Canad. Math. Bull. Vol. **22** (3), 1979

BOUNDS ON POSITIVE INTEGRAL SOLUTIONS OF LINEAR DIOPHANTINE EQUATIONS II

BY

I. BOROSH* AND L. B. TREYBIG

Abstract. Let A be an $m\times n$ matrix of rank r and B an $m\times 1$ matrix, both with integer entries. Let M_2 be the maximum of the absolute values of the $r\times r$ minors of the augmented matrix $(A \mid B)$. Suppose that the system $Ax=B$ has a non-trivial solution in non-negative integers. We prove (1) If $r=n-1$ then the system $Ax=B$ has a non-negative non-trivial solution with entries bounded by M_2. (2) If A has a $r\times n$ submatrix such that none of its $r\times r$ minors is 0 and $x\geq 0$ is a solution of $Ax=B$ in integers such that $\sum_{i=1}^{n} x_i$ is minimal, then $\sum_{i=1}^{n} x_i \leq (nr+n-r^2)M_2$.

Introduction. In [1] the following problem was considered: Let A be an $m\times n$ matrix, B an $m\times 1$ matrix, both with integral entries, and consider the system of equations:

$$Ax=B. \tag{1}$$

Suppose (1) has a non-zero solution in non-negative integers. The problem is to find a bound $K=K(A,B)$ such that the existence of such a solution with entries bounded by K is always guaranteed. The problem first arose in a topological setting [3, 4, 5] and a bounding function $K=K(A,B)$ was found inductively in [5]. Let $(A \mid B)$ denote the augmented matrix of (1); r denote the rank of A; M_1, M_2 denote, respectively, the maximum of the absolute values of all the minors of order r of A and $(A \mid B)$; and M the maximum of the absolute values of all the minors of $(A \mid B)$.

It was conjectured in [1] that if (1) has a non-trivial solution in non-negative integers, then it has one whose entries are bounded by M_2. The above bound is clearly sharp. The conjecture was proved in [1] only for the case $m=1$ and the case in which the homogeneous system $Ax=0$ has no non-trivial non-negative solutions. In [2] the conjecture was proved for the homogeneous case. Also in [1], the bound $M_2(1+1/M_1)$ was obtained for the case $r=n-1$ and a bound of the order of M^2 was obtained in the general case.

In this paper the conjecture will be proved for the case $r=n-1$, and a better bound of the order of M_2 will be obtained in the general case. The main results are:

THEOREM 1. *If $r=n-1$ and (1) has a non-trivial non-negative solution then it has such a solution with* $\max_i x_i \leq M_2$.

Received by the editors August 3, 1977 and in revised form, September 12, 1978.

* This author was partially supported by NSF grant MCS 76–06092.

THEOREM 2. *If none of the minors of order r of A is 0 and $x = (x_1, \ldots x_n)$ is a non-trivial non-negative solution in integers of (1) such that $\sum_{i=1}^n x_i$ is minimum, then $\sum_{i=1}^n x_i \le (nr + n - r^2)M_2$.*

Proof of Theorem 1. We may suppose without loss of generality that $m = r$ and that $\max x_i$ is a minimum over all such solutions. Also, assume that the variables have been renamed so that $x_1 \le x_2 \le \cdots \le x_{r+1}$ and that $M_2 < x_{r+1}$. We multiply both sides of $Ax = B$ by the adjoint of the matrix A' whose columns are the first r columns of A, and easily derive:

$$cx_i = -n_i + p_i x_{r+1}, \qquad 1 \le i \le r, \tag{2}$$

where each $-n_i$, p_i, c or its negative is the determinant of an $r \times r$ submatrix of $(A \mid B)$. We may assume without loss of generality that $c \ge 0$.

If $c = 0$, then $-n_i + p_i x_{r+1} = 0$, $1 \le i \le r$. But $p_i = 0$, $1 \le i \le r$, implies the rows of A are not linearly independent, and some $p_i \ne 0$ implies $x_{r+1} \le |n_i|$. Thus $c > 0$. Also notice that if $x_i = x_{r+1}$ for $i \le r$, then $p_i \ne 0$ since $p_i = 0$ would imply $x_i \le M_2 < x_{r+1}$.

