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A linear recurrence sequence of composite numbers

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Abstract

We prove that for each positive integer k in the range $2 \leqslant k \leqslant 10$ and for each positive integer $k \equiv 79 \pmod{120}$ there is a k-step Fibonacci-like sequence of composite numbers and give some examples of such sequences. This is a natural extension of a result of Graham for the Fibonacci-like sequence.

1. *Introduction*

For each integer $k \geqslant 2$ one can define a k-step Fibonacci-like sequence, that is, the sequence of integers $(x_n)_{n=0}^{\infty}$ satisfying the following relation

$$x_n = \sum_{i=1}^{k} x_{n-i}$$

for $n = k,\ k+1,\ k+2, \ldots$. Since the values of $x_0, x_1, \ldots, x_{k-1}$ determine the k-step Fibonacci-like sequence we denote it by $S_k(x_0,\ x_1, \ldots, x_{k-1})$. The terms of the sequence $S_k(0, 0, \ldots, 0, 1)$ are well-known Fibonacci k-step numbers. Flores [4] developed the calculation of Fibonacci k-step numbers without recursion. Noe and Post [9] showed that Fibonacci k-step numbers are nearly devoid of primes in the first 10 000 terms for $k \leqslant 100$.

The aim of this paper is to prove the following theorem.

Theorem 1.1. *For each positive integer k in the range $2 \leqslant k \leqslant 10$ and for each positive integer $k \equiv 79 \pmod{120}$ there are positive integers $a_0, a_1, \ldots, a_{k-1}$ such that $\gcd(a_0, a_1, \ldots, a_{k-1}) = 1$ and the sequence $S_k(a_0, a_1, \ldots, a_{k-1})$ consists of composite numbers only.*

Graham [5] proved Theorem 1.1 for $k = 2$ in 1964. He showed that the sequence

$$S_2(331\,635\,635\,998\,274\,737\,472\,200\,656\,430\,763,\ 1\,510\,028\,911\,088\,401\,971\,189\,590\,305\,498\,785)$$

contains no prime numbers. Several authors (see [7, 8, 15]) made progress in finding smaller initial values. Currently, the smallest known sequence (in the sense that the maximum of the first two elements is the smallest positive integer) is due to Vsemirnov [14]:

$$S_2(106\,276\,436\,867,\ 35\,256\,392\,432).$$

The complete analysis of a binary linear recurrence sequence of composite numbers is given in [12] and independently in [3]. If $(c_1, c_2) \in \mathbb{Z}^2$, where $c_2 \neq 0$ and $(c_1, c_2) \neq (\pm 2, -1)$, then there exist two positive relatively prime composite integers a_0, a_1 such that the sequence given by $a_{n+1} = c_1 a_n + c_2 a_{n-1}$, $n = 1, 2, \ldots$, consists of composite numbers only. Alternatively, it is easily seen that for $(c_1, c_2) = (\pm 2, -1)$ every non-periodic sequence $a_{n+1} = c_1 a_n + c_2 a_{n-1}$, $n = 1, 2, \ldots$, with $\gcd(a_0, a_1) = 1$ contains infinitely many prime numbers. Somer [12] was using deep results of Bilu *et al.* [1], Choi [2], and also the theorem of Parnami and Shorey [11]

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in his proof, while [3] contains some explicit calculations, and examples of sequences for fixed c_1, c_2 and the proof do not use external results.

All proofs of Theorem 1.1 for $k = 2$ are based on the fact that the Fibonacci sequence is a *regular divisibility sequence*, that is, $F_0 = 0$ and $F_n \mid F_m$ if $n \mid m$. However, by a result of Hall [6], there are no regular divisibility sequences in the case $S(0, a_1, a_2)$ for any $a_1, a_2 \in \mathbb{Z}$. These difficulties have recently been overcome, and Theorem 1.1 was proved for $k = 3$ in [13], where we constructed the sequence

$$S_3(99\,202\,581\,681\,909\,167\,232, 67\,600\,144\,946\,390\,082\,339, 139\,344\,212\,815\,127\,987\,596)$$

of composite numbers.

Section 3 of this paper is devoted to the generalisation of the proof developed in [13]. We will describe the set of positive integer triples and show how to prove Theorem 1.1 if this set is given. In § 4 we will prove Theorem 1.1 for all $k \equiv 79 \pmod{120}$ and construct corresponding sequences for these cases. Finally, we will give an algorithm for the construction of the set of positive integer triples, and list examples of k-step Fibonacci-like sequences for k in the range $4 \leqslant k \leqslant 10$.

2. *Auxiliary lemmas*

We start with the following elementary property of the k-step Fibonacci-like sequence.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{Z}^k$. Define $S_k(\mathbf{a}) = S_k(a_0, a_1, \ldots, a_{k-1})$. We will denote by $\mathcal{F}_k$ the set of all k-step Fibonacci-like sequences.

LEMMA 2.1. *We have that $\mathcal{F}_k$ is a free abelian group of rank k, and the map*

$$\mathbb{Z}^k \to \mathcal{F}_k, \quad \mathbf{a} \to S_k(\mathbf{a})$$

is an isomorphism of abelian groups.

The proof of this fact is straightforward.

