. Proofs

The proofs of the theorems are based on the next lemma, coming from Bell and Shallit's paper [2] although not explicitly stated there.

LEMMA 2.1. *For any positive integers n and k with $k \geq 3$,*

$$R_{\mathbb{N}\setminus A,k}(n) - R_{\mathbb{N}\setminus A,k}(n-1) = \binom{n+k-2}{k-2} + \sum_{i=1}^{k-2}\binom{k}{i}(-1)^i\Big(\sum_{m=0}^{n}\binom{m+k-i-2}{k-i-2}R_{A,i}(n-m)\Big)$$
$$+ (-1)^{k-1}kR_{A,k-1}(n) + (-1)^k(R_{A,k}(n) - R_{A,k}(n-1)).$$

PROOF. Observe that

$$(1-x)(G_{\mathbb{N}\setminus A}(x))^k = \sum_{n=0}^{\infty} R_{\mathbb{N}\setminus A,k}(n)x^n - \sum_{n=0}^{\infty} R_{\mathbb{N}\setminus A,k}(n)x^{n+1}$$
$$= R_{\mathbb{N}\setminus A,k}(0) + \sum_{n=1}^{\infty}(R_{\mathbb{N}\setminus A,k}(n) - R_{\mathbb{N}\setminus A,k}(n-1))x^n.$$

However,

$$(1-x)((G_{\mathbb{N}\setminus A})(x))^k = (1-x)\Big(\frac{1}{1-x} - G_A(x)\Big)^k = (1-x)\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{(1-x)^{k-i}}G_A(x)^i$$
$$= \frac{1}{(1-x)^{k-1}} + \sum_{i=1}^{k-2}\binom{k}{i}\frac{(-1)^i}{(1-x)^{k-i-1}}G_A(x)^i + (-1)^{k-1}kG_A(x)^{k-1} + (-1)^k(1-x)G_A(x)^k.$$

It is well known that

$$\frac{1}{(1-x)^m} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}x^n.$$

It follows that

$$R_{\mathbb{N}\setminus A,k}(0) + \sum_{n=1}^{\infty}(R_{\mathbb{N}\setminus A,k}(n) - R_{\mathbb{N}\setminus A,k}(n-1))x^n$$
$$= \sum_{n=0}^{\infty}\binom{n+k-2}{k-2}x^n + \sum_{i=1}^{k-2}(-1)^i\binom{k}{i}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{m+k-i-2}{k-i-2}R_{A,i}(n-m)\Big)x^n$$
$$+ (-1)^{k-1}k\sum_{n=0}^{\infty}R_{A,k-1}(n)x^n + (-1)^kR_{A,k}(0) + (-1)^k\sum_{n=0}^{\infty}(R_{A,k}(n) - R_{A,k}(n-1))x^n.$$

By comparing the coefficient of x^n on both sides of this equation, Lemma 2.1 follows immediately. □

PROOF OF THEOREM 1.1. Clearly,

$$R_{A,i}(n) = |\{(a_1, a_2, \ldots, a_i) \in A^i : a_1 + a_2 + \cdots + a_i = n\}|$$
$$\leq |\{(a_1, a_2, \ldots, a_{i-1}) \in A^{i-1} : a_1, a_2, \ldots, a_{i-1} \leq n\}| = A(n)^{i-1}.$$

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By Lemma 2.1, there exist constants c_1, c_2, c_3, c_4 only depending on k such that