If $p_i < 0$ for some i, then (2) implies $cx_i - p_i x_{r+1} = -n_i$, which is impossible since $x_{r+1} > |n_i|$. Thus $p_i \ge 0$, $1 \le i \le r$. If $p_i \le x_i$, $1 \le i \le r$, then $(x_1 - p_1, \ldots, x_r - p_r, x_{r+1} - c)$ is a non-trivial solution with a smaller maximum, a contradiction. Suppose then that $x_i < p_i$.

Now, $x_{r+1} > c$, $x_i < p_i$, and (2) imply:

$$n_i = x_{r+1}p_i - x_i c = x_i(x_{r+1} - c) + (p_i - x_i)x_{r+1} \ge x_{r+1},$$

a contradiction.

COROLLARY. *If $r = n - 1$ and (1) has a solution in integers, then it has a solution in integers such that $\max_i |x_i| \le M_2$.*

Proof. Suppose that (1) has a solution in integers $x_1, \ldots x_n$ and assume $x_1, \ldots, x_k \ge 0$ and $x_{k+1}, \ldots, x_n < 0$. Define

$$y_i = \begin{cases} x_i & i \le k \\ -x_i & i > k \end{cases}$$

Let $\bar{A}$ be the matrix obtained from A by changing the signs of the columns $k+1, \ldots, n$ of A. Then, the system $\bar{A}x = B$ has the non-trivial solution y and, therefore, by Theorem 1, has a solution bounded by M_2 (since the absolute value of the minors of A and $\bar{A}$ are equal). A solution to (1) is then easily obtained by adjusting the signs.

Proof of Theorem 2. We may assume without loss of generality that $r = m$ and that $x_1 \ge x_2 \ge \cdots \ge x_n$. For $i = 0, 1, \ldots r$, let S_i denote the set of all $r \times r$

https://doi.org/10.4153/CMB-1979-045-2 Published online by Cambridge University Press

submatrices of A whose first i columns coincide with the first i columns of A. In particular, S_0 is the set of all $r\times r$ submatrices of A. In particular, S_0 is the set of all $r\times r$ submatrices of A. Let D_i denote the maximum of the absolute values of the determinants of S_i for $i=0,\ldots,r$. We have obviously: $M_1=D_0\geq D_1\geq\cdots\geq D_r$. We now distinguish between two cases:

CASE 1. For every j, $j=0,\ldots,r-1$, we have

$$x_{j+1}\geq D_j. \tag{3}$$

Let $D=D_r\neq 0$, and assume, without loss of generality that $D>0$. Let A' be the submatrix whose columns are the first r columns of A. Solving for $x_1,\ldots,x_r$ we get:

$$Dx_i=\sum_{k=r+1}^{n} a'_{ik}x_k+b'_i \tag{4}$$

where a'_{ik} is the determinant of the $r\times r$ submatrix of A obtained from A' by replacing the ith column of A by kth column. This submatrix belongs to S_{i-1} and therefore $|a'_{ik}|\leq D_{i-1}$. The term b'_i is the minor obtained by replacing the ith column of A' by B. Let v be the largest integer j where $x_{j-1}\geq D$. From (3) we see that $v\geq r+1$. If $v>r+1$ let p be any integer such that $r+1\leq p<v$, and let $m_p=\sum_{i=1}^{r} a'_{ip}+D$. If $m_p\neq 0$, define a new solution y of (1) as follows:

$$y_j=\begin{cases} x_j & \text{if} \quad j\geq r+1, \qquad j\neq p\\ x_p-(\operatorname{sgn} m_p)D & \qquad\qquad\qquad j=p\\ x_j-(\operatorname{sgn} m_p)a'_{jp} & \text{for} \quad j=1,\ldots,r\end{cases} \tag{5}$$

In (5) $\operatorname{sgn} m_p=+1$ if $m_p>0$ and -1 if $m_p<0$. It is easily seen from (4) that y is a solution to (1). Since $p<v$, $x_p\geq D$. So $y_p=x_p-\operatorname{sgn} m_pD\geq 0$. Since $|a'_{jp}|\leq D_{j-1}$ and from (3), $x_j\geq D_{j-1}$ we have for $j=1,\ldots,r$ $y_j=x_j-(\operatorname{sgn} m_p)a'_{jp}\geq 0$.

