§ 5. Orthocentre.

*The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.**

One of the earliest demonstrations occurs in Pierre Herigone's *Cursus Mathematicus*, I. 318 (1634). Three cases are considered, when the triangle is right-angled, acute-angled, obtuse-angled.

From the various proofs that have been published, the following are selected.

First Demonstration.†

Figure 36.

Let AX, BY which are perpendicular to BC, CA meet at H, and let CH be joined and produced to meet AB at Z.

Join XY.

Because ∠ AXC and ∠ BYC are right,

therefore C, X, H, Y are concyclic, as well as A, Y, X, B;

therefore ∠ ACZ = ∠ AXY,

= ∠ ABY.

Now ∠ ZAY is common to triangles ACZ, ABY;

therefore ∠ AZC = ∠ AYB,

= a right angle.

Second Demonstration.‡

Figure 37.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Through A, B, C draw B_1C_1, C_1A_1, A_1B_1 respectively parallel to BC, CA, AB.

---

* This theorem occurs without proof in the fifth of the *Lemmas* ascribed to Archimedes, and also in Pappus's *Mathematical Collection*, VII. 62. In Commandino's editions of Pappus, which were published after his death, the proof supplied is erroneous. The mistake has been noticed by several mathematical writers.

† Robert Simson's *Opera Quaedam Reliqua*, p. 171 (1776).

‡ This mode of proof is given by F. J. Servois in his *Solutions peu connues de différens problèmes de Géométrie-pratique*, p. 15 (1804). It was also given by Gauss, and will be found in Schumacher's translation into German of Carnot's *Géométrie de Position*, II. 363 (1810).

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Then $ABCB_1$, $ACBC_1$ are parallelograms,
and A is the mid point of B_1C_1.
Hence also B and C are the mid points of C_1A_1 and A_1B_1.
But AX, BY, CZ are respectively perpendicular to BC, CA, AB ;
therefore they must be respectively perpendicular to B_1C_1, C_1A_1, A_1B_1.
If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the mid points of the sides are concurrent,

AX, BY, CZ are concurrent.

Third Demonstration.*

Figure 38.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Join YZ, ZX, XY.

Since the points A, Z, X, C are concyclic,

therefore $\angle BXZ = \angle BAC$.

Since the points A, Y, X, B are concyclic,

therefore $\angle CXY = \angle BAC$;

therefore $\angle BXZ = \angle CXY$.

Now $\angle BXA = \angle CXA$;

therefore AX bisects $\angle ZXY$.

Hence BY „ $\angle XYZ$,

and CZ „ $\angle YZX$.

If therefore it be assumed as true that the internal angular bisectors of a triangle are concurrent

AX, BY, CZ are concurrent.

Fourth Demonstration.†

" If three straight lines drawn through the vertices of a triangle are concurrent, their isogonals with respect to the angles of the triangle are also concurrent."

This theorem, which is due to Steiner,‡ taken along with the property, which is established in the proof of Brahmegupta's theorem, namely,

* Mr Bernh. Möllmann in Grunert's *Archiv*, XVII., 376 (1851).

† Dr James Booth's *New Geometrical Methods*, II. 260-1 (1877).

‡ Gergonne's *Annales*, XIX. 37-64 (1828), or Steiner's *Gesammelte Werke*, I. 193 (1881).

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"The perpendicular from any vertex of a triangle to the opposite side and the diameter of the circumcircle drawn from that vertex are isogonal with respect to the vertical angle" furnishes a ready proof. For the diameters of the circumcircle are concurrent.

The point H, where AX, BY, CZ are concurrent, is now generally called the orthocentre* of ABC; and the triangle XYZ is called sometimes the orthic,† sometimes the orthocentric,‡ and sometimes the pedal, triangle.

It may be noted that H is the initial letter in English, French, and German of the names for AX, BY, CZ (*Heights*, *Hauteurs*, *Höhen*).

(1) If in Fig. 37 ABC be considered the fundamental triangle, $A_1B_1C_1$ is anticomplementary to it, and hence the orthocentre of any triangle is the circumcentre of the anticomplementary triangle.

If however $A_1B_1C_1$ be considered the fundamental triangle, ABC is complementary to it, and hence the circumcentre of any triangle is the orthocentre of the complementary triangle.

(2) The four points A, B, C, H, taken three by three form four triangles ABC, HCB, CHA, BAH; of these four triangles the fourth points H, A, B, C are the respective orthocentres, and in all the four cases the orthic triangle is XYZ. "The figure is therefore a system of four points joined two and two by straight lines such that each of them passing through two of these points cuts perpendicularly that which passes through the two others." §

In naming the four triangles the order of the letters is such that X is the foot of the perpendicular from the vertex first named, Y the foot of that from the second named vertex, and Z the foot of that from the third. This is a matter of much more importance than appears at first sight.

It may be convenient to call a set of four points such as A, B, C, H an *orthic tetrastigm*.

---

* This useful expression was suggested by Dr Ferrers and Dr W. H. Besant in 1866-7. It is introduced in Dr Besant's *Conic Sections*, § 138 (1869).

† Mr Emile Vigarié in *Mathesis*, VII. 61 (1887).

‡ Dr James Booth in his *New Geometrical Methods*, II. 261 (1877).

§ Carnot, *Corrélation des Figures de Géométrie*, § 143 (1801).

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(3) The angles of the triangles HCB, CHA, BAH expressed in terms of A, B, C are

$$\angle BHC = 180° - A, \quad \angle HCB = 90° - B, \quad \angle CBH = 90° - C$$
$$\angle ACH = 90° - A, \quad \angle CHA = 180° - B, \quad \angle HAC = 90° - C$$
$$\angle HBA = 90° - A, \quad \angle BAH = 90° - B, \quad \angle AHB = 180° - C.$$

(4) *The fundamental triangle is inversely similar to the triangles "cut off" from it by the sides of the orthic triangle.*

FIGURE 38.