Define

$$(s_n^{(i)})_{n=0}^{\infty} = S_k(\delta_0^i, \delta_1^i, \ldots, \delta_{k-1}^i)$$

for $i = 1, 2, \ldots, k-1$, where δ_j^i is Kronecker's delta symbol. Let p be a prime number and let $(y_n)_{n=0}^{\infty} \equiv S_k(0, a_1, a_2, \ldots, a_{k-1}) \pmod{p}$ for $a_1, a_2, \ldots, a_{k-1} \in \mathbb{Z}$. Lemma 2.1 implies

$$y_n \equiv \sum_{i=1}^{k-1} a_i s_n^{(i)} \pmod{p}. \tag{2.1}$$

LEMMA 2.2. *Fix $k \geqslant 3$. Let p be a prime number and let $(y_n)_{n=0}^{\infty} \equiv S_k(0, a_1, a_2, \ldots, a_{k-1}) \pmod{p}$ with some $a_i \in \mathbb{Z}$ for i in the range $1 \leqslant i \leqslant k-1$. Suppose that $m \geqslant 2$ is an integer. If $y_{im} \equiv 0 \pmod{p}$ for i satisfying $1 \leqslant i \leqslant k-1$, then $y_{lm} \equiv 0 \pmod{p}$ for $l = 0, 1, 2, \ldots$.*

Proof. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 1 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

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be a $k \times k$ matrix and

$$Y_n = (y_{n+k-1}, y_{n+k-2}, \ldots, y_{n+1}, y_n).$$

Then the recurrence relation $y_{n+k} = y_{n+k-1} + y_{n+k-2} + \ldots + y_{n+1} + y_n$ can be rewritten in the matrix form $Y_{n+1} = Y_n A$, for $n = 0, 1, 2 \ldots$. In particular, $Y_n = Y_0 A^n$ and

$$Y_{lm} = (y_{lm+k-1}, y_{lm+k-2}, \ldots, y_{lm+1}, y_{lm}) = (y_{k-1}, y_{k-2}, \ldots, y_1, y_0)(A^m)^l. \tag{2.2}$$

Let $B = A^m$. This is a $k \times k$ matrix with integer coefficients. By the Cayley–Hamilton theorem,

$$B^k = b_0 I + b_1 B + b_2 B^2 + \ldots + b_{k-1} B^{k-1},$$

for some integers $b_0, b_1, \ldots, b_{k-1}$. Since $Y_{lm} = Y_0 B^l$ we find that

$$Y_{lm} = b_0 Y_{(l-k)m} + b_1 Y_{(l-k+1)m} + \ldots + b_{k-1} Y_{(l-1)m}$$

for $l \geqslant k$. Considering the last entries for these vectors, we have that

$$y_{lm} = b_0 y_{(l-k)m} + b_1 y_{(l-k+1)m} + \ldots + b_{k-1} y_{(l-1)m}.$$

The lemma follows by induction. □

Define the matrix

$$B_{k,m} = \begin{pmatrix} s_m^{(1)} & s_{2m}^{(1)} & \ldots & s_{(k-1)m}^{(1)} \\ s_m^{(2)} & s_{2m}^{(2)} & \ldots & s_{(k-1)m}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ s_m^{(k-1)} & s_{2m}^{(k-1)} & \ldots & s_{(k-1)m}^{(k-1)} \end{pmatrix} \tag{2.3}$$

for each positive integer m. Let $|B_{k,m}|$ be the determinant of the matrix (2.3).

LEMMA 2.3. *Let $m \geqslant 2$ be an integer. If p is prime number and*

$$|B_{k,m}| \equiv 0 \pmod p$$

then there exist $a_1, a_2, \ldots, a_{k-1} \in \mathbb{Z}$ such that a_i is not divisible by p for at least one $i = 1, 2, \ldots k-1$, and

$$\sum_{i=1}^{k-1} a_i s_{lm}^{(i)} \equiv 0 \pmod p$$

for $l = 0, 1, 2, \ldots$.

Proof. Set $y_n = \sum_{i=1}^{k-1} a_i s_n^{(i)}$. Since $y_0 = \sum_{i=1}^{k-1} a_i s_0^{(i)} = 0$, by Lemma 2.2, it suffices to show that there exist suitable $a_1, a_2, \ldots, a_{k-1} \in \mathbb{Z}$ such that $y_{lm} \equiv 0 \pmod p$ for $l = 1, 2, \ldots, k-1$. Our aim is to solve the following system of linear equations:

$$(a_1, a_2, \ldots, a_{k-1}) B_{k,m} \equiv (0, 0, \ldots, 0) \pmod p. \tag{2.4}$$

Let us consider system (2.4) as a homogeneous linear system over the finite field $\mathbb{Z}/p\mathbb{Z}$. The assumption $|B_{k,m}| \equiv 0 \pmod p$ implies that the rank of the system (2.4) is at most $k-2$. Therefore, the system has a non-trivial solution in $\mathbb{Z}/p\mathbb{Z}$. In other words, there exist $a_1, a_2, \ldots, a_{k-1} \in \mathbb{Z}$ such that a_i is not divisible by p for at least one $i = 1, 2, \ldots k-1$. □

3. *General case*

Let I be a positive integer (to be defined later). Our goal is to find a finite set $\mathfrak{S}_k(N)$ of positive integer triples (p_i, m_i, r_i) $(i = 1, 2, \ldots, I)$ with the following properties.

(1) Each p_i is a prime number and $p_i \neq p_j$ if $i \neq j$.
(2) The positive integer p_i divides the determinant $|B_{k,m_i}|$, where B_{k,m_i} is the matrix (2.3).
(3) The congruences

$$x \equiv r_i \pmod{m_i} \tag{3.1}$$

cover the integers; that is, for any integer x there is some index i, $1 \leqslant i \leqslant I$, such that $x \equiv r_i \pmod{m_i}$.

