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ON PROPERTIES OF FINITE-ORDER MEROMORPHIC SOLUTIONS OF A CERTAIN DIFFERENCE PAINLEVÉ I EQUATION

MEI-RU CHEN✉ and ZONG-XUAN CHEN

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Abstract

In this paper, we investigate properties of finite-order transcendental meromorphic solutions, rational solutions and polynomial solutions of the difference Painlevé I equation

$$f(z+1)+f(z)+f(z-1)=\frac{az+b}{f(z)}+c,$$

where a, b and c are constants, $|a|+|b|\neq 0$.

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1. Introduction and results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [14, 19, 24]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of $f(z)$; and $\lambda(f)$, $\lambda(1/f)$ and $\tau(f)$ to denote, respectively, the exponents of convergence of zeros, poles and fixed points of $f(z)$. Note that the exponent of convergence of fixed points of f is defined as

$$\tau(f)=\limsup_{r\to\infty}\frac{\log N(r,1/(f-z))}{\log r}.$$

Painlevé and his colleagues [21] classified all equations of Painlevé type of the form

$$\frac{d^2y}{dz^2}=F\left(z;y,\frac{dy}{dz}\right),$$

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where F is rational in y and dy/dz and (locally) analytic in z. The first of these is P_I:

$$\frac{d^2y}{dz^2} = 6y^2 + z.$$

Differential Painlevé equations were discovered at the beginning of the last century. Although they were discovered from mathematical considerations, they occur in many physical situations—plasma physics, statistical mechanics, nonlinear waves, quantum gravity, general relativity, quantum field theory, nonlinear optics and fibre optics. Painlevé equations have attracted much interest as the reduction of soliton equations which are solvable by inverse scattering transformations (see [1, 3, 17]), and so on. In particular, Kudryashov [18] showed that the second Painlevé equation can be used as a model for describing the electric field in a semiconductor.

In the past 20 years, discrete Painlevé equations have become important research problems (see [9]). For example, the equation

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \gamma,$$

where α, β and γ are constants, $n \in \mathbb{N}$, is known as the special discretization of P_I.

Recently, a number of papers (including [2, 4, 6, 8, 10–13, 15, 16, 20]) have focused on complex difference equations and difference analogues of Nevanlinna's theory. While investigating difference analogues of Nevanlinna's theory, many results on complex difference equations were rapidly obtained.

Some results on the existence of meromorphic solutions for certain difference equations were obtained by Shimomura [22] and Yanagihara [23] during the early 1980s.

Ablowitz *et al.* [2] looked at difference equations of the type

$$w(z+1) + w(z-1) = R(z, w), \tag{1.1}$$

where R is rational in both of its arguments, and showed the following theorem.

THEOREM A (See [2]). *If the second-order difference equation*

$$y(z+1) + y(z-1) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + a_q(z)y^q},$$

where a_i and b_i are polynomials, admits a nonrational meromorphic solution of finite order, then $\max(p, q) \le 2$.

Halburd and Korhonen [11–13] used value distribution theory and reasoning related to singularity confinement to single out difference Painlevé I equations from the difference equation (1.1). They obtained that if (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed by a linear change in w to some classical difference equations, which

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include the difference Painlevé I equations

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.2}$$

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+\frac{c}{f(z)^2}, \tag{1.3}$$

$$f(z+1)+f(z)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.4}$$

where a, b and c are constants.

From the above, we see that the difference Painlevé I equation is the development of the differential and discrete Painlevé I equation. So it is an important class of difference equation. Besides, the Painlevé I equation is important because it creates a new transcendental function. Therefore, it is important to investigate properties of finite-order transcendental meromorphic solutions and rational solutions and polynomial solutions of this difference Painlevé I equation.

Chen and Shon [7] investigated some properties of meromorphic solutions of the difference Painlevé I equation, and proved the following theorem.

THEOREM B. *Let a, b, c be constants with $a \neq 0$. If $f(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.2), then:*

(1) *f has at most one nonzero finite Borel exceptional value;*
(2) *$\lambda(1/f)=\lambda(f)=\sigma(f)$;*
(3) *$f(z)$ has infinitely many fixed points and satisfies $\tau(f)=\sigma(f)$.*

In [7], they also consider properties of rational solutions of (1.2).

