Fourier coefficients of $\frac{1}{\sin\theta}$ and $\frac{\theta}{\sin\theta}$

G. J. O. JAMESON

Introduction

Recall that for a 2π-periodic function f, the Fourier coefficients on the interval $(-\pi, \pi)$ are

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta,$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta,$$

and the Fourier series for f is

$$\tfrac{1}{2}a_0(f) + \sum_{n=1}^{\infty}\left(a_n(f)\cos n\theta + b_n(f)\sin n\theta\right).$$

The Riemann-Lebesgue lemma says that if f is bounded and integrable, then $a_n(f)$ and $b_n(f)$ tend to 0 as $n \to \infty$: for f with bounded derivative, this is easily proved by integration by parts. Under suitable (stronger) conditions, the Fourier series converges to $f(\theta)$; it is more than enough if f is bounded and differentiable on $(-\pi, \pi)$. Of course, if the coefficients do not tend to 0, then the series does not converge.

If f is *even*, then $b_n(f) = 0$ and $a_n(f) = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$, and the series has only cosine terms. If f is *odd*, then $a_n(f) = 0$ and $b_n(f) = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$. If f is only specified on $(0, \pi)$, these are the 'half-range' cosine and sine coefficients, corresponding to even and odd extensions of f defined by taking either $f(-\theta) = f(\theta)$ or $f(-\theta) = -f(-\theta)$ on $(-\pi, 0)$.

Here we will describe the half-range Fourier coefficients of $1/\sin\theta$ and $\theta/\sin\theta$, first on the interval $(0, \pi)$, then, by suitable scaling, on the interval $(0, \frac{1}{2}\pi)$. On $(0, \pi)$ the coefficients do not even tend to zero, but on $(0, \frac{1}{2}\pi)$ they generate rather interesting cosine and sine series converging to $\theta/\sin\theta$, nicely exhibiting the connection with the known integral in terms of Catalan's constant. On the way we will need to investigate the integrals of the products of these functions with $\sin n\theta$ and $\cos n\theta$ or $(1 - \cos n\theta)$. These integrals are of some interest in themselves, and have further applications. A simple variation delivers corresponding results for $\cos\theta/\sin\theta$ and $\theta\cos\theta/\sin\theta$. The resulting catalogue of integrals is quite lengthy, but hopefully not quite so formidable when viewed as a systematic scheme.

Integrals on $(0, \pi)$

We start by recording the integrals of $\cos n\theta$ and $\sin n\theta$ themselves,

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

distinguishing even and odd n where necessary. They are completely elementary, but it will be useful to have the list available for reference. For positive integers n and r, we have:

$$\int_0^\pi \cos n\theta\, d\theta = \frac{1}{n}\Big[\sin n\theta\Big]_0^\pi = 0, \tag{1}$$

$$\int_0^\pi \sin 2r\theta\, d\theta = -\frac{1}{2r}\Big[\cos 2r\theta\Big]_0^\pi = 0, \tag{2}$$

$$\int_0^\pi \sin(2r-1)\theta\, d\theta = -\frac{1}{2r-1}\Big[\cos(2r-1)\Big]_0^\pi = \frac{2}{2r-1}, \tag{3}$$

$$\int_0^\pi \theta\sin n\theta\, d\theta = -\frac{1}{n}\Big[\theta\cos n\theta\Big]_0^\pi + \frac{1}{n}\int_0^\pi \cos n\theta\, d\theta = (-1)^{n-1}\frac{\pi}{n}, \tag{4}$$

$$\int_0^\pi \theta\cos 2r\theta\, d\theta = \frac{1}{2r}\Big[\theta\sin 2r\theta\Big]_0^\pi - \frac{1}{2r}\int_0^\pi \sin 2r\theta\, d\theta = 0, \tag{5}$$

$$\int_0^\pi \theta\cos(2r-1)\theta\, d\theta = \frac{1}{2r-1}\Big[\theta\sin(2r-1)\theta\Big]_0^\pi - \frac{1}{2r-1}\int_0^\pi \sin(2r-1)\theta\, d\theta$$
$$= -\frac{2}{(2r-1)^2}, \tag{6}$$

Identities (4), (5) and (6) lead to the well-known Fourier series for θ and $|\theta|$, but that is not our topic here.