Now, suppose that we already found the set $\mathfrak{S}_k(N)$ and that I is a fixed positive integer. Choose i, where $1 \leqslant i \leqslant I$. Since $B_{k,m_i} \equiv 0 \pmod{p_i}$, by Lemma 2.3, there exist $a_{i,1},\ a_{i,2}, \ldots, a_{i,k-1} \in \mathbb{Z}$ such that $a_{i,j}$ is not divisible by p_i for at least one $j = 1, 2, \ldots k-1$, and

$$\sum_{j=1}^{k-1} a_{i,j} s_{lm_i}^{(j)} \equiv 0 \pmod{p_i} \tag{3.2}$$

for $l = 0, 1, 2, \ldots$.

We shall construct the sequence $(x_n)_{n=0}^{\infty} = S_k(x_0, x_1, \ldots, x_{k-1})$ satisfying

$$x_n \equiv \sum_{j=1}^{k-1} s_{m_i - r_i + n}^{(j)}\, a_{i,j} \pmod{p_i} \quad i = 1, 2, \ldots I \tag{3.3}$$

for $n = 0, 1, 2, \ldots$. Set

$$\begin{aligned} A_{i,0} &= \sum_{j=1}^{k-1} s_{m_i - r_i}^{(j)}\, a_{i,j}, \\ A_{i,1} &= \sum_{j=1}^{k-1} s_{m_i - r_i + 1}^{(j)}\, a_{i,j}, \\ &\vdots \\ A_{i,k-1} &= \sum_{j=1}^{k-1} s_{m_i - r_i + k-1}^{(j)}\, a_{i,j} \end{aligned} \tag{3.4}$$

for $i = 1, 2, \ldots, I$. Since the sequence $(x_n)_{n=0}^{\infty}$ is defined by its first k terms, it suffices to solve the following equations:

$$\begin{aligned} x_0 &\equiv A_{i,0} \pmod{p_i}, \\ x_1 &\equiv A_{i,1} \pmod{p_i}, \\ &\vdots \\ x_{k-1} &\equiv A_{i,k-1} \pmod{p_i} \end{aligned} \tag{3.5}$$

for $i = 1, 2, \ldots, I$. By the Chinese remainder theorem, the system of congruences (3.5) has the positive integer solution $x_0 = X_0,\ x_1 = X_1, \ldots, x_{k-1} = X_{k-1}$. It is assumed that $\gcd(X_0, X_1, \ldots, X_{k-1}) = 1$.

By (3.2) and (3.3), p_i divides x_n if $n \equiv r_i \pmod{m_i}$, where $i \in \{1, 2, \ldots, I\}$. Since congruences (3.1) cover the integers, we see that for every non-negative integer n there is some i, $1 \leqslant i \leqslant I$, such that p_i divides x_n. The sequence $(x_n)_{n=I}^{\infty}$ is strictly increasing, so x_n must be composite for $n \geqslant I$. In this way, we can construct the k-step Fibonacci-like sequence of composite numbers $(x_n)_{n=I}^{\infty}$ if the set $\mathfrak{S}_k(N)$ is given.

Note that the assumption $\gcd(X_0, X_1, \ldots, X_{k-1}) = 1$ is unnecessarily restrictive. We can always construct the solution of (3.5) with this property. Indeed, let $\gcd(X_1, \ldots, X_{k-1}) = d_1$, $\gcd(X_0, d_1) = d_0 > 1$, and $P = \prod_{i=1}^{I} p_i$. Suppose that p is a prime number and $p \mid d_0$. If $p \mid P$,

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then, by (3.5),

$$\begin{aligned} A_{i,0} &\equiv 0 \pmod{p}, \\ A_{i,1} &\equiv 0 \pmod{p}, \\ &\vdots \\ A_{i,k-1} &\equiv 0 \pmod{p}. \end{aligned} \tag{3.6}$$

Let

$$C = \begin{pmatrix} s^{(1)}_{m_i-r_i} & s^{(1)}_{m_i-r_i+1} & \dots & s^{(1)}_{m_i-r_i+k-1} \\ s^{(2)}_{m_i-r_i} & s^{(2)}_{m_i-r_i+1} & \dots & s^{(2)}_{m_i-r_i+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ s^{(k-1)}_{m_i-r_i} & s^{(k-1)}_{m_i-r_i+1} & \dots & s^{(k-1)}_{m_i-r_i+k-1} \end{pmatrix}$$

be a $(k-1)\times k$ matrix over the finite field $\mathbb{Z}/p\mathbb{Z}$. By (3.4) and (3.6), we get

$$(a_{i,1}, a_{i,2}, \dots, a_{i,k-1})C \equiv (0, 0, \dots, 0) \pmod{p}. \tag{3.7}$$

The system of equations (3.7) has a non-trivial solution if $\operatorname{rank}(C) \leqslant k-2$. But

$$\begin{aligned} \operatorname{rank}(C) &= \operatorname{rank}\begin{pmatrix} s^{(1)}_{m_i-r_i-1} & s^{(1)}_{m_i-r_i} & \dots & s^{(1)}_{m_i-r_i+k-2} \\ s^{(2)}_{m_i-r_i-1} & s^{(2)}_{m_i-r_i} & \dots & s^{(2)}_{m_i-r_i+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ s^{(k-1)}_{m_i-r_i-1} & s^{(k-1)}_{m_i-r_i} & \dots & s^{(k-1)}_{m_i-r_i+k-2} \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} s^{(1)}_0 & s^{(1)}_1 & \dots & s^{(1)}_{k-1} \\ s^{(2)}_0 & s^{(2)}_1 & \dots & s^{(2)}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ s^{(k-1)}_0 & s^{(k-1)}_1 & \dots & s^{(k-1)}_{k-1} \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} = k-1, \end{aligned}$$

which is a contradiction. From this if follows that

$$\gcd(X_0, d_1, P) = 1. \tag{3.8}$$

It is easy to check that if $(X_0, X_1, \dots, X_{k-1})$ is a solution of (3.5), then $(X_0 + lP, X_1, \dots, X_{k-1})$ is also a solution for all integers l. Let $\gcd(X_0, P) = d$; then, by Dirichlet's theorem on prime numbers in arithmetic progression, we conclude that $X_0/d + lP/d$ is a prime number for infinitely many integers l. So, $\gcd(X_0/d + lP/d, d_1) = 1$ for some l. It follows from (3.8) that $\gcd(X_0 + lP, d_1) = 1$ for some l, which is the desired conclusion.