THEOREM C. *Let a, b, c be constants where a, b are not both equal to zero.*

(1) *If $a \neq 0$, then (1.2) has no rational solution.*
(2) *If $a=0$, and $b \neq 0$, then (1.2) has the nonzero constant solution $f(z)=A$, where A satisfies $2A^2-cA-b=0$. The other rational solution $f(z)$ satisfies $f(z)=A+P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy* $\deg P(z) < \deg Q(z)$.

Chen [5] considered the other form of the difference Painlevé I equation, that is, Equation (1.3), and proved the following two theorems.

THEOREM D. *Let a, b, c be constants with $ac \neq 0$. Suppose that $f(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.3). Then:*

(1) *f has no any Borel exceptional value;*
(2) *if $p(z)$ is a nonconstant polynomial, then $f(z)-p(z)$ has infinitely many zeros and $\lambda(f-p)=\sigma(f)$.*

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THEOREM E. *Let a, b, c be constants with $c \neq 0$ and $|a| + |b| \neq 0$. Then:*

(1) *if $a = 0$, then (1.4) has nonzero constant solution $f(z) = B$, where B satisfies*

$$2B^2 - bB - c = 0;$$

the other rational solution $f(z)$ satisfies $f(z) = B + S(z)/H(z)$, where $S(z)$ and $H(z)$ are relatively prime polynomials satisfying $\deg S(z) < \deg H(z)$*;*

(2) *if $a \neq 0$ and $f(z) = S(z)/H(z)$ is a rational solution of (1.3), where*

$$S(z) = sz^k + s_{k-1}z^{k-1} + \cdots + s_0, \quad H(z) = hz^t + h_{t-1}z^{t-1} + \cdots + h_0,$$

where s, $s_{k-1}, \ldots, s_0$ and h, $h_{t-1}, \ldots, h_0$ are constants such that $sh \neq 0$, then

$$t = k + 1 \quad \text{and} \quad s = -\frac{ch}{a}.$$

The main aims of this paper are to consider the properties of finite-order transcendental meromorphic solutions, rational solutions and polynomial solutions of the difference Painlevé I equation (1.4), and obtain the following results.

THEOREM 1.1. *Let a, b, c be constants such that $|a| + |b| \neq 0$. If $f(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.4), then:*

(1) $\lambda(1/f) = \lambda(f) = \sigma(f)$*;*

(2) *if $p(z)$ is a nonconstant polynomial, then $f(z) - p(z)$ has infinitely many zeros and $\lambda(f - p) = \sigma(f)$;*

(3) *if $a \neq 0$, then $f(z)$ has no Borel exceptional value; if $a = 0$, then the Borel exceptional value of $f(z)$ can only come from a set $E = \{z \mid 3z^2 - cz - b = 0\}$.*

REMARK 1.2. Taking $P(z) = z$ in Theorem 1.1, we immediately obtain the corresponding result that $f(z)$ has infinitely many fixed points and satisfies $\tau(f) = \sigma(f)$.

THEOREM 1.3. *Let a, b, c be constants such that $|a| + |b| \neq 0$.*

(1) *If $a \neq 0$, then (1.4) has no rational solution.*

(2) *If $a = 0$, then (1.4) has nonzero constant solution $f(z) = B$, where B satisfies $3B^2 - cB - b = 0$. Furthermore:*

  (i) *if $c^2 + 12b \neq 0$, then (1.4) has no nonconstant rational solution;*

  (ii) *if $c^2 + 12b = 0$, and $f(z) = B + S(z)/H(z)$ is a nonconstant rational solution of (1.4), where $S(z)$ and $H(z)$ are relatively prime polynomials satisfying* $\deg S(z) < \deg H(z)$*,*

$$S(z) = sz^m + s_{m-1}z^{m-1} + \cdots + s_0, \quad H(z) = hz^n + h_{n-1}z^{n-1} + \cdots + h_0,$$

where s, $s_{m-1}, \ldots, s_0$ and h, $h_{n-1}, \ldots, h_0$ are constants such that $sh \neq 0$, then

$$n = m + 2 \quad \text{and} \quad s = -\frac{ch}{3}.$$

REMARK 1.4. From Theorem 1.3, we see that if a, b, c are constants and $|a| + |b| \neq 0$, then (1.4) has no nonconstant polynomial solution.