We turn to the integral of $\sin(2n+1)\theta/\sin\theta$, and the same multiplied by θ. Viewed purely as an integral, there is no problem about the integrand: the zeros of $\sin\theta$ at 0 and π are cancelled by zeros of $\sin(2n+1)\theta$. A simple trigonometric identity makes this point more explicit, and also enables the evaluation. In fact, adding the identities

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\cos 2r\theta\sin\theta$$

for $1 \leqslant r \leqslant n$, we obtain, for $0 < \theta < \pi$,

$$\frac{\sin(2n+1)\theta}{\sin\theta} = 1 + 2\sum_{r=1}^{n}\cos 2r\theta. \tag{7}$$

(With the substitution $\theta = \frac{1}{2}\phi$, this is the 'Dirichlet kernel'. As the reader may know, it is an essential tool in the proof of convergence of Fourier series.) By (1) and (5), only the term 1 contributes to the integrals: we obtain

$$\int_0^\pi \frac{\sin(2n+1)\theta}{\sin\theta}\, d\theta = \pi, \tag{8}$$

$$\int_0^\pi \theta\frac{\sin(2n+1)\theta}{\sin\theta}\, d\theta = \frac{\pi^2}{2}. \tag{9}$$

These integrals, multiplied by $2/\pi$, give the Fourier sine coefficients of

$1/\sin\theta$ and $\theta/\sin\theta$. Clearly they do not satisfy the Riemann-Lebesgue Lemma. This was not to be expected, since both functions tend to infinity as θ tends to π, and also $1/\sin\theta$ as θ tends to 0.

A variant of (7) is as follows. Since

$$\sin(2n+1)\theta = \sin 2n\theta \cos\theta + \cos 2n\theta \sin\theta,$$

we have

$$\frac{\sin 2n\theta \cos\theta}{\sin\theta} = \frac{\sin(2n+1)\theta}{\sin\theta} - \cos 2n\theta$$

$$= 1 + 2\sum_{r=1}^{n-1}\cos 2r\theta + \cos 2n\theta. \qquad (10)$$

So the integrals in (8) and (9) retain the same value if $\sin(2n+1)\theta$ is replaced by $\sin 2n\theta \cos\theta$, thereby describing even-numbered Fourier coefficents of $\cos\theta/\sin\theta$ and $\theta\cos\theta/\sin\theta$. We will see an application of this variant shortly.

Now let us replace $2n+1$ by $2n$ in these integrals. Adding the equalities $\sin 2r\theta - \sin(2r-2)\theta = 2\cos(2r-1)\theta\sin\theta$ for $1 \leqslant r \leqslant n$, we obtain

$$\frac{\sin 2n\theta}{\sin\theta} = 2\sum_{r=1}^{n}\cos(2r-1)\theta. \qquad (11)$$

So by (1) and (6) we have

$$\int_0^{\pi}\frac{\sin 2n\theta}{\sin\theta}\,d\theta = 0, \qquad (12)$$

$$\int_0^{\pi}\theta\frac{\sin 2n\theta}{\sin\theta}\,d\theta = -4\sum_{r=1}^{n}\frac{1}{(2r-1)^2}. \qquad (13)$$

In the same way as for (8) and (9), there are variants with $\sin 2n\theta$ replaced by $\sin(2n-1)\theta\cos\theta$; in (13), the final term in the sum is halved.