If ABC be the fundamental triangle, H is its orthocentre, XYZ its orthic triangle, and the triangles cut off from ABC and similar to it are AYZ, XBZ, XYC.

If HCB be taken as the fundamental triangle, A is its orthocentre, XYZ its orthic triangle, and the triangles "cut off" from HCB and similar to it are HYZ, XCZ, XYB.

Similarly for CHA and triangles CYZ, XHZ, XYA
and for BAH „ „ BYZ, XAZ, XYH.

(5) ABC is the orthic triangle not only of $I_1I_2I_3$, but also of II_3I_2, I_3II_1, I_2I_1I.

FIGURE 28.

Hence the sides of ABC "cut off" from these four triangles four triads of triangles which are respectively similar to them. They are

To $I_1I_2I_3$; I_1BC, AI_2C, ABI_3
„ $I\,I_3I_2$; $I\,BC$, AI_3C, ABI_2
„ $I_3I\,I_1$; I_3BC, $AI\,C$, ABI_1
„ I_2I_1I ; I_2BC, AI_1C, ABI .

(6) The following triads of lines form by their intersections four triangles which are similar and oppositely situated to the four triangles of the orthic tetrastigm $II_1I_2I_3$.

FIGURE 28.

Lines.	Triangles.
E_1F_1, F_2D_2, D_3E_3	$I_1I_2I_3$
$E\,F$, F_3D_3, D_2E_2	$I\,I_3I_2$
E_3F_3, $F\,D$, D_1E_1	$I_3I\,I_1$
E_2F_2, F_1D_1, $D\,E$	I_2I_1I

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Compare the subscripts in the naming of the lines with the subscripts in the naming of the triangles.

(7) *The sides of the orthic triangle are respectively antiparallel* * *to those of the fundamental triangle with respect to the angles of the fundamental triangle.*

FIGURE 38.

If ABC be taken as the fundamental triangle,

YZ	is	antiparallel	to	BC	with	respect	to ∠CAB,
ZX	„	„	„	CA	„	„	„ ∠ABC
XY	„	„	„	AB	„	„	„ ∠BCA.

If HCB be taken as the fundamental triangle,

YZ	is	antiparallel	to	CB	with	respect	to ∠BHC
ZX	„	„	„	BH	„	„	„ ∠HCB
XY	„	„	„	HC	„	„	„ ∠CBH.

Similarly for the triangles CHA, BAH.

(8) The angles of triangle XYZ expressed in terms of A, B, C are:

$$\angle X = 180^\circ - 2A = -A + B + C$$

$$\angle Y = 180^\circ - 2B = A - B + C$$

$$\angle Z = 180^\circ - 2C = A + B - C.$$

(9) If ABC, XYZ, $X_1Y_1Z_1$, $X_2Y_2Z_2$ ...... be a series of triangles such that each is the orthic triangle of the preceding, the following tabular statements of their angles may be given.*

* Carnot's *Géométrie de Position*, § 151 (1803). The term antiparallel was first used by Antoine Arnauld. See *Nouveaux Eléments de Géométrie*, par Messrs de Port-Royal, p. 212, or livre onzième (1667). Further information regarding the use of the word will be found in two letters from Mr E. M. Langley to *Nature*, XL., 460-1 (1889), and XLI., 104-5 (1889).

* These are taken from an article by Mr H. Brocard in the *Nouvelle Correspondance Mathématique*, VI. 145-151 (1880).

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TRIANGLES.	ANGLES.		
A B C	A	B	C
X Y Z	$-A+B+C$	$A-B+C$	$A+B-C$
$X_1Y_1Z_1$	$3A-B-C$	$-A+3B-C$	$-A-B+3C$
$X_2Y_2Z_2$	$-5A+3B+3C$	$3A-5B+3C$	$3A+3B-5C$
$X_3Y_3Z_3$	$11A-5B-5C$	$-5A+11B-5C$	$-5A-5B+11C$
......	......	......	......

Consider the coefficients (all taken with the positive sign) of the angle A in the first column of angles. They form the series

$$\begin{matrix} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ 1 & 3 & 5 & 11 & 21 & 43 & 85 & 171 \end{matrix}$$

where the law of recurrence is

$$u_{n+1} = u_n + 2u_{n-1}$$

with the initial conditions $u_0 = 1$, $u_1 = 3$.

TRIANGLES.	ANGLES.		
A B C	A	B	C
X Y Z	$\pi - 2A$	$\pi - 2B$	$\pi - 2C$
$X_1Y_1Z_1$	$4A - \pi$	$4B - \pi$	$4C - \pi$
$X_2Y_2Z_2$	$3\pi - 8A$	$3\pi - 8B$	$3\pi - 8C$
$X_3Y_3Z_3$	$16A - 5\pi$	$16B - 5\pi$	$16C - 5\pi$
......	......	......	......

In these expressions the coefficient of A, B, or C is a power of 2, and the coefficient of π is one term of the series $u_0\ u_1\ u_2\ u_3 \ldots$

The angle $X_{2n} = u_{2n-1}\pi - 2^{2n+1}A$;

,, ,, $X_{2n-1} = 2^{2n} A - u_{2n-2}\pi$.

(10) *The orthocentre and the vertices of the fundamental triangle are the incentre and the excentres of the orthic triangle.**

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 24 (1822).

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FIGURE 38.

In Möllmann's demonstration of the concurrency of the perpendiculars, it was shown that, if ABC be taken as the fundamental triangle, H is the incentre of XYZ.

Now since BC, CA, AB are respectively perpendicular to AX, BY, CZ, therefore BC, CA, AB are the bisectors of the external angles of XYZ;
therefore A, B, C are the excentres of XYZ.