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Example 1.5. The nonzero constant solution $f(z) = B = 1/3$ satisfies the difference Painlevé I equation (1.4)

$$f(z+1) + f(z) + f(z-1) = \frac{1/6}{f(z)} + \frac{1}{2}$$

and

$$a = 0, \quad 3B^2 - cB - b = 0.$$

Example 1.6. The nonconstant rational solution

$$f(z) = B + \frac{S(z)}{H(z)} = 1 + \frac{-2}{z^2+3z+2} = \frac{z^2+3z}{z^2+3z+2}$$

satisfies the difference Painlevé I equation (1.4)

$$f(z+1) + f(z) + f(z-1) = \frac{-3}{f(z)} + 6$$

and

$$a = 0, \quad 3B^2 - cB - b = 0, \quad c^2 + 12b = 0,$$
$$B = \frac{p}{q}, \quad n = m+2 \quad \text{and} \quad s = -\frac{ch}{3}.$$

2. Lemmas for proofs of theorems

Lemma 2.1 (Valiron–Mohon'ko lemma (see [19])). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_j(z), b_j(z)$ $(a_m b_n \not\equiv 0)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + o(\Psi(r)),$$

where $d = \max\{m, n\}$, and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$. In particular, if

$$T(r, a_i) = S(r, f)\ (i = 0, 1, \ldots, m), \quad T(r, b_j) = S(r, f)\ (j = 0, 1, \ldots, n),$$

then we have

$$T(r, R(z, f(z))) = dT(r, f) + S(r, f).$$

Lemma 2.2 (See [20]). *Let $w(z)$ be a nonconstant finite-order meromorphic solution of*

$$P(z, w) = 0,$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function a satisfying $\lim_{r\to\infty}(T(r, a)/T(r, w)) = 0$, then

$$m\left(r, \frac{1}{w-a}\right) = S(r, w)$$

outside of a possible exceptional set of finite logarithmic measure.

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LEMMA 2.3 (See [20]). *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z, f)$, $Q(z, f)$ are difference polynomials so that the total degree of $Q(z, f)$ is at most n. Then, for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma-1+\varepsilon}) + S(r, f),$$

possibly outside of an exceptional set of finite logarithmic measure.

LEMMA 2.4 (See [12, 13, 20]). *Let $f(z)$ be a nonconstant finite-order meromorphic function. Then*

$$N(r + 1, f) = N(r, f) + S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

3. Proofs of theorems

3.1. Proof of Theorem 1.1. Assume that $f(z)$ is a finite-order transcendental meromorphic solution of (1.4).

(1) First, we prove that $\lambda(f) = \sigma(f)$. By (1.4), we obtain that

$$P(z, f) := f(z)(f(z + 1) + f(z) + f(z - 1)) - (az + b) - cf(z) = 0. \tag{3.1}$$

By (3.1) and $|a| + |b| \neq 0$, we see that

$$P(z, 0) := -(az + b) \not\equiv 0. \tag{3.2}$$

Then by Lemma 2.2 and (3.2), we get that

$$m\left(r, \frac{1}{f}\right) = S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure. Thus

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f) \tag{3.3}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by (3.3), we have $\lambda(f) = \sigma(f)$.