4. *Proof of Theorem 1.1 for* $k \equiv 79 \pmod{120}$

In this section we will show that if $k \equiv 79 \pmod{120}$, then there exist a k-step Fibonacci-like sequence of composite numbers. We will need the following lemma.

LEMMA 4.1. *Suppose that the numbers k, p and the sequence $(y_n)_{n=0}^{\infty}$ are defined as in Lemma 2.2. If there is a positive integer l such that*

$$\sum_{n=0}^{l-1} y_n \equiv 0 \pmod p \tag{4.1}$$

and

$$y_n \equiv y_{n-l} \pmod p \quad \text{for } n = l,\ l+1,\ l+2, \ldots, \tag{4.2}$$

then for every non-negative integer t the sequence

$$(y_n^{(t)})_{n=0}^{\infty} \equiv S_{tl+k}(y_0, y_1, \ldots, y_{tl+k-1}) \pmod p$$

has the following property:

$$y_n^{(t)} \equiv y_n \pmod p \tag{4.3}$$

for $n = 0, 1, 2, \ldots$.

Proof. If $t = 0$, then the statement of the lemma is trivial. Let $t \geqslant 1$. By the definition of the sequence $y_n^{(t)}$, $n = 0, 1, 2, \ldots$, (4.3) is true for $n = 0, 1, \ldots, tl + k - 1$. Let $r \geqslant k$ be an integer and suppose (4.3) is true for $n = 0, 1, \ldots tl + r - 1$. By (4.1) and (4.2), $\sum_{n=r}^{r+l-1} y_n \equiv 0 \pmod p$ for any positive integer r. Thus we have

$$y_{tl+r}^{(t)} \equiv \sum_{i=r-k}^{tl+r-1} y_i \equiv \sum_{i=r-k}^{r-1} y_i \equiv y_r \equiv y_{tl+r} \pmod p.$$

By induction, (4.3) is true for $n = 0, 1, 2, \ldots$. □

Assume that $k = 4$ and $B_{4,3}$ is the matrix defined in (2.3). It is easy to check that

$$|B_{4,3}| = \begin{vmatrix} 0 & 3 & 23 \\ 0 & 4 & 27 \\ 1 & 4 & 29 \end{vmatrix} = -11,$$

and

$$(1,\ 2,\ 0)B_{4,3} = (0,\ 0,\ 0) \qquad \pmod{11}.$$

By Lemma 2.3, the sequence $(y_n)_{n=0}^{\infty} \equiv S_4(0, 1, 2, 0) \pmod{11}$ has the following property:

$$11 \mid y_{3n} \tag{4.4}$$

for $n = 0, 1, 2, \ldots$. We calculate the first elements of sequence $(y_n)_{n=0}^{\infty} \pmod{11}$:

$$0,\ 1,\ 2,\ 0,\ 3,\ 6,\ 0,\ 9,\ 7,\ 0,\ 5,\ 10,\ 0,\ 4,\ 8,\ 0,\ 1,\ 2,\ 0, \ldots.$$

By a simple induction, one can prove that the sequence $(y_n)_{n=0}^{\infty} \pmod{11}$ is periodic. The length of the period is 15 and $\sum_{i=0}^{14} y_i \equiv 0 \pmod{11}$. By Lemma 4.1 applied to $k = 4$, $l = 15$, $p = 11$ and to the sequence $(y_n)_{n=0}^{\infty}$, we conclude that the sequence

$$(y_n^{(t)})_{n=0}^{\infty} \equiv S_{15t+4}(y_0, y_1, \ldots, y_{15t+3}) \pmod{11}$$

satisfies the property (4.3) for $t = 0, 1, 2, \ldots$. It follows that $(y_n^{(t)})_{n=0}^{\infty}$ satisfies the property (4.4) for $t = 0, 1, 2, \ldots$.

Now, let $k = 7$. It is easy to check that

$$|B_{7,3}| = \begin{vmatrix} 0 & 0 & 3 & 24 & 191 & 1508 \\ 0 & 0 & 4 & 28 & 223 & 1761 \\ 1 & 0 & 4 & 30 & 239 & 1888 \\ 0 & 0 & 4 & 31 & 247 & 1952 \\ 0 & 0 & 4 & 32 & 251 & 1984 \\ 0 & 1 & 4 & 32 & 253 & 2000 \end{vmatrix} = -5 \cdot 17$$

and

$$(1, 2, 0, 2, 4, 0)B_{7,3} = (0, 0, 0, 0, 0, 0) \pmod 5,$$
$$(1, 2, 0, 9, 1, 0)B_{7,3} = (0, 0, 0, 0, 0, 0) \pmod{17}.$$