If HCB be taken as the fundamental triangle, its vertices, B, C and its orthocentre A are the excentres of XYZ, and the vertex H is the incentre.

Similarly for the triangles CHA, BAH.

(11) *If from the mid points of YZ, ZX, XY perpendiculars be drawn to BC, CA, AB, these perpendiculars are concurrent.**

If X′, Y′, Z′ be the mid points,
then triangle X′Y′Z′ is similar and oppositely situated to XYZ;
therefore the respective perpendiculars are the bisectors of the angles of X′Y′Z′, and consequently concurrent at the incentre of X′Y′Z′.

(12) The perpendiculars from X′, Y′, Z′ respectively to

CB, BH, HC	are concurrent at the	first excentre of X′Y′Z′
HA, AC, CH		second „ „ „
AH, HB, BA		third „ „ „

These four points, the incentre and the excentres of triangle X′Y′Z′, will be considered again, in connection with the Taylor circles.

(13) *If the perpendiculars of a triangle meet the circumcircle again in R, S, T, then R, S, T are the images of the orthocentre in the sides.*

FIGURE 39.

Let ABC be the triangle, H its orthocentre.
Join BR.

Then $\angle CBY = \angle CAX = \angle CBR$;

therefore the right-angled triangles BXH, BXR are congruent,

and $HX = RX$.

Similarly $HY = SY$ and $HZ = TZ$.

* Édouard Lucas in *Nouvelle Correspondance Mathématique*, II. 95, 218 (1876).

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If HCB be taken as the triangle instead of ABC, then A is its orthocentre, HX, CY, BZ its perpendiculars. Let a circle be circumscribed about HCB, and let the perpendiculars meet it again at R_1, S_1, T_1.

FIGURE 40.

Then it may be shown as before that

$$AX = R_1X, \quad AY = S_1Y, \quad AZ = T_1Z.$$

Similarly for the triangles CHA, BAH.

(14) *The triangles RST, XYZ are similar and similarly situated; H is their homothetic centre, and their ratio of similitude is* 2 : 1.

FIGURE 40.

Since X, Y, Z are the mid points of HR, HS, HT, therefore the sides of XYZ are respectively parallel to those of RST, and equal to the halves of them.

In like manner the triangles $R_1S_1T_1$, XYZ are similar and similarly situated; A is their homothetic centre, and their ratio of similitude is 2 : 1.

(15) H is the incentre of R S T,
A „ „ first excentre „ $R_1S_1T_1$.

Similarly for B and C.

(16) *The circumcircle of ABC is equal* to the circumcircles of HCB, CHA, BAH.*

FIGURE 39.

For triangle HCB is congruent to RCB;
and the circumcircle of RCB is the circumcircle of ABC.

(17) *If O_a, O_b, O_c be the centres of the circumcircles of HCB, CHA, BAH, then triangle $O_aO_bO_c$ is congruent, and oppositely situated, to ABC.*

FIGURE 41.

For O_bO_c, O_cO_a, O_aO_b are perpendicular to HA, HB, HC
and BC, CA, AB „ „ „ „ „ „.

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* Carnot's *Corrélation des Figures de Géométrie*, § 146 (1801), or *Géométrie de Position*, § 130 (1803).

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(18) *H is the circumcentre of $O_aO_bO_c$,*
O ,, ,, orthocentre ,, ,, .

Since the circles O_b, O_c are equal,
therefore O_bO_c bisects, and is bisected by, their common chord HA perpendicularly;

therefore $HO_b = HO_c$.
Similarly $HO_c = HO_a$.

Again, since the circles O, O_a are equal,
therefore OO_a bisects, and is bisected by, their common chord BC perpendicularly;

therefore O_aO is perpendicular to O_bO_c.
Similarly O_bO ,, ,, ,, O_cO_a.

(19) The points O_a, O_b, O_c, O form an orthic tetrastigm, congruent and oppositely situated to the orthic tetrastigm A, B, C, H.

(20) If through A any straight line be drawn meeting the circles O_b, O_c in M, N, then MC, NB will meet on the circumference of O_a.

(21) If any point L be taken on the circumference of O_a, and LC, LB meet the circumferences of O_b, O_c again in M, N, then the points M, A, N are collinear, and triangle LMN is directly similar to ABC.

(22) Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles O_a, O_b, O_c, that triangle $A_1B_1C_1$ is a maximum whose sides are perpendicular to AH, BH, CH.

Compare § 2, (15) – (19).

(23) *Triangle $A_1B_1C_1$ is the anticomplementary triangle of ABC; it has H for its circumcentre, and its circumcircle touches the circles O_a, O_b, O_c at the points A_1, B_1, C_1.*

For A, B, C are the mid points of B_1C_1, C_1A_1, A_1B_1;
and H, O_a, A_1; H, O_b, B_1; H, O_c, C_1 are collinear.

(24) What has been already proved with regard to the triangle ABC, its orthocentre H, its circumcentre O, and the circles O_a, O_b, O_c may be applied, with the necessary modifications, to the triangle HCB, its orthocentre A, its circumcentre O_a, and the circles O, O_c, O_b; and to the triangles CHA, BAH.

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(25) *If* RT, RS *meet* BC *at* D, D' ; SR, ST *meet* CA *at* E, E' ; TS, TR *meet* AB *at* F, F', *then*

$HDRD'$, $HESE'$, $HFTF'$ *are rhombi,*

and D, H, E' ; E, H, F' ; F, H, D' ;

D', H, F ; E', H, D ; F', H, E *are collinear.**

FIGURE 42.

Since HR is bisected perpendicularly by DD′,

therefore HD = RD and HD′ = RD′.