Secondly, we show that $\lambda(1/f) = \sigma(f)$. By (1.4), we obtain that

$$f(z)G(z, f) = H(z, f), \tag{3.4}$$

where

$$G(z, f) = f(z + 1) + f(z) + f(z - 1), \tag{3.5}$$

$$H(z, f) = az + b + cf(z). \tag{3.6}$$

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By (3.4)–(3.6) and Lemma 2.3, we see that for any given $\varepsilon > 0$, there is a subset $H \subset (1, \infty)$ having finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup H$,

$$m(r, G) = O(r^{\sigma-1+\varepsilon}) + S(r, f), \tag{3.7}$$

where $\sigma = \sigma(f)$. By (1.4),

$$G(z, f) = f(z+1) + f(z) + f(z-1) = \frac{az+b}{f(z)} + c. \tag{3.8}$$

By Lemma 2.1 (Valiron–Mohon'Ko lemma), and (3.8), we see that

$$T(r, G) = T(r, f) + S(r, f). \tag{3.9}$$

Thus, (3.7) and (3.9) give

$$N(r, G) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f). \tag{3.10}$$

On the other hand, if $f(z)$ has a pole of multiplicity m at z_0 and $|z_0| \le r - 1$, then $f(z+1)$ and $f(z-1)$ have poles $z_0 - 1, z_0 + 1$ of multiplicity m, respectively. So

$$N(r, G) \le 3N(r+1, f). \tag{3.11}$$

By Lemma 2.4,

$$N(r+1, f) = N(r, f) + S(r, f). \tag{3.12}$$

Hence, by (3.10)–(3.12) we obtain that

$$\lambda\left(\frac{1}{f}\right) = \sigma(f). \tag{3.13}$$

(2) Suppose that $p(z)$ is a nonconstant polynomial. We use a similar method to the above to prove that $\lambda(f - p) = \sigma(f)$. Set

$$p(z) = d_k z^k + \cdots + d_0,$$

where $d_k, \ldots, d_0$ are constants, $d_k \neq 0$ and $k \ge 1$, and

$$g_1(z) = f(z) - p(z).$$

Then $\lambda(g_1) = \lambda(f - p)$, $\sigma(g_1) = \sigma(f - p)$. Substituting $f(z) = g_1(z) + p(z)$ into (1.4), we obtain that

$$g_1(z+1) + g_1(z) + g_1(z-1) + p(z+1) + p(z) + p(z-1) = \frac{az+b}{g_1(z) + p(z)} + c. \tag{3.14}$$

By (3.14), we obtain that

$$\begin{aligned} P_1(z, g_1) := {} & (g_1(z+1) + g_1(z) + g_1(z-1) \\ & + p(z+1) + p(z) + p(z-1))(g_1(z) + p(z)) \\ & - (az+b) - c(g_1(z) + p(z)) = 0. \end{aligned} \tag{3.15}$$

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By (3.15), we see that

$$\begin{aligned} P_1(z,0) &:= (p(z+1)+p(z)+p(z-1))p(z)-(az+b)-cp(z)\\ &= 3d_k^2 z^{2k}+\cdots \not\equiv 0. \end{aligned}$$

Continuing to use the same method as above, we obtain that $\lambda(f-p)=\sigma(f)$.

(3) We prove that if $a\neq 0$, then $f(z)$ has no Borel exceptional value.

We show that for any finite value α, we have $\lambda(f-\alpha)=\sigma(f)$. Set $g_2(z)=f(z)-\alpha$. Substituting $f(z)=g_2(z)+\alpha$ into (1.4), we obtain that

$$g_2(z+1)+g_2(z)+g_2(z-1)+3\alpha=\frac{az+b}{g_2(z)+\alpha}+c. \tag{3.16}$$

It follows from (3.16) that

$$\begin{aligned} P_2(z,g_2) &:= (g_2(z+1)+g_2(z)+g_2(z-1)+3\alpha)(g_2(z)+\alpha)\\ &\quad -(az+b)-c(g_2(z)+\alpha)=0. \end{aligned}$$

By (3.16) and $a\neq 0$,

$$P_2(z,0) := 3\alpha^2-(az+b)-c\alpha \not\equiv 0. \tag{3.17}$$

Thus by Lemma 1.3 and (3.17), we get that

$$m\Big(r,\frac{1}{g_2}\Big)=S(r,g_2)$$

outside of a possible exceptional set of finite logarithmic measure. Thus

$$\begin{aligned} N\Big(r,\frac{1}{f-\alpha}\Big) &= N\Big(r,\frac{1}{g_2}\Big)\\ &= T(r,g_2)+S(r,g_2)=T(r,f)+S(r,f), \end{aligned} \tag{3.18}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by (3.18), we have $\lambda(f-\alpha)=\sigma(f)$. Combining with (3.13), we see that $f(z)$ has no Borel exceptional value.