$$
\begin{aligned}
&R_{\mathbb{N}\setminus A,k}(n) - R_{\mathbb{N}\setminus A,k}(n-1)\\
&= \binom{n+k-2}{k-2} + \sum_{i=1}^{k-2}\binom{k}{i}(-1)^i\Big(\sum_{m=0}^{n}\binom{m+k-i-2}{k-i-2}R_{A,i}(n-m)\Big)\\
&\quad + (-1)^{k-1}kR_{A,k-1}(n) + (-1)^k(R_{A,k}(n) - R_{A,k}(n-1))\\
&\geq \frac{n^{k-2}}{(k-2)!} - \sum_{i=1}^{k-2}2^k\sum_{m=0}^{n}\binom{m+k-i-2}{k-i-2}A(n)^{i-1} - kR_{A,k-1}(n) - A(n)^{k-1}\\
&\geq \frac{n^{k-2}}{(k-2)!} - \sum_{i=1}^{k-2}2^kA(n)^{i-1}\binom{n+k-i-1}{k-i-2}\\
&\quad - k\Big(\frac{n^{(k-2)/(k-1)}}{\sqrt[k-1]{(k-2)!}}\Big)^{k-2} - \Big(\frac{n^{(k-2)/(k-1)}}{\sqrt[k-1]{(k-2)!}} - 2\Big)^{k-1}\\
&\geq \frac{n^{k-2}}{(k-2)!} - c_1\sum_{i=1}^{k-2}A(n)^{i-1}n^{k-i-2} - k\cdot\frac{n^{(k-2)^2/(k-1)}}{((k-2)!\,)^{(k-2)/(k-1)}}\\
&\quad - \Big(\frac{n^{k-2}}{(k-2)!} - 2(k-1)\frac{n^{(k-2)^2/(k-1)}}{((k-2)!\,)^{(k-2)/(k-1)}} + c_2n^{(k-2)(k-3)/(k-1)}\Big)\\
&\geq \frac{n^{k-2}}{(k-2)!} - c_3n^{k-3} - k\frac{n^{(k-2)^2/(k-1)}}{((k-2)!\,)^{(k-2)/(k-1)}}\\
&\quad - \Big(\frac{n^{k-2}}{(k-2)!} - \frac{2(k-1)n^{(k-2)^2/(k-1)}}{((k-2)!\,)^{(k-2)/(k-1)}} + c_2n^{(k-2)(k-3)/(k-1)}\Big)\\
&= \frac{k-2}{((k-2)!\,)^{(k-2)/(k-1)}}\cdot n^{(k-2)^2/(k-1)} - c_4n^{k-3}.
\end{aligned}
$$

Hence, $R_{\mathbb{N}\setminus A,k}(n) - R_{\mathbb{N}\setminus A,k}(n-1) > 0$ when n is large enough. □

LEMMA 2.2. *For any set A of natural numbers and for any natural number n, one has $R_{A,3}(n) \leqslant \frac{3}{4}A(n)^2 + \{\frac{1}{4}A(n)^2\}$, where $\{x\}$ denotes the fractional part of x.*

Note that Lemma 2.2 is sharp: if $A = \{0, 1, \ldots, m\}$, then

$$R_{A,3}\Big(\Big\lfloor\frac{3m}{2}\Big\rfloor\Big) = \frac{3}{4}A\Big(\Big\lfloor\frac{3m}{2}\Big\rfloor\Big)^2 + \Big\{\frac{A(\lfloor 3m/2\rfloor)^2}{4}\Big\},$$

where $\lfloor y\rfloor$ denotes the maximal integer not greater than y.

PROOF OF LEMMA 2.2. Fix a natural number n. Let $A \cap [1, n] = \{a_1 < a_2 < \cdots < a_m\}$ and $\overline{A} = \{n - a_m < n - a_{m-1} < \cdots < n - a_1\}$. For $i = 1, 2, \ldots, m$, we define

$$A_i = \{a_i + a_1 < a_i + a_2 < \cdots < a_i + a_{m+1-i} < a_{i+1} + a_{m+1-i} < \cdots < a_m + a_{m+1-i}\}.$$

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Clearly,

$$\begin{aligned} R_{A,3}(n) = \sum_{i=1}^{m} |A_i \cap \overline{A}| \leqslant \sum_{i=1}^{m} \min\{2m - 2i + 1, m\} \\ = \sum_{i=1}^{\lfloor m/2 \rfloor} m + \sum_{i=\lfloor m/2 \rfloor + 1}^{m} (2m - 2i + 1) \\ = m\left\lfloor \frac{m}{2} \right\rfloor + \left(m - \left\lfloor \frac{m}{2} \right\rfloor\right)^2 = \frac{3}{4}m^2 + \left\{\frac{m^2}{4}\right\}. \end{aligned}$$