But since XY, XZ make equal angles with BC,

and RS, RT are respectively parallel to XY, XZ ;

therefore RD = RD′, and HDRD′ is a rhombus.

Again since DH is parallel to RD′

and HE′ ,, ,, ,, ES ;

therefore D, H, E′ are collinear.

(26) *If* R_1T_1, R_1S_1 *meet* CB *at* D, D' ; S_1R_1, S_1T_1 *meet* BH *at* E', E ; T_1S_1, T_1R_1 *meet* HC *at* F', F, *then*

ADR_1D', AES_1E', AFT_1F' *are rhombi,*

and D, A, E ; E', A, F ; F', A, D', *etc., are collinear.*

FIGURE 40.

Two other triads of rhombi, and of collinear points may be obtained from triangles CHA, BAH.

(27) *If* U, V, W *be the mid points of* AH, BH, CH, *then* U, V, W *are the orthocentres of triangles* $AC'B'$, $C'BA'$, $B'A'C$.

FIGURE 43.

For the perpendicular from B′ to AC′ is parallel to CH ;
and since B′ is the mid point of AC, this perpendicular passes through the mid point of AH, that is U ;
and AU is perpendicular to C′B′.

(28) *The points* U, V, W, H *form an orthic tetrastigm, where* H *is the orthocentre of* UVW.

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* *Nouvelles Annales*, 2nd series, XIX. 176 (1880) and 3rd series, I. 186–9 (1882).

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If the triangle UVW be translated so that U moves along UA and VW remains parallel to BC, it will coincide with triangle AC′B′.

Similarly the triangle UVW may be made to coincide with C′BA′ and B′A′C.

(29) FIGURE 43.

U, B′, C′ A′, V, C′ A′, B′, W	are orthocentres of	HWV, CA′W, BVA′ CWB′, HUW, AB′U BC′V, AUC′, HVU.

(30) *The point H may be the orthocentre of an infinite number of triangles inscribed in the circle ABC.*

FIGURE 39.

For, take any point A on the circumference;
and draw the chord AHR.
Bisect HR at X, and through X draw the chord BC perpendicular to AR.

Then ABC is a triangle whose orthocentre is H.

(31) *The point A may be the orthocentre of an infinite number of triangles inscribed in the circle HCB.*

FIGURE 40.

For, take any point H on the circumference;
and draw the secant AHR_1.
Bisect AR_1 at X, and through X draw the chord BC perpendicular to AR_1.

Then HCB is a triangle whose orthocentre is A.

Similarly for B and C.

(32) *The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the orthic triangle; and the straight lines joining the orthocentre to the vertices are perpendicular to the sides of the complementary triangle; and conversely.*

FIGURE 44.

Let OA meet YZ at X′;
from O draw O B′ perpendicular to CA;
from B′ draw B′C′ parallel to BC.

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Then B′ is the mid point of CA,
and B′C′ is a side of the complementary triangle.

Hence $\angle AOB' = \angle ABC = \angle AYX'$;
therefore $\angle AX'Y = \angle AB'O =$ a right angle;
and HA is perpendicular to B′C′.

This theorem will be found to be a particular case of a more general one regarding isogonal lines.

(33) The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to all straight lines which are anti-parallel to the sides with respect to the opposite angles; and the straight lines joining the orthocentre to the vertices are perpendicular to all straight lines which are parallel to the sides.

(34) *The straight lines AX and AO are isogonals* with respect to angle BAC.*

FIGURE 44.

For $\angle ABX = \angle AOB'$,
$\angle AXB = \angle AB'O$;
therefore $\angle XAB = \angle OAC$.

Similarly BY, BO are isogonals with respect to $\angle B$.
and CZ, CO ,, ,, ,, ,, ,, $\angle C$.

The theorem may be stated and proved otherwise, thus:

The straight lines joining the incentre with the vertices of a triangle bisect the angles between the radii of the circumcircle drawn to the vertices and the perpendiculars.

FIGURE 45.

Produce AI to meet the circumcircle in U, and join OU.

Because AU bisects $\angle BAC$,
therefore U is the mid point of the arc BUC;
therefore OU is perpendicular to BC;
therefore $\angle XAU = \angle OUA = \angle OAU$.

* This corollary is established in the proof of the theorem known as Brahmegupta's.

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(35) *The straight lines joining the mid points of*

AH, BC ; BH, CA ; CH, AB

make with

AB , BC , CA

angles complementary to*

C , A , B.

FIGURE 43.

Let A′, U be the mid points of BC, AH, and O the circumcentre. Join OA.

Then OA′ is equal and parallel to AU ,
therefore OA is parallel to A′U.
Now OA makes with AB an angle equal to CAH,
that is, an angle complementary to C ;
therefore A′U makes with AB an angle complementary to C.

(36) *The same straight lines make with AX, BY, CZ angles equal to B ~ C, C ~ A, A ~ B.*

For $$\angle A'UX = \angle OAX$$
$$= \angle BAX - \angle CAX$$
$$= C - B$$

(37) *The angle between*†

B′Z and C′Y = 3A, C′X and A′Z = 3B, A′Y and B′X = 3C.

FIGURE 46.

Produce BY to B_1 so that $B_1Y = BY$,
and „ CZ „ C_1 „ „ $C_1Z = CZ$,
and join AB_1, AC_1.

Then $\angle B_1AY$ and $\angle C_1AZ$ are each equal to A ;
therefore $\angle B_1AC_1 = 3A$.
But since C′ and Y are the mid points of BA and BB_1,
therefore C′Y is parallel to AB_1.
Similarly B′Z „ „ „ AC_1 ;

* Dr C. Taylor in *Mathematical Questions from the Educational Times*, XVIII. 65 (1872).

† This property and the demonstration of it are due to Professor R. E. Allardice.