We now consider companion integrals involving $\cos n\theta$ instead of $\sin n\theta$. Adding the identities $\cos(2r-1)\theta - \cos(2r+1)\theta = 2\sin 2r\theta\sin\theta$, we obtain

$$\frac{\cos\theta - \cos(2n+1)\theta}{\sin\theta} = 2\sum_{r=1}^{n}\sin 2r\theta. \qquad (14)$$

(This is the 'conjugate Dirichlet kernel'.) So by (2) and (4) (with $n = 2r$),

$$\int_0^{\pi}\frac{\cos\theta - \cos(2n+1)\theta}{\sin\theta}\,d\theta = 0, \qquad (15)$$

$$\int_0^{\pi}\theta\,\frac{\cos\theta - \cos(2n+1)\theta}{\sin\theta}\,d\theta = -\pi\sum_{r=1}^{n}\frac{1}{r}. \qquad (16)$$

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

Note that for these integrals to exist, it is essential to have the difference of two cosines in the numerator, in order to cancel the zeros of $\sin\theta$ at 0 and π. In other words, Fourier cosine coefficients of $1/\sin\theta$ and $\theta/\sin\theta$ on $(0, \pi)$ do not exist.

Corresponding to (10), we have the following variant of (14) derived from the identity $\cos(2n + 1)\theta = \cos 2n\theta \cos\theta - \sin 2n\theta \sin\theta$:

$$\frac{(1 - \cos 2n\theta)\cos\theta}{\sin\theta} = 2\sum_{r=1}^{n-1} \sin 2r\theta + \sin 2n\theta. \tag{17}$$

So variants of (15) and (16) apply with $\cos\theta - \cos(2n + 1)\theta$ replaced by $(1 - \cos 2n\theta)\cos\theta$; in (16), the final term is halved. This variant will be needed for a later application.

Now, replacing $2r$ by $2r - 1$, we have the following companion to (14):

$$\frac{1 - \cos 2n\theta}{\sin\theta} = 2\sum_{r=1}^{n} \sin(2r - 1)\theta, \tag{18}$$

so by (3) and (4),

$$\int_0^\pi \frac{1 - \cos 2n\theta}{\sin\theta}\, d\theta = 4\sum_{r=1}^{n} \frac{1}{2r - 1}, \tag{19}$$

$$\int_0^\pi \theta\, \frac{1 - \cos 2n\theta}{\sin\theta}\, d\theta = 2\pi \sum_{r=1}^{n} \frac{1}{2r - 1}. \tag{20}$$

Of course, in these integrals we can write $2\sin^2 n\theta$ instead of $1 - \cos 2n\theta$.

Again, variants apply with $1 - \cos 2n\theta$ replaced by $1 - \cos(2n - 1)\theta\cos\theta$.

Application to $\log\sin\theta$. (This is a digression which the reader is at liberty to ignore.)

Although $\log\sin\theta$ is unbounded on $(0, \pi)$, it has a finite integral. In fact, it can be shown quite simply that $\int_0^\pi \log\sin\theta\, d\theta = -\pi\log 2$ (see [1] or [2]). Our earlier results enable us to evaluate the Fourier cosine coefficients. Let

$$I_n = \int_0^\pi (\log\sin\theta)\cos n\theta\, d\theta.$$

Integrating by parts, we have

$$I_n = \frac{1}{n}\Big[(\log\sin\theta)\sin n\theta\Big]_0^\pi - \frac{1}{n}\int_0^\pi \frac{\cos\theta}{\sin\theta}\sin n\theta\, d\theta.$$

From the fact that $\theta\log\theta \to 0$ as $\theta \to 0^+$, we can show that the first bracket equals 0. By the variant forms of (8) and (12), we deduce that $I_{2n} = -\pi/(2n)$ and $I_{2n-1} = 0$.

If we assume that the Fourier series converges to the function, we conclude that

$$\log\sin\theta = -\log 2 - \sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\theta \tag{21}$$

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

for $0 < \theta < \pi$. However, because the function is unbounded, the general theorem on convergence of Fourier series, under the conditions usually stated, does not apply. Convergence of the series, at least, is assured by Dirichlet's test. In fact (21) can be seen as a case of the complex logarithmic series $\log(1 - z) = -\sum_{n=1}^{\infty} z^n/n$, with $z = e^{2i\theta}$, since $\left|1 - e^{2i\theta}\right| = 2\sin\theta$ for $0 < \theta < \pi$. The proof of this identity for $z \neq 1$ with $|z| = 1$ is by no means trivial, but that is not our concern here.