□

PROOF OF THEOREM 1.3. Applying Lemma 2.1 for $k = 3$,

$$\begin{aligned} &R_{\mathbb{N}\setminus A,3}(n) - R_{\mathbb{N}\setminus A,3}(n-1) \\ &\quad = n + 1 - 3\sum_{m=0}^{n} R_{A,1}(n-m) + 3R_{A,2}(n) - (R_{A,3}(n) - R_{A,3}(n-1)) \\ &\quad = n + 1 - 3A(n) + 3R_{A,2}(n) - (R_{A,3}(n) - R_{A,3}(n-1)). \end{aligned}$$

Hence, by Lemma 2.2,

$$\begin{aligned} R_{\mathbb{N}\setminus A,3}(n) - R_{\mathbb{N}\setminus A,3}(n-1) &\geq n + 1 - 3A(n) - R_{A,3}(n) \\ &\geq n + 1 - 3\left(\frac{2}{\sqrt{3}}\sqrt{n} - 2\right) - \frac{3}{4}\left(\frac{2}{\sqrt{3}}\sqrt{n} - 2\right)^2 - \frac{1}{4} = \frac{15}{4} > 0, \end{aligned}$$

which completes the proof. □

PROOF OF THEOREM 1.4. We may suppose that $f(n) < \sqrt[k-1]{k-1} \cdot n^{(k-2)/(k-1)}$. We define an infinite sequence of natural numbers $N_1, N_2, \ldots$ by induction. Let $N_1 = 100k^4$. Assume that $N_1, \ldots, N_j$ are already defined. Let N_{j+1} be an even number with $N_{j+1} > 100k^4 N_j^{k-1}$ and $f(n) > (k-1)(N_1^{k-2} + \cdots + N_j^{k-2})$ for every $n \geq N_{j+1}$. We define the set A by

$$A = \bigcup_{j=1}^{\infty} \{N_j, 2N_j, 3N_j, \ldots, (k-1)N_j^{k-1}\}.$$

First, we give an upper estimation for $A(n)$. Let $n \geq 100k^4$. Then there exists an index j such that $N_j \leq n < N_{j+1}$. Define l as the largest integer with $l \leq (k-1)N_j^{k-2}$ and $lN_j \leq n$. Then,

$$\begin{aligned} &A(n) - \sqrt[k-1]{k-1}\, n^{(k-2)/(k-1)} \\ &\quad \leq (k-1)(N_1^{k-2} + \cdots + N_j^{k-2}) + l - \sqrt[k-1]{k-1}\,(lN_j)^{(k-2)/(k-1)} \\ &\quad = (k-1)(N_1^{k-2} + \cdots + N_j^{k-2}) + l^{(k-2)/(k-1)}(l^{1/(k-1)} - (k-1)^{1/(k-1)} N_j^{(k-2)/(k-1)}) \\ &\quad \leq f(n), \end{aligned}$$

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which implies that

$$A(n) < \sqrt[k-1]{k-1} \cdot n^{(k-2)/(k-1)} + f(n).$$

Next, we shall prove that there exist infinitely many positive integers n such that $R_{\mathbb{N}\setminus A,k}(n) < R_{\mathbb{N}\setminus A,k}(n-1)$. To prove this, we divide into two cases according to the parity of k.

Suppose that k is an odd integer. For $j = 1, 2, \ldots$, we define

$$u_j = (k-1)N_j^{k-1} + 100(k-2)(k-1)^3 N_j^{k-2}.$$

Now, we show that $R_{\mathbb{N}\setminus A,k}(u_j) < R_{\mathbb{N}\setminus A,k}(u_j - 1)$ when j is large enough.

Since all the elements of A are even and $u_j - 1$ is odd, it follows that $R_{A,k}(u_j - 1) = 0$. By Lemma 2.1,

$$\begin{aligned}
&R_{\mathbb{N}\setminus A,k}(u_j) - R_{\mathbb{N}\setminus A,k}(u_j - 1)\\
&\quad= \binom{u_j + k - 2}{k-2} + \sum_{i=1}^{k-2} \binom{k}{i}(-1)^i \Big(\sum_{m=0}^{u_j} \binom{m + k - i - 2}{k - i - 2} R_{A,i}(u_j - m) \Big)\\
&\qquad + (-1)^{k-1} k R_{A,k-1}(u_j) + (-1)^k (R_{A,k}(u_j) - R_{A,k}(u_j - 1))\\
&\quad\le \binom{u_j + k - 2}{k-2} + k^2 \Big(\sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} R_{A,2}(u_j - m) \Big)\\
&\qquad + \sum_{i=3}^{k-2} 2^k \sum_{m=0}^{u_j} \binom{m + k - i - 2}{k - i - 2} A(u_j)^{i-1} + k R_{A,k-1}(u_j) - R_{A,k}(u_j).
\end{aligned} \tag{2.1}$$