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therefore the angle between B′Z and C′Y is equal to the angle between AB_1 and AC_1.

(38) *The straight lines drawn from the orthocentre of a triangle through the mid points of the sides and terminated by the circumcircle are bisected by the sides.*

FIGURE 47.

Let ABC be the triangle, H its orthocentre.

Draw CL parallel to HB and terminated by the circumcircle. Join BL.

Because CL is parallel to AB,
therefore $\angle$ ACL is right;
therefore $\angle$ ABL „ „ ;
therefore BL is parallel to HC.
Hence HBLC is a parallelogram, and its diagonals bisect each other;
that is, HL drawn through A′, the mid point of BC, is bisected by BC.

This corollary may be used to prove part of the characteristic property of the nine-point circle.

(39) $A'Y = A'Z,\ B'Z = B'X,\ C'X = C'Y.$

FIGURE 47.

For B, Z, Y, C are situated on the circumference of a circle whose centre is A′.

(40) *If on each side of a triangle as diagonal two parallelograms be constructed, the one having a vertex at the opposite angle of the triangle, the other at the centre of the circumcircle, then the straight lines which join the other vertices of these three pairs of parallelograms will pass through the orthocentre.**

FIGURE 48.

FIRST DEMONSTRATION.

Let H be the orthocentre, O the circumcentre; and let O′ and A′ be the vertices opposite to O and A of the parallelograms of which BC is the common diagonal.

* Mr W. J. C. Miller in the *Lady's and Gentleman's Diary* for 1862, p. 74.

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Since $\angle$ BHC is supplementary to $\angle$ A,
therefore $\angle$ BHC „ „ „ $\angle$ A′;
therefore H is on the circumcircle of A′BC.
Now $\angle$ A′BH is right;
therefore A′H is a diameter of the circle A′BC;
therefore A′H passes through O′ its centre.

SECOND DEMONSTRATION.*

Draw AA_1 parallel to BC;
join A′O and produce it to meet the circumcircle in R;
join AR meeting BC at X.

Then AR is perpendicular to BC,
and if we imagine the whole figure reflected in BC,
A_1 and O will reflect into the vertices of the two parallelograms on BC as diagonal.
Hence the line joining these vertices will meet AX at the point H, the reflection of R.
But since $\angle BHC = \angle BRC = 180^\circ - \angle BAC$,
therefore H is the orthocentre of ABC;
therefore the straight line joining the two vertices of the parallelogram on BC as diameter passes through the orthocentre.

(41) *If through A, B, C there be drawn AC_1, BA_1, CB_1 making equal angles respectively with HA, HB, HC, a new triangle $A_1B_1C_1$ is formed, which is similar to ABC, and whose circumcentre† is H.*

FIGURE 49.

Because $\angle HCB_1 = \angle HBA_1$,
therefore the points H, C, A_1, B are concyclic;
therefore $\angle A_1 = 180^\circ - BHC = \angle A$.
Similarly $\angle B_1 = \angle B, \quad \angle C_1 = \angle C$.

Join HB_1, HC_1.

---

* "Conic" of St John's College, Cambridge, in the *Lady's and Gentleman's Diary* for 1863, p. 51.

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 34 (1825).

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Because $\angle ACH = \angle ABH$,

and $\angle ACH = \angle AB_1H$,

$\angle ABH = \angle AC_1H$;

therefore $\angle AB_1H = \angle AC_1H$;

therefore $HB_1 = HC_1$.

Similarly $HC_1 = HA_1$;

therefore H is the circumcentre of $A_1B_1C_1$.

(42) Since HA, HB, HC are respectively perpendicular to BC, CA, AB, the theorem of the preceding corollary is equivalent to the following:

If through the vertices of a triangle straight lines be drawn making equal angles with the opposite sides, they will form by their intersection a new triangle, which is similar to the original triangle, and which has for circumcentre the orthocentre of the original triangle.

A particular case of this theorem has already been given, that, namely, where the straight lines drawn through the vertices are parallel to the opposite sides. The triangle $A_1B_1C_1$ so formed, the anticomplementary triangle of ABC, is the maximum triangle that can be constructed under such conditions, and it it is equal to four times ABC.

(43) *Triangle XYZ is the triangle of minimum perimeter* inscribed in ABC.*

It is usually considered that this statement is proved † when it is shown that XY and XZ make equal angles with BC

„ YZ „ YX „ „ „ „ CA

„ ZX „ ZY „ „ „ „ AB.

No objection can be taken to the following proof ‡:

FIGURE 50.

Produce YZ both ways, making ZX_1 equal to ZX, YX_2 equal to YX; then X_1X_2 is the perimeter of XYZ.

Join BX_1, CX_2.

Because $\angle XZB = \angle AZY = \angle X_1ZB$

* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum anni* 1775, p. 296.

† See Prof. R. E. Allardice's paper "On a property of odd and even polygons" in the *Proceedings of the Edinburgh Mathematical Society*, VIII. 23 (1890).

‡ Marsano, *Considerazioni sul Triangolo Rettilineo*, pp. 18, 19 (1863).

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therefore triangles XZB and X_1ZB are congruent.
Similarly „ XYC „ X_2YC „ „ .

If now DEF be any other triangle inscribed in ABC, and along BX_1 there be taken BD_1 equal to BD, and along CX_2 there be taken CD_2 equal to CD, and FD_1, ED_2 be joined, it may be proved that $FD_1 = FD$, $ED_2 = ED$, and that consequently the line D_1FED_2 is the perimeter of DEF.

If D_1FED_2 is not straight, join D_1D_2 and join the vertex A with X, D, X_1, D_1, X_2, D_2.