Integrals on $\left(0, \frac{1}{2}\pi\right)$ *involving* $1/\sin\theta$

We now consider corresponding integrals on the interval $\left(0, \frac{1}{2}\pi\right)$. Those involving $1/\sin\theta$ are a good deal simpler, and we deal with them first. Corresponding to (1) and (3), we have

$$\int_0^{\pi/2} \cos 2r\theta \, d\theta = -\frac{1}{2r}\Big[\sin 2r\theta\Big]_0^{\pi/2} = 0 \qquad (r \geqslant 1), \tag{22}$$

$$\int_0^{\pi/2} \cos(2r - 1)\theta \, d\theta = \frac{1}{2r - 1}\Big[\sin(2r - 1)\theta\Big]_0^{\pi/2} = \frac{(-1)^{r-1}}{2r - 1}, \tag{23}$$

$$\int_0^{\pi/2} \sin(2r - 1)\theta \, d\theta = -\frac{1}{2r - 1}\Big[\cos(2r - 1)\theta\Big]_0^{\pi/2} = \frac{1}{2r - 1}. \tag{24}$$

By (7) and (22), we have

$$\int_0^{\pi/2} \frac{\sin(2n + 1)\theta}{\sin\theta} \, d\theta = \frac{\pi}{2}, \tag{25}$$

while by (11) and (23), we have

$$\int_0^{\pi/2} \frac{\sin 2n\theta}{\sin\theta} \, d\theta = 2\sum_{r=1}^{n} \frac{(-1)^{r-1}}{2r - 1}. \tag{26}$$

Also, by (18) and (24),

$$\int_0^{\pi/2} \frac{1 - \cos 2n\theta}{\sin\theta} \, d\theta = 2\sum_{r=1}^{n} \frac{1}{2r - 1} \tag{27}$$

(this also follows easily from (19)). Mindful that our catalogue of integrals is getting rather long, we have left out the analogues of (2) and (15), which would logically belong here. Any sufficiently determined reader will be able to supply these.

We now show how with the help of the Riemann-Lebesgue Lemma, we can deduce the 'sine integral'

$$\int_0^{\infty} \frac{\sin\theta}{\theta} \, d\theta = \frac{\pi}{2}. \tag{28}$$

Write

$$F(\theta) = \frac{1}{\sin\theta} - \frac{1}{\theta}.$$

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

Then $F(\theta) \to 0$ as $\theta \to 0$, since the series for $\sin\theta$ gives

$$F(\theta) = \frac{\theta - \sin\theta}{\theta\sin\theta} = \frac{\theta^3/3! - \theta^5/5! + \ldots}{\theta^2 - \theta^4/3! + \ldots} = \frac{\theta/3! - \theta^3/5! + \ldots}{1 - \theta^2/3! + \ldots}.$$

So $F(\theta)$ is bounded and continuous on $[0, \frac{\pi}{2}]$ when assigned the value 0 at 0. Hence the Riemann-Lebesgue lemma applies to give

$$\int_0^{\pi/2} F(\theta)\sin(2n+1)\theta\,d\theta \to 0 \qquad \text{as } n \to \infty.$$

Now

$$\int_0^{\pi/2} \frac{\sin(2n+1)\theta}{\theta}\,d\theta = \int_0^{(n+\frac{1}{2})\pi} \frac{\sin\phi}{\phi}\,d\phi.$$

Denote this by J_{2n+1}. With (25) we have shown that $\frac{\pi}{2} - J_{2n+1} \to 0$, so that $J_{2n+1} \to \frac{\pi}{2}$ as $n \to \infty$, which proves (28).