Lemma 2.3 implies that the sequence

$$(u_n)_{n=0}^{\infty} \equiv S_7(0, 1, 2, 0, 2, 4, 0) \pmod 5$$

has the property

$$5 \mid u_{3n} \tag{4.5}$$

for $n = 0, 1, 2, \ldots$ and the sequence

$$(v_n)_{n=0}^{\infty} \equiv S_7(0, 1, 2, 0, 9, 1, 0) \pmod{17}$$

has the property

$$17 \mid v_{3n} \tag{4.6}$$

for $n = 0, 1, 2, \ldots$. The first members of the sequence $(u_n)_{n=0}^{\infty} \pmod 5$ are

$$0,\ 1,\ 2,\ 0,\ 2,\ 4,\ 0,\ 4,\ 3,\ 0,\ 3,\ 1,\ 0,\ 1,\ 2,\ 0,\ 2,\ 4,\ 0, \ldots,$$

and those of the sequence $(v_n)_{n=0}^{\infty} \pmod{17}$ are

$$0,\ 1,\ 2,\ 0,\ 9,\ 1,\ 0,\ 13,\ 9,\ 0,\ 15,\ 13,\ 0,\ 16,\ 15,\ 0,\ 8,\ 16,\ 0,\ 4,\ 8,\ 0,\ 2,\ 4,$$
$$0,\ 1,\ 2,\ 0,\ 9,\ 1,\ 0 \ldots.$$

By induction, one can prove that the sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$ are periodic with the length of the period 12 and 24, respectively. Since $\sum_{i=0}^{11} u_i \equiv 0 \pmod 5$ and $\sum_{i=0}^{23} v_i \equiv 0 \pmod{17}$, by Lemma 4.1 applied to $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$, we derive that the sequences

$$(u_n^{(t)})_{n=0}^{\infty} \equiv S_{12t+7}(u_0, u_1, \ldots, u_{12t+6})$$

and

$$(v_n^{(t)})_{n=0}^{\infty} \equiv S_{24t+7}(v_0, v_1, \ldots, v_{24t+6})$$

satisfy the property (4.3) for $t = 0, 1, 2, \ldots$. Hence, the sequence $u_n^{(t)}$ for $n = 0, 1, 2, \ldots$ satisfies the property (4.5), and the sequence $v_n^{(t)}$ for $n = 0, 1, 2, \ldots$ satisfies the property (4.6) for $t = 0, 1, 2, \ldots$.

Set $t_1 = 8t + 5$, $t_2 = 10t + 6$, $t_3 = 5t + 3$ for some positive integer t. Our goal is to find a sequence $x_n^{(t)}$ for $n = 0, 1, 2, \ldots$ satisfying the following conditions for every positive integer n:

$$\begin{aligned} x_n^{(t)} &\equiv y_n^{(t_1)} \pmod{11}, \\ x_n^{(t)} &\equiv u_{n+1}^{(t_2)} \pmod 5, \\ x_n^{(t)} &\equiv v_{n+2}^{(t_3)} \pmod{17}. \end{aligned} \tag{4.7}$$

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Using the definition of the sequences $y_n^{(t)}$, $u_n^{(t)}$, and $v_n^{(t)}$ for $n = 0, 1, 2, \ldots$ we can rewrite (4.7) as

$$\begin{aligned}(x_n^{(t)})_{n=0}^{\infty} &\equiv S_{120t+79}(y_0, y_1, \ldots, y_{120t+78}) \pmod{11},\\ (x_n^{(t)})_{n=0}^{\infty} &\equiv S_{120t+79}(u_1, u_2, \ldots, u_{120t+78}, u_7) \pmod{5},\\ (x_n^{(t)})_{n=0}^{\infty} &\equiv S_{120t+79}(v_2, v_3, \ldots, v_{120t+78}, v_7, v_8) \pmod{17}.\end{aligned}$$

By the Chinese remainder theorem, the system of equations (4.7) has a solution for every non-negative integer t. For $t = 0$ we find that

$$\begin{aligned}(x_n^{(0)})_{n=0}^{\infty} = S_{79}(&121, 782, 145, 902, 289, 710, 264, 493, 865, 693, 731, 560, 66, 697, 195,\\ &407, 34, 310, 484, 663, 325, 803, 306, 205, 121, 357, 230, 902, 884, 30, 264,\\ &408, 695, 693, 476, 50, 66, 867, 535, 407, 544, 395, 484, 323, 580, 803, 221,\\ &35, 121, 102, 655, 902, 119, 370, 264, 918, 780, 693, 136, 305, 66, 782, 365,\\ &407, 289, 820, 484, 493, 920, 803, 731, 120, 121, 697, 910, 902, 34, 200, 264),\end{aligned}$$

and for $t > 0$ we define

$$(x_n^{(t)})_{n=0}^{\infty} = S_{120t+79}(x_0^{(0)}, x_1^{(0)}, \ldots, x_{120t+78}^{(0)}).$$

By (4.3), (4.7) and by the properties (4.4)–(4.6), it follows immediately that the following hold.

- If $n \equiv 0 \pmod{3}$, then $x_n^{(t)} \equiv 0 \pmod{11}$.
- If $n \equiv 1 \pmod{3}$ then $x_n^{(t)} \equiv 0 \pmod{17}$.
- If $n \equiv 2 \pmod{3}$ then $x_n^{(t)} \equiv 0 \pmod{5}$.

Since $x_n^{(0)} > 17$ for $n = 0, 1, 2, \ldots,$ we conclude that $x_n^{(t)}$ for $n = 0, 1, 2, \ldots$ is a k-step Fibonacci-like sequence of composite numbers for $k = 120t + 79$ and $t = 0, 1, 2, \ldots$.

5. *An algorithm for the construction of the set* $\mathfrak{S}_k(N)$

The construction of the set $\mathfrak{S}_k(N)$ splits into two parts. We first generate the finite set $\mathfrak{s}_k(N) = \{(p_1, m_1), (p_2, m_2), \ldots, (p_I, m_I), \}$, where p_i is a prime number and m_i is a positive integer (Algorithm 1). Then we try to construct the covering system $\{r_1 \pmod{m_1}, r_2 \pmod{m_2}, \ldots, r_{I'} \pmod{m_{I'}}\}$ for $I' \leqslant I$. Algorithm 2 gives the answer 'I can't construct a covering system' or returns a covering system. In the second case, we construct the set $\mathfrak{S}_k(N) = \{(p_i, m_i, r_i)\}$. These algorithms were implemented using a computer algebra system PARI/GP [**10**].