Then $$AX_1 = AX = AX_2,$$
$$AD_1 = AD = AD_2;$$
therefore the triangles AX_1X_2, AD_1D_2 are isosceles.
And their vertical angles X_1AX_2, D_1AD_2 are equal,
since each is double of angle BAC;
therefore the triangles AX_1X_2, AD_1D_2 are similar.
Now AX_1 is less than AD_1, since AX is less than AD;
therefore X_1X_2 is less than D_1D_2, and consequently less than D_1FED_2.

If the triangle ABC be right-angled at A, the points Y, Z coalesce with A, X_1X_2 and D_1D_2 pass through A and are respectively double of AX and AD.

If the triangle ABC be obtuse-angled at A, the points Y, Z fall outside the triangle ABC (*Figure* 51) and X_1X_2 is now equal to $XY - YZ + ZX$. If therefore the preceding statements and proof are to hold good, the side YZ must be considered negative.

(44) *If XX_1, XX_2 be joined cutting AB, AC in P, Q, then PQ is the semiperimeter* of triangle XYZ.*

FIGURES 50, 51.

For P is the mid point of XX_1 and Q the mid point of XX_2;
therefore $PQ = \frac{1}{2}X_1X_2 =$ semiperimeter of XYZ.

P and Q are the feet of the perpendiculars from X on AB and AC.

If triangle ABC be obtuse-angled, the perimeter of XYZ must be understood with the qualification of the preceding corollary.

* Lhuilier, *Élémens d'Analyse*, p. 231 (1809). The proof in the text is given by Feuerbach, *Eigenschaften ..des...Dreiecks*, Section VI., Theorem 8 (1822).

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(45) *If two triangles ABC, A′B′C′ have their sides parallel, and one of them is circumscribed about and the other is inscribed in the same triangle DEF, the area of this last triangle is a mean proportional between the areas of the two others.**

FIGURE 52.

Let AB′, AC′ meet BC at P and Q. Through A′ draw A′A″ parallel to B′C′ or BC and meeting AC′ at A″.

Join A″B′, AA′, B′Q.

Then A′B′F = A′B′A, A′C′E = A′C′A, B′C′D = B′C′Q;

therefore DEF = AB′Q, A′B′C′ = A″B′C′.

Now A″B′C′ : AB′Q = A″C′ : AQ;

and A″C′ : AQ is the ratio of the altitudes of the similar triangles A′B′C′, ABC.

Hence A″C′ : AQ = B′C′ : BC;

therefore A″B′C′ : AB′Q = B′C′ : BC.

Again A B′Q : APQ = AB′ : AP
= B′C′ : PQ;

and A PQ : AB C = PQ : BC;

therefore A B′Q : AB C = B′C′ : BC;

therefore A″B′C′ : AB′Q = AB′Q : ABC

or A′B′C′ : DEF = DE F : ABC.

The terms *inscribed* and *circumscribed* have the following signification,

One triangle is inscribed in a second triangle when the vertices of the first are situated on the sides or the sides produced of the second; and in either case the second triangle is circumscribed about the first.

---

* This theorem is due to Mr Rochat of Saint-Brieux, and is thus stated in Gergonne's *Annales de Mathématiques* II. 93 (1811-2).

If to any triangle T there be circumscribed another T′, and to T′ a third T″ having its sides respectively parallel to those of T; then to T″ a new triangle T‴ having its sides respectively parallel to those of T′, and so on: the triangles T, T′, T″, T‴ will be similar in pairs and form a geometrical progression.

The demonstration in the text is given by Mr Léon Anne, in the *Nouvelles Annales*, III. 27 (1844).

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(46) *If $I_1I_2I_3$ be the fundamental triangle, ABC its orthic triangle, and DEF the triangle formed by joining the points of contact of the incircle of ABC, then**

$$I_1I_2I_3 : ABC = ABC : DEF.$$

FIGURE 28.

In the same way if II_3I_2 be the fundamental triangle, ABC its orthic triangle, and $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, then

$$II_3I_2 : ABC = ABC : D_1E_1F_1\,;$$

and so on.

(47) $$ABC : DEF = 2R : r$$

$$ABC : D_1E_1F_1 = 2R : r_1$$

and so on.

For $$(ABC)^2 : (DEF)^2 = I_1I_2I_3 : DEF$$
$$= 4R^2 : r^2$$

since 2R and r are the radii of the circumcircles of the similar triangles $I_1I_2I_3$ and DEF.

(48) *If ABC be the fundamental triangle, DEF the triangle formed by joining the points of contact of the incircle of ABC, and X'Y'Z' the orthic triangle of DEF, then*

$$ABC : DEF = DEF : X'Y'Z'.$$

FIGURE 53.

For $\angle BDF = \angle DEF = \angle DY'Z'$;

therefore Y'Z' is parallel to BC.

Hence Z'X' „ „ CA

and X'Y' „ „ AB.

In the same way if ABC be the fundamental triangle, $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, and the orthic triangle of $D_1E_1F_1$ be constructed, it will be found that this orthic triangle has its sides parallel to those of ABC, and that $D_1E_1F_1$ is a mean proportional between it and ABC.

* The theorems (46)-(48) are given by Feuerbach, *Eigenschaften...des...Dreiecks*, §§ 61, 8, 63 (1822).

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(49) Hence $I_1I_2I_3$, ABC, DEF, X′Y′Z′, ...
are a series of triangles whose areas form a geometrical progression, the alternate terms being similar.

Other series may be obtained from

$$II_3I_2,\ ABC,\ D_1E_1F_1,\ \ldots,\ \text{etc.}$$

(50) $$ABC : X'Y'Z' = 4R^2 : r^2.$$

Def. If P be any point in the plane of ABC, and D, E, F be the projections of P on BC, CA, AB, then DEF is called the *pedal* triangle of P with respect to ABC.