Of course, this also shows that $J_{2n} \to \frac{\pi}{2}$ as $n \to \infty$. Since $\int_0^{\pi/2} F(\theta)\sin 2n\theta\,d\theta$ also tends to 0 as $n \to \infty$, we can deduce from (26) that $\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{2r-1} = \frac{\pi}{4}$ (however, this is more easily deduced from the Fourier series for θ itself).

Integrals on $(0, \frac{\pi}{2})$ *involving* $\theta/\sin\theta$

A new ingredient emerges when we come to $\theta/\sin\theta$. While this function is not integrable on $(0, \pi)$, it is bounded and continuous on $(0, \frac{\pi}{2})$, so it will have convergent Fourier sine and cosine series on this interval. We describe how Fourier coefficients and series translate to this interval. Given a function $f(\phi)$ on $(0, \pi)$, substitute $\phi = 2\theta$ to obtain

$$a_n(f) = \frac{2}{\pi}\int_0^{\pi} f(\phi)\cos n\phi\,d\phi = \frac{4}{\pi}\int_0^{\pi/2} f(2\theta)\cos 2n\theta\,d\theta.$$

In the Fourier cosine series, the term $a_n(f)\cos n\phi$ becomes $a_n(f)\cos 2n\theta$. Similar adjustments apply to the sine series. To obtain $f(2\theta) = \theta/\sin\theta$, we take $f(\phi)$ to be $\phi/\left(2\sin\frac{1}{2}\phi\right)$. We will determine the coefficients: they are hardly transparent in advance. We start by recording the integrals corresponding to (4) and (6):

$$\int_0^{\pi/2} \theta\sin(2r-1)\theta\,d\theta = -\frac{1}{2r-1}\Big[\theta\cos(2r-1)\theta\Big]_0^{\pi/2} + \frac{1}{2r-1}\int_0^{\pi/2}\cos(2r-1)\theta\,d\theta$$
$$= \frac{(-1)^{r-1}}{(2r-1)^2}, \tag{29}$$

$$\int_0^{\pi/2} \theta\cos(2r-1)\theta\,d\theta = \frac{1}{2r-1}\Big[\theta\sin(2r-1)\theta\Big]_0^{\pi/2} - \frac{1}{2r-1}\int_0^{\pi/2}\sin(2r-1)\theta\,d\theta$$
$$= \frac{(-1)^{r-1}}{2r-1}\frac{\pi}{2} - \frac{1}{(2r-1)^2}. \tag{30}$$

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

We also need the integral of $\theta / \sin\theta$ itself, which can be evaluated in terms of Catalan's constant. This constant, commonly denoted by G, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots .$$

By termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

one finds that

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = G.$$

Substituting $x = \tan\phi$ and then $2\phi = \theta$, we deduce

$$G = \int_0^{\pi/4} \frac{\phi}{\tan\phi} \sec^2\phi \, d\phi = \int_0^{\pi/4} \frac{\phi}{\sin\phi\cos\phi} d\phi = \int_0^{\pi/2} \frac{\theta}{2\sin\theta} d\theta. \quad (31)$$

(There are numerous other integrals evaluated in terms of G; again see [1, 2].)

We consider the cosine series first. By (18) and (29), we have

$$\int_0^{\pi/2} \theta \frac{1 - \cos 2n\theta}{\sin\theta} d\theta = 2 \sum_{r=1}^{n} \frac{(-1)^{r-1}}{(2r-1)^2}. \quad (32)$$

In (32), unlike (20) or (27), we can separate the terms. By (31),

$$\int_0^{\pi/2} \frac{\theta}{\sin\theta} d\theta = 2G = 2 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^2}.$$

Hence

$$\int_0^{\pi/2} \theta \frac{\cos 2n\theta}{\sin\theta} d\theta = 2p_n,$$

where

$$p_n = \sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^2}.$$

(Note that $p_0 = G$.) Of course, this shows that $p_n \to 0$ as $n \to \infty$, in accordance with the Riemann-Lebesgue lemma. Conversely, if we assume the Riemann-Lebesgue lemma, we can deduce (31) from (32), though admittedly this is rather a tortuous route to this integral.