If $a=0$ and $\alpha\notin E$, then

$$P_2(z,0) := 3\alpha^2-c\alpha-b\not\equiv 0.$$

Using a similar method to the above we obtain $\lambda(f-\alpha)=\sigma(f)$. Hence, the Borel exceptional value of $f(z)$ can only come from a set $E=\{z\mid 3z^2-cz-b=0\}$.

Thus Theorem 1.1 is proved. □

3.2. Proof of Theorem 1.3. Assume that $f(z)$ is a nonzero rational solution of (1.4), and has poles $t_1,\dots,t_k$. Consequently, we suppose that

$$\frac{d_{js_j}}{(z-t_j)^{s_j}}+\cdots+\frac{d_{j1}}{z-t_j}\quad (j=1,\dots,k)$$

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are the principal parts of $f(z)$ at t_j respectively, where $d_{js_j}, \dots, d_{j1}$ are constants, $d_{js_j} \neq 0$. Thus, $f(z)$ can be represented as

$$f(z) = \sum_{j=1}^{k}\left[\frac{d_{js_j}}{(z-t_j)^{s_j}} + \cdots + \frac{d_{j1}}{z-t_j}\right] + a_0 + a_1 z + \cdots + a_m z^m, \tag{3.19}$$

where $a_0, a_1, \dots, a_m$ are constants.

We affirm that $a_m = \cdots = a_1 = 0$. Assume that $a_m \neq 0$ $(m \geq 1)$. For sufficiently large z, by (3.19),

$$f(z) = a_m z^m (1 + o(1)); \tag{3.20}$$

$$f(z \pm 1) = a_m (z \pm 1)^m (1 + o(1)). \tag{3.21}$$

By (1.4), we obtain that

$$(f(z+1) + f(z) + f(z-1))f(z) = az + b + cf(z). \tag{3.22}$$

Substituting (3.20), (3.21) into (3.22), we obtain that

$$3a_m^2 z^{2m}(1 + o(1)) = az + b + ca_m z^m (1 + o(1)). \tag{3.23}$$

Clearly, (3.23) is a contradiction since $a_m \neq 0$. Hence $a_m = 0$ $(m \geq 1)$.

(1) Suppose that $a \neq 0$. Above we have $a_m = 0$ $(m \geq 1)$. Now assume that $a_0 \neq 0$. Then for sufficiently large z, by (3.19), we see that

$$f(z) = a_0 + o(1), \quad f(z \pm 1) = a_0 + o(1). \tag{3.24}$$

By (1.4) and (3.24), we obtain that

$$(3a_0 + o(1))(a_0 + o(1)) = az + b + c(a_0 + o(1)).$$

This is a contradiction since $a \neq 0$. Hence $a_0 = 0$.

Thus, $f(z)$ can be rewritten by (3.19) as

$$f(z) = \frac{P(z)}{Q(z)}, \tag{3.25}$$

where

$$P(z) = pz^k + p_{k-1}z^{k-1} + \cdots + p_0, \quad Q(z) = qz^t + q_{t-1}z^{t-1} + \cdots + q_0, \tag{3.26}$$

where $p, p_{k-1}, \dots, p_0$ and $q, q_{t-1}, \dots, q_0$ are constants such that $pq \neq 0$ and $k < t$. Then substituting (3.25) into (1.4), we obtain that

$$\begin{aligned} &P(z+1)P(z)Q(z-1)Q(z) + P(z)^2 Q(z+1)Q(z-1) + P(z-1)P(z)Q(z+1)Q(z) \\ &\quad = (az+b)Q(z+1)Q(z)^2 Q(z-1) + cP(z)Q(z+1)Q(z)Q(z-1). \end{aligned} \tag{3.27}$$

Thus, since $k < t$ and $a \neq 0$, we see that the degree of the left-hand side of (3.27) does not exceed $2k + 2t$, but the degree of the right-hand side of (3.27) is equal to $1 + 4t$. This is a contradiction.