Next we shall give a bound for each term of the right-hand side of (2.1). There exists a constant c_5 only depending on k such that

$$\binom{u_j + k - 2}{k-2} \le \frac{(k-1)^{k-2} N_j^{k^2 - 3k + 2} + 100(k-2)^2 (k-1)^k N_j^{k^2 - 3k + 1} + c_5 N_j^{k^2 - 3k}}{(k-2)!} \tag{2.2}$$

and

$$\begin{aligned}
&k^2 \sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} R_{A,2}(u_j - m) \le k^2 \sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} A(u_j - m)\\
&\quad\le k^2 \sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} A(kN_j^{k-1}) \le k^2 \sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} 2 \sqrt[k-1]{k-1} (kN_j^{k-1})^{(k-2)/(k-1)}\\
&\quad\le k^2 \sum_{m=0}^{u_j} \binom{m + k - 4}{k-4} 2kN_j^{k-2} = 2k^3 N_j^{k-2} \binom{u_j + k - 3}{k-3}\\
&\quad\le 2k^3 N_j^{k-2} \binom{kN_j^{k-1}}{k-3} \le \frac{2k^k}{(k-3)!} N_j^{k^2 - 3k + 1}.
\end{aligned} \tag{2.3}$$

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Furthermore,

$$\begin{aligned}
&\sum_{i=3}^{k-2} 2^k \sum_{m=0}^{u_j} \binom{m+k-i-2}{k-i-2} A(u_j)^{i-1} \\
&\quad \le c_6 \sum_{i=3}^{k-2} 2^k \sum_{m=0}^{u_j} \binom{m+k-i-2}{k-i-2} ((N_j^{k-1})^{(k-2)/(k-1)})^{i-1} \\
&\quad \le c_6 \sum_{i=3}^{k-2} 2^k N_j^{(k-2)(i-1)} \sum_{m=0}^{u_j} \binom{m+k-i-2}{k-i-2} \\
&\quad = c_6 \sum_{i=3}^{k-2} 2^k N_j^{(k-2)(i-1)} \binom{u_j+k-i-1}{k-i-1} \\
&\quad \le c_7 \sum_{i=3}^{k-2} N_j^{(k-2)(i-1)} \cdot N_j^{(k-1)(k-i-1)} = c_7 \sum_{i=3}^{k-2} N_j^{k^2-3k-i+3} \le c_8 N_i^{k^2-3k},
\end{aligned} \tag{2.4}$$

where c_6, c_7 and c_8 are constants only depending on k. Moreover,

$$\begin{aligned}
R_{A,k-1}(u_j)) &\le A(u_j)^{k-2} \le A(kN_j^{k-1})^{k-2} \\
&\le (2\sqrt[k-1]{k-1}(kN_j^{k-1})^{(k-2)/(k-1)})^{k-2} \le (2k)^{k-2} N_j^{(k-2)^2}.
\end{aligned} \tag{2.5}$$

Obviously,

$$\begin{aligned}
&R_{A,k}(u_j) \\
&\ge \left|\left\{(x_1,\dots,x_k) \in (\mathbb{Z}^+)^k : \sum_{t=1}^{k} x_t = u_j, N_j \mid x_t, x_t \le (k-1)N_j^{k-1} \text{ for } t = 1,\dots,k\right\}\right| \\
&= \left|\left\{(y_1,\dots,y_k) \in (\mathbb{Z}^+)^k : \sum_{t=1}^{k} y_t = \frac{u_j}{N_j}, y_t \le (k-1)N_j^{k-2} \text{ for } t = 1,\dots,k\right\}\right| \\
&= \left|\left\{(y_1,\dots,y_k) \in (\mathbb{Z}^+)^k : \sum_{t=1}^{k} y_t = \frac{u_j}{N_j}\right\}\right| \\
&\quad - \left|\left\{(y_1,\dots,y_k) \in (\mathbb{Z}^+)^k : \sum_{t=1}^{k} y_t = \frac{u_j}{N_j},\ y_t > (k-1)N_j^{k-2} \text{ for some } t \in \{1,\dots,k\}\right\}\right|.
\end{aligned}$$