The only thing we can control in the construction of the set $\mathfrak{S}_k(N)$ is the integer N. If Algorithm 2 gives an answer 'I can't construct a covering system', then we can try to choose a different N and try again. We can have different sets $\mathfrak{S}_k(N)$ for different values of N. The implementation of these algorithms takes less than one minute to give an answer on a modestly powered computer (Athlon XP 2100+) for $3 \leqslant k \leqslant 10$ and for good choice of N.

Define $A_N = \{1, 2, \ldots, N\}$ for some positive integer N and let $A_N(m, r) = \{a | a \in A_N, a \equiv r \pmod{m}\}$.

Empirical results suggest that we can choose suitable a N for any positive integer $k \geqslant 2$, so we state a following conjecture.

CONJECTURE 1. Let $k \geqslant 2$ be some fixed positive integer. Then there exist positive integers $a_0, a_1, \ldots, a_{k-1}$ such that $\gcd(a_0, a_1, \ldots, a_{k-1}) = 1$ and the sequence $S_k(a_0, a_1, \ldots, a_{k-1})$ contains no prime numbers.

Algorithm 1 Construct the set $\mathfrak{s}_k(N)$

Require: $k \geqslant 2$, $N \geqslant 2$.
Ensure: The set $\mathfrak{s}_k(N)$.

primes_list $\leftarrow \{\}$
$\mathfrak{s}_k(N) \leftarrow \{\}$
divisors_list $\leftarrow$ list of N divisors
for $d \in$ divisors_list **do**
  Construct the matrix $B_{k,d}$ {see § 2}
  determinant $\leftarrow |B_{k,d}|$
  factors_list $\leftarrow$ prime factors of determinant
  for factor $\in$ factors_list **do**
    if factor $\notin$ primes_list **then**
      Put factor in primes_list
      Put $(factor, divisor)$ in $\mathfrak{s}_k(N)$
    end if
  end for
end for
return $\mathfrak{s}_k(N)$

Algorithm 2 Construct a covering system

Require: A finite set of positive integers $\{m_1, m_2, \ldots, m_I\}$.
Ensure: The covering system $\{r_1 \pmod{m_1}, r_2 \pmod{m_2}, \ldots, r_{I'} \pmod{m_{I'}}\}$.

$N \leftarrow \mathrm{lcm}(m_1, m_2, \ldots, m_I)$
Covering_set $\leftarrow \{\}$
$B \leftarrow A_N$
for i from 1 to I **do**
  MAX $\leftarrow 0$
  for r from 0 to $m_i - 1$ **do**
    if MAX $< |A_N(m_i, r) \cap B|$ **then**
      $r_i \leftarrow r$
    end if
    Put $r_i \pmod{m_i}$ in Covering_set
    $B \leftarrow B \backslash A_N(m_i, r_i)$
    if $B = \{\}$ **then**
      return Covering_set
    end if
  end for
end for
print 'I can't construct a covering system'

6. *Examples of sequences for* $k = 4, 5, \ldots, 10$

Since the case $k = 2$ is proved in [**5**] and the case $k = 3$ in [**13**], in this section we will prove Theorem 1.1 for $k = 4, 5, \ldots, 10$. As was noticed in § 3, we only need to construct the set $\mathfrak{S}_k(N)$. Below we list some examples of sequences $(x_n)_{n=0}^{\infty}$ for each k in the range $4 \leqslant k \leqslant 10$.

$$(x_n)_{n=0}^{\infty} = S_4(6\,965\,341\,197\,997\,216\,603\,441\,345\,255\,549\,082\,199\,598,\ 10\,958\,188\,570\,324\,452\,297\,588\,339\,728\,720\,332\,112\,233,\ 3\,338\,506\,596\,043\,156\,696\,233\,507\,996\,784\,908\,854\,102,\ 11\,794\,350\,400\,878\,505\,028\,751\,078\,386\,520\,701\,499\,400).$$

$$(x_n)_{n=0}^{\infty} = S_5(1\,670\,030, 2\,329\,659, 907\,322, 2\,009\,158, 580\,558).$$

$$(x_n)_{n=0}^{\infty} = S_6(14\,646\,825\,659\,441\,969\,908\,161\,645\,620, 17\,528\,323\,654\,959\,029\,482\,507\,167\,866,\ 34\,890\,970\,296\,357\,954\,582\,882\,737\,564, 26\,873\,338\,145\,021\,062\,044\,773\,578\,613,\ 51\,550\,231\,534\,183\,425\,910\,033\,499\,205, 42\,628\,449\,155\,999\,760\,197\,422\,601\,556).$$

$$(x_n)_{n=0}^{\infty} = S_7(49\,540, 32\,691, 13\,932, 18\,650, 9962, 31\,004, 21\,990).$$

$$(x_n)_{n=0}^{\infty} = S_8(4\,540\,180\,821\,663\,595\,548\,672, 4\,698\,078\,862\,727\,331\,233\,761,\ 6\,155\,103\,797\,589\,406\,562\,086, 6\,372\,283\,045\,103\,453\,008\,950,\ 2\,279\,826\,085\,324\,947\,150\,546, 1\,997\,011\,623\,084\,108\,165\,756,\ 2\,558\,082\,925\,488\,023\,201\,996, 1\,574\,529\,020\,466\,071\,641\,536).$$

$$(x_n)_{n=0}^{\infty} = S_9(56\,233\,156\,963\,124, 2\,686\,035\,354\,591, 59\,483\,968\,596\,828,\ 9\,266\,206\,975\,260, 5\,763\,383\,142\,928, 2\,968\,317\,519\,550,\ 56\,580\,150\,371\,822, 38\,270\,799\,500\,006, 16\,687\,306\,893\,378).$$

$$(x_n)_{n=0}^{\infty} = S_{10}(2\,757\,357, 684\,913, 197\,119, 5\,440\,883, 4\,628\,571,\ 6\,208\,094, 871\,487, 2\,421\,952, 1\,064\,430, 5\,329\,024).$$