Hence, if $a \neq 0$, then (1.4) has no nonzero rational solution.

(2) Suppose that $a = 0$. By observation for (1.4), we see that $f(z) = B$ is a nonzero rational solution of (1.4), where B satisfies $3B^2 - cB - b = 0$. Since $a_m = 0$ $(m \geq 1)$,

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we now suppose that $f(z)$ is a nonconstant rational solution of (1.4). Then $f(z)$ can be rewritten by (3.19) as the form (3.25), where $P(z)$ and $Q(z)$ satisfy (3.26) with $k \le t$. Suppose that $k < t$. Then, substituting (3.25) into (1.4), we obtain that

$$\begin{aligned} &P(z+1)P(z)Q(z-1)Q(z) + P(z)^2Q(z+1)Q(z-1) + P(z-1)P(z)Q(z+1)Q(z) \\ &\quad = bQ(z+1)Q(z)^2Q(z-1) + cP(z)Q(z+1)Q(z)Q(z-1). \end{aligned} \tag{3.28}$$

Thus, in (3.28) there only exists one term $bQ(z+1)Q(z)^2Q(z-1)$ such that its degree is highest. This contradiction shows that $k = t$. Then, by (3.26) and (3.28), we obtain that

$$\begin{aligned} &\frac{p(z+1)^k + p_{k-1}(z+1)^{k-1} + \cdots + p_0}{q(z+1)^t + q_{t-1}(z+1)^{t-1} + \cdots + q_0} + \frac{pz^k + p_{k-1}z^{k-1} + \cdots + p_0}{qz^t + q_{t-1}z^{t-1} + \cdots + q_0} \\ &\quad + \frac{p(z-1)^k + p_{k-1}(z-1)^{k-1} + \cdots + p_0}{q(z-1)^t + q_{t-1}(z-1)^{t-1} + \cdots + q_0} \\ &\quad = \frac{b(qz^t + q_{t-1}z^{t-1} + \cdots + q_0)}{pz^k + p_{k-1}z^{k-1} + \cdots + p_0} + c. \end{aligned} \tag{3.29}$$

By (3.29), we obtain that as $z \to \infty$,

$$3B^2 - cB - b = 0,$$

where $B = p/q$. Hence, $f(z)$ can be rewritten as

$$f(z) = B + \frac{S(z)}{H(z)} \tag{3.30}$$

where $S(z)$ and $H(z)$ are relatively prime polynomials and satisfy $\deg S(z) < \deg H(z)$, B is a constant satisfying $3B^2 - cB - b = 0$,

$$S(z) = sz^m + s_{m-1}z^{m-1} + \cdots + s_0, \quad H(z) = hz^n + h_{n-1}z^{n-1} + \cdots + h_0,$$

where $s, s_{m-1}, \ldots, s_0$ and $h, h_{n-1}, \ldots, h_0$ are constants such that $sh \neq 0$.

Furthermore, we consider the existence of nonconstant rational solutions for (1.4).

(i) We prove that if $c^2 + 12b \neq 0$, then (1.4) has no nonconstant rational solution. Suppose that $f(z)$ is a nonconstant rational solution of (1.4). Then $f(z)$ can be rewritten by (3.30), where $S(z)$ and $H(z)$ are relatively prime polynomials and satisfy $m = \deg S(z) < \deg H(z) = n$. Substituting (3.30) into (1.4), we obtain that

$$\begin{aligned} &(4B-c)S(z)H(z)H(z+1)H(z-1) + BS(z+1)H(z)^2H(z-1) \\ &\quad + BS(z-1)H(z)^2H(z+1) \\ &\quad = -(S(z)^2H(z+1)H(z-1) + S(z)S(z+1)H(z)H(z-1) \\ &\qquad + S(z-1)S(z)H(z)H(z+1)). \end{aligned} \tag{3.31}$$

Comparing coefficients and degrees of all terms of (3.31), by $n > m$, we get that $(6B - c)sh^3z^{m+3n}$ is the highest-degree term of (3.31).