We see that

$$\begin{aligned}
&\left|\left\{(y_1,\dots,y_k) \in (\mathbb{Z}^+)^k : y_1 + \cdots + y_k = \frac{u_j}{N_j}\right\}\right| \\
&\quad = \binom{u_j/N_j - 1}{k-1} \ge \frac{N_j^{k^2-3k+2} + 100(k-2)(k-1)^{k+2} N_j^{k^2-3k+1} + c_9 N_j^{k^2-3k}}{(k-1)!},
\end{aligned}$$

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where c_9 is a constant only depending on k, and

$$\begin{aligned}&\left|\left\{(y_1,\dots,y_k)\in(\mathbb{Z}^+)^k : \sum_{t=1}^{k} y_t=\frac{u_j}{N_j},\ y_t>(k-1)N_j^{k-2} \text{ for some } t\in\{1,\dots,k\}\right\}\right|\\ &\quad = k|\{(z_1,\dots,z_k)\in(\mathbb{Z}^+)^k : z_1+\cdots+z_k=100(k-2)(k-1)^3N_j^{k-3}\}|\\ &\quad \le k(100(k-2)(k-1)^3)^kN_j^{k^2-3k}.\end{aligned}$$

The last equality holds because if $y_1+\cdots+y_k=u_j/N_j$ with $y_t>(k-1)N_j^{k-1}$, then

$$y_1+\cdots+y_{t-1}+(y_t-(k-1)N_j^{k-2})+y_{t+1}+\cdots+y_k=100(k-2)(k-1)^3N_j^{k-3},$$

where every term is positive. Furthermore, if $z_1+\cdots+z_k=100(k-2)(k-1)^3N_j^{k-3}$, $z_i\in\mathbb{Z}^+$, then one can create k different sums of the form $y_1+\cdots+y_k=u_j/N_j$ with $y_i=z_i$ if $i\neq t$ and $y_t=z_t+(k-1)N_j^{k-2}$. Therefore,

$$R_{A,k}(u_j)\ge\frac{(k-1)^{k-1}N_j^{k^2-3k+2}+100(k-2)(k-1)^{k+2}N_j^{k^2-3k+1}+c_{10}N_j^{k^2-3k}}{(k-1)!},\tag{2.6}$$

where c_{10} is a constant. In view of (2.1)–(2.6),

$$\begin{aligned}&R_{\mathbb{N}\setminus A,k}(u_j)-R_{\mathbb{N}\setminus A,k}(u_j-1)\\ &\quad\le\frac{(k-1)^{k-2}N_j^{k^2-3k+2}+100(k-2)^2(k-1)^kN_j^{k^2-3k+1}+c_5N_j^{k^2-3k}}{(k-2)!}\\ &\qquad+\frac{2k^k}{(k-3)!}N_j^{k^2-3k+1}+c_8N_i^{k^2-3k}+(2k)^{k-2}N_j^{(k-2)^2}\\ &\qquad-\frac{(k-1)^{k-1}N_j^{k^2-3k+2}+100(k-2)(k-1)^{k+2}N_j^{k^2-3k+1}+c_{10}N_j^{k^2-3k}}{(k-1)!}\\ &\quad=\left(\frac{2k^k}{(k-3)!}-100\frac{(k-1)^k}{(k-3)!}\right)N_j^{k^2-3k+1}+(2k)^{k-2}N_j^{(k-2)^2}+c_{11}N_j^{k^2-3k},\end{aligned}$$

where c_{11} is a constant. Thus, we have $R_{\mathbb{N}\setminus A,k}(u_j)<R_{\mathbb{N}\setminus A,k}(u_j-1)$ when j is large enough.

If k is even, then the same argument shows that $R_{\mathbb{N}\setminus A,k}(u_j+1)<R_{\mathbb{N}\setminus A,k}(u_j)$ when j is large enough. □

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