$$\sum_{j=1}^{n} y_j=\sum_{j=1}^{n} x_j-\operatorname{sgn} m_p\left(\sum_{j=1}^{r} a'_{jp}+D\right)=\sum_{j=1}^{n} x_j-|m_p|$$
$$<\sum_{j=1}^{n} x_j.$$

This contradicts the minimality of $\sum_1^n x_j$. We have therefore $m_p=0$ and:

$$\sum_{i=1}^{r} a'_{ip}=-D, \qquad p=r+1,\ldots,v-1 \tag{6}$$

Summing (4) for $i=1,\ldots,r$ we get:

$$D\sum_{i=1}^{r} x_i=\sum_{p=r+1}^{v-1}\left(\sum_{i=1}^{r} a'_{ip}\right)x_p+\sum_{p=v}^{n}\left(\sum_{i=1}^{r} a'_{ip}\right)x_p+\sum_{i=1}^{r} b'_i. \tag{7}$$

https://doi.org/10.4153/CMB-1979-045-2 Published online by Cambridge University Press

Using (6) we get:

$$D\sum_{i=1}^{v-1} x_i = \sum_{p=v}^{n}\left(\sum_{i=1}^{r} a'_{ip}\right)x_p + \sum_{i=1}^{r} b'_i$$

and since $x_p < D$ for $p \geq v$;

$$\sum_{i=1}^{v-1} x_i \leq (n-v+1)rM_1 + rM_2$$

$$\begin{aligned}\sum_{i=1}^{n} x_i &\leq ((n-v+1)r + (n-v+1))M_1 + rM_2\\ &\leq (n-v+1)(r+1)M_1 + rM_2\\ &\leq (n-r)(r+1)M_1 + rM_2\\ &\leq (nr+n-r^2)M_2.\end{aligned}$$

If $v = r+1$, then the first term on the right side of (7) may be replaced by zero.

CASE 2. There exists j, $0 \leq j < r$ such that $x_{j+1} < D_j$. We rename the variables $x_{j+1}, \ldots, x_n$ in such a way that the matrix A' whose columns are the first r-columns, has determinant $D = \pm D_j$; and we may assume $D = D_j$. We solve for $x_1, \ldots, x_r$:

$$Dx_i = \sum_{p=r+1}^{n} a'_{ip}x_p + b'_i$$

where a'_{ip} and b'_i are minors of A and (A/B) as in Case 1. Since $x_p < D$ for $p = r+1, \ldots, n$ we have $x_i \leq (n-r)M_1 + M_2$, $i = 1, \ldots, j$

$$\begin{aligned}\sum_{i=1}^{n} x_i &\leq j(n-r)M_1 + (n-j)M_1 + jM_2\\ &\leq (j(n-r-1)+n)M_1 + jM_2\\ &\leq (r+1)(n-r)M_1 + rM_2\\ &\leq (nr+n-r^2)M_2.\end{aligned}$$

REMARK. In the proofs of Theorems 1 and 2 we assume that $r = m$. In fact, we can choose among the r-tuples of rows of $(A \mid B)$ the one for which $M_2 \neq 0$ is minimal.

REFERENCES

1. I. Borosh and L. B. Treybig, *Bounds on positive integral solutions of linear diophantine equations*, Proceedings of the A.M.S. Vol. **55** Number 2 March 1976, 299–304.

2. I. Borosh, *A sharp bound for positive solutions of homogeneous linear diophantine equations*, Proceedings of the A.M.S. Vol. **60** October 1976, 19–21.

https://doi.org/10.4153/CMB-1979-045-2 Published online by Cambridge University Press

3. W. Haken, *Theorie der Normal flacken Acta. Math.* **105** (1961), 245–375.

4. H. Schubert, *Bestimmung der Primfaktorzerlegung von Verkettungen*, Math. Zeit., **76** (1961), 116–148.

5. L. B. Treybig, *Bounds in piecewise linear topology*, Trans. Amer. Math. Soc., **201** (1975), 383–405.

DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY
COLLEGE OF SCIENCE
COLLEGE STATION, TEXAS 77840

https://doi.org/10.4153/CMB-1979-045-2 Published online by Cambridge University Press