So we have obtained the Fourier cosine series (valid for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$)

$$\frac{\theta}{\sin\theta} = \frac{4G}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} p_n \cos 2n\theta.$$

The sum of this series has period π, so (for example) for $\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$ it equals $(\pi - \theta) / \sin\theta$.

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

Since $\theta/\sin\theta \to 1$ as $\theta \to 0$, we can evaluate at 0 to deduce that $\sum_{n=1}^{\infty} p_n + \frac{1}{2}G = \frac{\pi}{8}$. Evaluation at $\frac{\pi}{2}$ gives $\sum_{n=1}^{\infty} (-1)^n p_n + \frac{1}{2}G = \frac{\pi^2}{16}$. One can verify these identities directly from the definition of p_n by reversal of summation in the implied double series.

We now turn to the sine series. By (11) and (30), we have

$$\int_0^{\pi/2} \theta \frac{\sin 2n\theta}{\sin\theta}\, d\theta = 2q_n,$$

where

$$q_n = \frac{\pi}{2}\sum_{r=1}^{n} \frac{(-1)^{r-1}}{2r-1} - \sum_{r=1}^{n} \frac{1}{(2r-1)^2}.$$

According to the Riemann-Lebesgue lemma, we should have $q_n \to 0$. This indeed happens, in a rather interesting way. Recall that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{2r-1} = \frac{\pi}{4} \qquad \text{and} \qquad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}.$$

It follows that $q_n \to \frac{\pi^2}{8} - \frac{\pi^2}{8} = 0$. Conversely, if we assume the Riemann-Lebesgue lemma, we can deduce either of these series from the other. Also, we can rewrite q_n as follows:

$$q_n = \sum_{r=n+1}^{\infty} \frac{1}{(2r-1)^2} - \frac{\pi}{2}\sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{2r-1}.$$

So we have the Fourier sine series (valid for $0 < \theta < \frac{\pi}{2}$)

$$\frac{\theta}{\sin\theta} = \frac{8}{\pi}\sum_{n=1}^{\infty} q_n \sin 2n\theta.$$

The sum of this series is an odd function, so for $-\frac{\pi}{2} < \theta < 0$, it equals $-\theta/\sin\theta$. Also, it has period π, so equals $(\theta - \pi)/\sin\theta$ for $\frac{\pi}{2} < \theta < \pi$. Clearly it is 0 at 0 and $\frac{\pi}{2}$, reflecting the fact that at discontinuities Fourier series converge to the average of the left and right limits.

Still missing are the analogues of (9) and (16) for $(0, \frac{\pi}{2})$. We will leave (9) for the sufficiently keen reader to investigate, and concentrate on (16), which opens the way to another interesting Fourier series, the cosine series for $\theta\cos\theta/\sin\theta$. For this we need

$$\int_0^{\pi/2} \theta\sin 2r\theta\, d\theta = -\frac{1}{2r}\Big[\theta\cos 2r\theta\Big]_0^{\pi/2} + \frac{1}{2r}\int_0^{\pi/2}\cos 2r\theta\, d\theta = (-1)^{r-1}\frac{\pi}{4r}.$$

So by (14) we have, for $n \geqslant 1$,

$$\int_0^{\pi/2} \theta\,\frac{\cos\theta - \cos(2n+1)\theta}{\sin\theta}\, d\theta = \pi\sum_{r=1}^{n} \frac{(-1)^{r-1}}{2r}.$$

This is the analogue of (16), but for the Fourier series in question, what we

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

actually want is the following variant, derived from (17) instead of (14):

$$\int_0^{\pi/2} \frac{\theta \cos\theta}{\sin\theta}(1 - \cos 2n\theta)\, d\theta = \pi \sum_{r=1}^{n-1} \frac{(-1)^{r-1}}{2r} + \pi \frac{(-1)^{n-1}}{4n}. \quad (33)$$