(51) *If* H_1 , H_2 , H_3

be the orthocentres of the triangles

AEF , BFD , CDE

cut off from ABC by the sides of the pedal triangle DEF of any point P, the triangle $H_1H_2H_3$ *is congruent and oppositely situated to DEF.*

Figure 54.

Since	PD, FH_2 are perpendicular to BC,
therefore	PD is parallel to FH_2.
Similarly	PF ,, ,, DH_2;
therefore	PDH_2F is a parallelogram,
and	$PD = FH_2$.
Hence also	$PD = EH_3$,
therefore	EFH_2H_3 is a parallelogram,
and	$H_2H_3 = EF$.

Figure 55.

The sides of the four triangles

$$DEF,\ D_1E_1F_1,\ D_2E_2F_2,\ D_3E_3F_3$$

make with the sides of ABC the following

twelve triangles	whose orthocentres are
AEF , BFD , CDE	H_1, H_2, H_3
AE_1F_1, BF_1D_1, CD_1E_1	H_1', H_2', H_3'
....	....
....	....

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(52) The twelve orthocentres are situated in pairs on the six lines

$$II_1,\ II_2,\ II_3,\ I_2I_3,\ I_3I_1,\ I_1I_2.$$

(53) The four triangles

$$H_1H_2H_3, \quad H_1'H_2'H_3', \text{ and so on,}$$

are congruent and oppositely situated to

$$D\ E\ F, \quad D_1\ E_1\ F_1, \text{ and so on.}$$

(54) The following figures are rhombi :

$$D H_3 E I\ ,\quad E H_1 F I\ ,\quad F H_2 D I$$
$$D_1H_3'E_1I_1\ ,\quad E_1H_1'F_1I_1\ ,\quad F_1H_2'D_1I_1$$

....

....

their sides being r, r_1, r_2, r_3 respectively.

(55) The following figures are equilateral hexagons :

$$DH_3EH_1FH_2,\ D_1H_3'E_1H_1'F_1H_2', \text{ ...., ...}$$

their perimeters being $6r$, $6r_1$, $6r_2$, $6r_3$ respectively.

(56) I , I_1 , I_2 , I_3

which are the circumcentres of the triangles

$$D\ E\ F, \quad D_1\ E_1\ F_1, \quad \textit{and so on,}$$

*are the orthocentres of the triangles**

$$H_1H_2H_3, \quad H_1'H_2'H_3', \quad \textit{and so on.}$$

Take, for example, the triangle $H_1H_2H_3$.

Because H_1I is perpendicular to $E\ F$,

therefore H_1I ,, ,, ,, H_2H_3.

Similarly for H_2I and H_3I.

(57) *If* H_0 , H_0' , H_0'' , H_0'''

be the orthocentres of the triangles

$$D\ E\ F, \quad D_1\ E_1\ F_1, \quad \textit{and so on,}$$

they will be the circumcentres of the triangles *

$$H_1H_2H_3, \quad H_1'H_2'H_3', \quad \textit{and so on.}$$

---

* The first parts of (56) and (57) are given by Feuerbach, *Eigenschaften...des... Dreiecks*, §§ 87, 88 (1822).

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FIGURE 56.

Take, for example, the triangle $H_1H_2H_3$.

Because	DH_0 is perpendicular to EF,
therefore	DH_0 ,, ,, H_2H_3.
And since	$DH_2 = DH_3$,
therefore	DH_0 bisects H_2H_3
therefore	DH_0 passes through the circumcentre of $H_1H_2H_3$

Similarly for EH_0 and FH_0.

(58) $$H_0H_1 = H_0H_2 = H_0H_3$$
$$= I\ D = I\ E = I\ F = r$$

For H_0 and I are the circumcentres of two congruent triangles $H_1H_2H_3$ and DEF.

Similarly for H_0', I_1, and so on.

(59) The following figures are parallelograms :

$$DIH_1H_0,\ EIH_2H_0,\ FIH_3H_0\ ;$$

they have a common diagonal IH_0;
their other diagonals intersect at the mid point of IH_0.

Similarly for I_1H_0', and so on.

(60) The homothetic centre of DEF, $H_1H_2H_3$ is the mid point of IH_0.

Similarly for $D_1E_1F_1$, $H_1'H_2'H_3'$, and so on.

(61) The following figures are rhombi :

$$DH_3H_0H_2,\ EH_1H_0H_3,\ FH_2H_0H_1,$$

and their sides are equal to r.

Three other triads of rhombi can be obtained by putting subscripts and accents to the preceding letters.

(62) If from H_1, H_2, H_3 perpendiculars be drawn to BC, CA, AB, these perpendiculars will be concurrent at H_0.

Since	$EH_1H_0H_3$	is a rhombus,
therefore	H_1H_0	is parallel to EH_3
therefore	H_1H_0	is perpendicular to BC.
Similarly for	H_2H_0 and H_3H_0.	

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(63) Since I and H_0 are the circumcentre and orthocentre of DEF and the orthocentre and circumcentre of $H_1H_2H_3$, these two triangles have the same nine-point circle,* and its centre is the mid point of IH_0.

FIGURE 56.

(64) In triangle DEF

$$\left.\begin{array}{l} DI,\ DH_0 \\ EI,\ EH_0 \\ FI,\ FH_0 \end{array}\right\} \text{ are isogonals with respect to } \left\{\begin{array}{l} D \\ E \\ F \end{array}\right.$$

(65) In triangle $H_1H_2H_3$

$$\left.\begin{array}{l} H_1I,\ H_1H_0 \\ H_2I,\ H_2H_0 \\ H_3I,\ H_3H_0 \end{array}\right\} \text{ are isogonals with respect to } \left\{\begin{array}{l} H_1 \\ H_2 \\ H_3 \end{array}\right.$$

(66) *Of the perpendiculars to BC from H_1, H_2, H_3, I the first is equal to the sum of the other three.*†

FIGURE 57.