Since the set $\mathfrak{S}_k(N)$ is essential in the construction of a k-step Fibonacci-like sequence $S_k(x_0, x_1, \ldots, x_{k-1})$, we give this set for each k in the range $4 \leqslant k \leqslant 10$ (see Tables 1–7).

TABLE 1. *The set* $\mathfrak{S}_4(360)$.

i	p_i	m_i	r_i	$\lvert B_{4,m_i}\rvert$
1	11	3	0	11
2	2	5	0	2^6
3	41	6	1	$11 \cdot 41$
4	1511	8	0	1511
5	521	9	2	$11 \cdot 521$
6	29	10	2	$2^{12} \cdot 29$
7	167	12	10	$11^2 \cdot 41 \cdot 167$
8	33 391	15	8	$2^6 \cdot 11 \cdot 33\,391$
9	73	18	5	$11 \cdot 41 \cdot 73 \cdot 251 \cdot 521$
10	251	18	17	$11 \cdot 41 \cdot 73 \cdot 251 \cdot 521$
11	10 399	20	4	$2^{18} \cdot 29 \cdot 10\,399$
12	13 177	24	4	$11^2 \cdot 41 \cdot 167 \cdot 1511 \cdot 13\,177$
13	6781	30	26	$2^{12} \cdot 11 \cdot 29 \cdot 41 \cdot 6781 \cdot 33\,391$
14	37	36	14	$11^2 \cdot 37 \cdot 41 \cdot 73 \cdot 167 \cdot 251 \cdot 521 \cdot 195\,407$
15	195 407	36	26	$11^2 \cdot 37 \cdot 41 \cdot 73 \cdot 167 \cdot 251 \cdot 521 \cdot 195\,407$

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TABLE 2. *The set* $\mathfrak{S}_5(16)$.

i	p_i	m_i	r_i	$\lvert B_{5,m_i}\rvert$
1	2	2	0	2^2
2	3	4	1	$2^4 \cdot 3^2$
3	71	8	3	$2^6 \cdot 3^2 \cdot 71$
4	47	16	7	$2^8 \cdot 3^2 \cdot 47 \cdot 71 \cdot 193$
5	193	16	15	$2^8 \cdot 3^2 \cdot 47 \cdot 71 \cdot 193$

TABLE 3. *The set* $\mathfrak{S}_6(32)$.

i	p_i	m_i	r_i	$\lvert B_{6,m_i}\rvert$
1	5	4	0	$5 \cdot 41$
2	41	4	1	$5 \cdot 41$
3	31	8	2	$5 \cdot 31 \cdot 41 \cdot 239$
4	239	8	3	$5 \cdot 31 \cdot 41 \cdot 239$
5	79	16	6	$5 \cdot 31 \cdot 41 \cdot 79 \cdot 239 \cdot 271 \cdot 1777$
6	271	16	7	$5 \cdot 31 \cdot 41 \cdot 79 \cdot 239 \cdot 271 \cdot 1777$
7	1777	16	14	$5 \cdot 31 \cdot 41 \cdot 79 \cdot 239 \cdot 271 \cdot 1777$
8	257	32	15	$5 \cdot 31 \cdot 41 \cdot 79 \cdot 239 \cdot 257 \cdot 271 \cdot 1777 \cdot 3\,827\,975\,948\,383$
9	3 827 975 948 383	32	31	$5 \cdot 31 \cdot 41 \cdot 79 \cdot 239 \cdot 257 \cdot 271 \cdot 1777 \cdot 3\,827\,975\,948\,383$

TABLE 4. *The set* $\mathfrak{S}_7(6)$.

i	p_i	m_i	r_i	$\lvert B_{7,m_i}\rvert$
1	2	2	0	2^3
2	5	3	0	$5 \cdot 17$
3	17	3	1	$5 \cdot 17$
4	337	6	5	$2^3 \cdot 5 \cdot 17 \cdot 337$

TABLE 5. *The set* $\mathfrak{S}_8(30)$.

i	p_i	m_i	r_i	$\lvert B_{8,m_i}\rvert$
1	2	3	0	2^7
2	3	5	0	$3^2 \cdot 7^2 \cdot 59$
3	7	5	1	$3^2 \cdot 7^2 \cdot 59$
4	59	5	2	$3^2 \cdot 7^2 \cdot 59$
5	41	6	1	$2^{15} \cdot 41$
6	586 919	10	4	$3^4 \cdot 7^2 \cdot 59 \cdot 586\,919$
7	151	15	8	$2^7 \cdot 3^4 \cdot 7^2 \cdot 59 \cdot 151 \cdot 25\,025\,941$
8	25 025 941	15	13	$2^7 \cdot 3^4 \cdot 7^2 \cdot 59 \cdot 151 \cdot 25\,025\,941$
9	31	30	29	$2^{15} \cdot 3^8 \cdot 7^2 \cdot 31 \cdot 41 \cdot 59 \cdot 151 \cdot 586\,919 \cdot 25\,025\,941 \cdot 38\,457\,989$

TABLE 6. *The set* $\mathfrak{S}_9(12)$.