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Since $3B^2 - cB - b = 0$ and $c^2 + 12b \neq 0$, it follows that $B \neq c/6$. Thus (3.31) is a contradiction for sufficiently large z. Therefore, if $a = 0$ and $c^2 + 12b \neq 0$, then (1.4) has no nonconstant rational solution.

(ii) If $c^2 + 12b = 0$, then by $3B^2 - cB - b = 0$, we have $B = c/6$. We divide this proof into three cases.

Case 1: $n = m + 1$. When $n = m + 1$, we have $3n + m - 1 = 2m + 2n$. Comparing coefficients of the highest-degree terms of (3.31), we obtain that the highest-degree term of (3.31) is

$$((18B - 3c)sh^2h_{n-1} + (6B - c)s_{m-1}h^3 + 3s^2h^2)z^{m+3n-1}.$$

Since $B = c/6$ and $sh \neq 0$, (3.31) is a contradiction. Therefore, if $a = 0$, $c^2 + 12b = 0$ and $n = m + 1$, (1.4) has no nonconstant rational solution.

Case 2: $n = m + 2$. When $n = m + 2$, we have $m + 3n - 2 = 2m + 2n$. Comparing coefficients of the highest-degree terms of (3.31), we obtain that

$$(2Bsh^3(n^2 - C_n^2 + C_m^2 - C_n^1C_m^1) + 3s^2h^2)z^{m+3n-2} = (6Bsh^3 + 3s^2h^2)z^{m+3n-2}$$

is the highest-degree term of (3.31). Since $c^2 + 12b = 0$, $a = 0$ and $|a| + |b| \neq 0$, it follows that $b \neq 0$ and $c \neq 0$. Then by $B = c/6$, $c \neq 0$ and $sh \neq 0$, we obtain that

$$\frac{s}{h} = -\frac{c}{3}.$$

Therefore, if $c^2 + 12b = 0$, (1.4) has a nonconstant rational solution $f(z)$ satisfying $f(z) = B + S(z)/H(z)$. Let

$$S(z) = sz^m + s_{m-1}z^{m-1} + \cdots + s_0, \quad H(z) = hz^n + h_{n-1}z^{n-1} + \cdots + h_0,$$

where $s, s_{m-1}, \ldots, s_0$ and $h, h_{n-1}, \ldots, h_0$ are constants such that $sh \neq 0$. Then

$$n = m + 2, \quad s = -\frac{ch}{3}.$$

Case 3: $n > m + 2$. When $n > m + 2$, again comparing coefficients of the highest-degree terms of (3.31), we obtain that

$$2Bsh^3(n^2 - C_n^2 + C_m^2 - C_n^1C_m^1)z^{m+3n-2}$$

is the highest-degree term of (3.31). Since

$$\begin{aligned} n^2 - C_n^2 + C_m^2 - C_n^1C_m^1 &= \frac{1}{2}((n-m)^2 + n - m) \\ &= \frac{1}{2}\left(\left(n - m + \frac{1}{2}\right)^2 - \frac{1}{4}\right) > 3 \neq 0 \end{aligned}$$

and $sh \neq 0$, it follows that $B = c/6 = 0$. So $c = 0$. If $c = 0$, by $c^2 + 12b = 0$, we have $b = 0$. This contradicts the fact that $|a| + |b| \neq 0$ since $a = 0$.

Therefore, if $a = 0$, $c^2 + 12b = 0$ and $n > m + 2$, (1.4) has no nonconstant rational solution.

Thus Theorem 1.3 is proved. □

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MEI-RU CHEN, School of Mathematical Sciences,
South China Normal University, Guangzhou, 510631, PR China
e-mail: chenmeiru2005@163.com

ZONG-XUAN CHEN, School of Mathematical Sciences,
South China Normal University, Guangzhou, 510631, PR China
e-mail: chzx@vip.sina.com

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