We now need to know the following integral:

$$\int_0^{\pi/2} \frac{\theta \cos\theta}{\sin\theta}\, d\theta = \frac{\pi}{2} \log 2. \quad (34)$$

Integration by parts shows that this is equivalent to the integral

$$\int_0^{\pi/2} \log \sin\theta = -\frac{\pi}{2} \log 2,$$

which we mentioned earlier. Alternatively, we can deduce (34) from (33), using the Riemann-Lebesgue lemma and the fact that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{2r} = \frac{1}{2} \log 2.$$

Either way, we can now deduce

$$\sum_{r=1}^{\infty} \frac{\theta \cos\theta}{\sin\theta} \cos 2n\theta\, d\theta = \frac{\pi}{2} t_n,$$

where

$$t_n = \frac{(-1)^{n-1}}{2n} + \sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{r}.$$

So we have the Fourier cosine series (valid for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

$$\frac{\theta \cos\theta}{\sin\theta} = \log 2 + 2 \sum_{n=1}^{\infty} t_n \cos 2n\theta.$$

A little more discussion of this series is illuminating. The half value for $r = n$ might appear to be a tiresome complication, but it actually makes good sense, for the following reason. Let us compare t_n with the simpler sum $\sum_{r=n}^{\infty} \frac{(-1)^{r-1}}{r}$, which we denote by t_n^*. Clearly $t_n^* = t_n + \frac{(-1)^{n-1}}{2n}$. Given any decreasing sequence (a_r) that tends to 0, let $\rho_n = \sum_{r=n}^{\infty} (-1)^{r-1} a_r$. We can estimate $(-1)^{n-1}\rho_n$ by bracketing it in two ways:

$$\begin{aligned}(-1)^{n-1}\rho_n &= (a_n - a_{n+1}) + (a_{n+2} - a_{n+3}) + \dots \\ &= a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) + \dots ,\end{aligned}$$

showing that $0 \leqslant (-1)^{n-1}\rho_n \leqslant a_n$. To apply this to (t_n), observe that

$$2(-1)^{n-1} t_n = \frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2} - \frac{2}{n+3} + \dots$$

https://doi.org/10.1017/mag.2023.93 Published online by Cambridge University Press

$$= \left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \ldots$$
$$= \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} - \ldots .$$

So $|t_n| = (-1)^{n-1} t_n < \dfrac{1}{2n(n+1)}$. Meanwhile, $|t_n^*| = (-1)^{n-1} t_n^* = \dfrac{1}{2n} + (-1)^{n-1} t_n$. Hence the series $\sum_{n=1}^{\infty} t_n \cos 2n\theta$ converges much more rapidly than $\sum_{n=1}^{\infty} t_n^* \cos 2n\theta$.

Having said this, let us identify the sum of $\sum_{n=1}^{\infty} t_n^* \cos 2n\theta$ (only to reinforce the conclusion that this is a less basic Fourier series than $\sum_{n=1}^{\infty} t_n \cos 2n\theta$). By (21), applied to $\theta + \frac{\pi}{2}$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos 2n\theta = \log \cos \theta + \log 2$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So

$$2 \sum_{n=1}^{\infty} t_n^* \cos 2n\theta = \frac{\theta \cos \theta}{\sin \theta} + \log \cos \theta .$$

References

1. Graham Jameson and Nick Lord, Integrals evaluated in terms of Catalan's constant, *Math. Gaz.* **101** (March 2017) pp. 38-49.
2. Seán M. Stewart, A Catalan constant inspired integral odyssey, *Math. Gaz.* **104** (November 2020) pp. 449-459.

10.1017/mag.2023.93
Published by Cambridge University Press
on behalf of The Mathematical Association

G. J. O. JAMESON
13 Sandown Road,
Lancaster LA1 4LN
e-mail: *pgjameson@talktalk.net*