i	p_i	m_i	r_i	$\lvert B_{9,m_i}\rvert$
1	2	2	0	2^4
2	31	4	1	$2^8 \cdot 31$
3	74 933	6	1	$2^4 \cdot 74\,933$
4	2927	12	3	$2^8 \cdot 31 \cdot 2927 \cdot 4957 \cdot 74\,933$
5	4957	12	11	$2^8 \cdot 31 \cdot 2927 \cdot 4957 \cdot 74\,933$

TABLE 7. *The set* $\mathfrak{S}_{10}(8)$.

i	p_i	m_i	r_i	$\lvert B_{10,m_i}\rvert$
1	3	4	0	$3 \cdot 17 \cdot 257$
2	17	4	1	$3 \cdot 17 \cdot 257$
3	257	4	2	$3 \cdot 17 \cdot 257$
4	7	8	3	$3^3 \cdot 7 \cdot 17 \cdot 71 \cdot 257 \cdot 3391$
5	71	8	7	$3^3 \cdot 7 \cdot 17 \cdot 71 \cdot 257 \cdot 3391$

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Finally, we give the coefficients of the system of equations (3.5) (see Tables 8–15). It is necessary because in Lemma 2.3 we prove only the existence of these coefficients; that is, with the same set $\mathfrak{S}_k(N)$ we can find the different k-step Fibonacci-like sequence $S_k(x_0, x_1, \ldots, x_{k-1})$.

TABLE 8. *Coefficients of* (3.5) *for* $k = 4$.

	i						
	1	2	3	4	5	6	7
$A_{i,0}$	0	0	21	0	421	7	124
$A_{i,1}$	1	1	0	1	128	7	64
$A_{i,2}$	2	0	35	4	0	0	22
$A_{i,3}$	0	0	5	1305	9	2	44

TABLE 9. *Coefficients of* (3.5) *for* $k = 4$.

	i							
	8	9	10	11	12	13	14	15
$A_{i,0}$	19 247	1	1	10 164	12 571	151	22	75 748
$A_{i,1}$	25 767	46	11	752	7342	302	5	105 421
$A_{i,2}$	2901	66	52	3340	5671	603	25	65 611
$A_{i,3}$	8709	70	64	6542	770	5420	11	100 766

TABLE 10. *Coefficients of* (3.5) *for* $k = 5$.

	i				
	1	2	3	4	5
$A_{i,0}$	0	2	39	26	1
$A_{i,1}$	1	0	7	10	149
$A_{i,2}$	0	2	13	34	29
$A_{i,3}$	0	1	0	2	28
$A_{i,4}$	0	1	62	14	14

TABLE 11. *Coefficients of* (3.5) *for* $k = 6$.

	i								
	1	2	3	4	5	6	7	8	9
$A_{i,0}$	0	8	8	51	25	3	1147	44	1
$A_{i,1}$	1	0	16	60	43	62	1159	123	1 671 520 683 283
$A_{i,2}$	4	18	0	120	35	126	353	123	1 187 982 745 969
$A_{i,3}$	3	31	11	0	49	93	940	187	2 373 684 950 413
$A_{i,4}$	0	21	3	37	56	45	46	116	1 575 934 864 371
$A_{i,5}$	1	0	23	65	29	79	92	206	2 981 147 295 654

TABLE 12. *Coefficients of* (3.5) *for* $k = 7$.

	i			
	1	2	3	4
$A_{i,0}$	0	0	2	1
$A_{i,1}$	1	1	0	2
$A_{i,2}$	0	2	9	115
$A_{i,3}$	0	0	1	115
$A_{i,4}$	0	2	0	189
$A_{i,5}$	0	4	13	0
$A_{i,6}$	0	0	9	85

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Table 13. *Coefficients of* (3.5) *for* $k = 8$.

	i = 1	2	3	4	5	6	7	8	9
$A_{i,0}$	0	0	1	35	9	506 111	92	14 176 025	1
$A_{i,1}$	1	1	0	11	0	249 334	80	6 652 214	12
$A_{i,2}$	0	1	2	0	14	146 730	9	1 932 056	17
$A_{i,3}$	0	0	6	51	1	293 460	17	15 861 862	13
$A_{i,4}$	0	0	2	40	2	0	18	16 528 118	12
$A_{i,5}$	0	0	4	15	3	8526	127	23 725 749	12
$A_{i,6}$	0	2	0	30	3	85 280	14	3 798 202	15
$A_{i,7}$	0	0	5	0	0	511 720	96	7 596 404	20

Table 14. *Coefficients of* (3.5) *for* $k = 9$.

	i = 1	2	3	4	5
$A_{i,0}$	0	2	33 332	143	1
$A_{i,1}$	1	0	0	286	1095
$A_{i,2}$	0	27	72 006	571	4380
$A_{i,3}$	0	23	63 225	0	3835
$A_{i,4}$	0	23	18 734	1286	405
$A_{i,5}$	0	0	37 468	2185	1364
$A_{i,6}$	0	16	2	2886	3240
$A_{i,7}$	0	1	0	92	2547
$A_{i,8}$	0	1	24 967	20	1996

Table 15. *Coefficients of* (3.5) *for* $k = 10$.

	i = 1	2	3	4	5
$A_{i,0}$	0	8	4	1	1
$A_{i,1}$	1	0	8	5	47
$A_{i,2}$	1	4	0	6	23
$A_{i,3}$	2	16	193	0	11
$A_{i,4}$	0	15	1	3	10
$A_{i,5}$	2	0	2	4	67
$A_{i,6}$	2	16	0	1	33
$A_{i,7}$	1	13	241	1	0
$A_{i,8}$	0	9	193	3	69
$A_{i,9}$	1	0	129	1	48

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