Let the feet of the perpendiculars on BC from H_1, H_2, H_3 be X_1, X_2, X_3;

let IH_1 meet EF at D′;

from D′ draw a perpendicular to BC, meeting BC at D″ and H_2H_3 at L.

Then	D′	is	the	mid	point	of	EF and IH_1;
therefore	D″	,,	,,	,,	,,	,,	X_2X_3 ,, X_1D;
Also	L	,,	,,	,,	,,	,,	H_2H_3;
therefore	L	,,	,,	,,	,,	,,	DH_0,

since $DH_3H_0H_2$ is a rhombus.

Hence $H_2X_2 + H_3X_3 = 2LD'' = H_0X_1$;

and $ID = H_0H_1$;

therefore $H_2X_2 + H_3X_3 + ID = H_1X_1$.

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* Feuerbach, § 89.

† Feuerbach, § 80. The mode of proof is not his.

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In triangle ABC, the perpendiculars AX, BY, CZ intersect at H the orthocentre, and XYZ is the orthic triangle.

FIGURE 58.

This figure has reference to the properties (67)–(82). The reader would find it convenient if he constructed a copy of it on a large scale.

Of the triangles	let the orthocentres be
AYZ, XBZ, XYC	H_1 , H_2 , H_3
HYZ, XCZ, XYB	H_1' , H_2' , H_3'
CYZ, XHZ, XYA	H_1'', H_2'', H_3''
BYZ, XAZ, XYH	H_1''', H_2''', H_3'''

(67) Of these H points, four pairs are collinear with X, four with Y, and four with Z, that is, through

X pass H_2H_2''', H_3H_3'', $H_2'H_2''$, $H_3'H_3'''$
Y ,, H_3H_3' , H_1H_1''', $H_3''H_3'''$, $H_1'H_1''$
Z ,, H_1H_1'', H_2H_2' , $H_1'H_1'''$, $H_2''H_2'''$

(68) The four* triangles $H_1H_2H_3$, $H_1'H_2'H_3'$ etc., are congruent and oppositely situated to XYZ.

(69) *The three triangles $H_1'H_1''H_1'''$, $H_2'H_2''H_2'''$, $H_3'H_3''H_3'''$ are congruent and oppositely situated to ABC ; and H_1, H_2, H_3 are their respective orthocentres.*

Take for example $H_3'H_3''H_3'''$.

Because $H_3''H_3'''$ passes through Y and is perpendicular to AX,
therefore $H_3''H_3'''$ is parallel to BC.
Similarly $H_3'''H_3'$ is parallel to CA.
Again $H_2'H_3'$ is equal and parallel to $H_2''H_3''$;
therefore $H_3'H_3''$,, ,, ,, ,, ,, $H_2'H_2''$.
Now $H_2'H_2''$ passes through X and is perpendicular to CZ;
therefore $H_3'H_3''$ is parallel to AB.
Hence $H_3'H_3''H_3'''$ is similar and oppositely situated to ABC.

* Feuerbach (§90) proves the congruency of XYZ, $H_1H_2H_3$

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Because $H_3' H_3$ passes through Y and is perpendicular to BC,
and $H_3'' H_3$ „ „ X „ „ „ „ CA;
therefore Y, X are the feet of two of the perpendiculars,
and H_3 is the orthocentre, of triangle $H_3'H_3''H_3'''$.

Lastly, since H, X in triangle ABC
correspond to H_3, Y „ „ $H_3'H_3''H_3'''$, and $HX = H_3Y$,
therefore triangles ABC, $H_3'H_3''H_3'''$ are congruent.

(70) *If* $H_1' H_1$ *meet* $H_1'' H_1'''$ *at* X_1,
$H_2'' H_2$ „ $H_2''' H_2'$ „ Y_1,
$H_3''' H_3$ „ $H_3' H_3''$ „ Z_1;

then the feet of the perpendiculars of triangle

$H_1'H_1''H_1'''$ *are* X_1, Z, Y,
$H_2'H_2''H_2'''$ „ Z, Y_1, X,
$H_3'H_3''H_3'''$ „ Y, X, Z_1;

and the sides of triangle $X_1Y_1Z_1$ *pass through* X, Y, Z *and are there bisected.*

Because triangles $H_1'H_1''H_1'''$ and ABC are congruent and oppositely situated,
therefore their orthic triangles X_1ZY and XYZ are congruent and oppositely situated.
Similarly ZY_1X and YXZ_1 are congruent and oppositely situated to XYZ;
therefore Y_1Z_1 passes through X and is bisected at X
Z_1X_1 „ „ Y „ „ „ „ Y
X_1Y_1 „ „ Z „ „ „ „ Z.

(71) *ABC,* $X_1Y_1Z_1$ *have the same nine-point circle.*

For X, Y, Z, the feet of the perpendiculars of ABC, are the mid points of the sides of $X_1Y_1Z_1$.

(72) *If* O, O_a, O_b, O_c *be the circumcentres of*

ABC, HCB, CHA, BAH;

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then the point of concurrency of

	AH_1 ,	BH_2 ,	CH_3	*is* O
	HH_1' ,	CH_2' ,	BH_3'	,, O_a
	CH_1'' ,	HH_2'' ,	AH_3''	,, O_b
	BH_1''' ,	AH_2''' ,	HH_3'''	,, O_c
and	O ,	O_a ,	O_b ,	O_c *are orthocentres*
of	$H_1H_2H_3$,	$H_1'H_2'H_3'$,	$H_1''H_2''H_3''$,	$H_1'''H_2'''H_3'''$.

For AH_1, BH_2, CH_3 are respectively perpendicular
to YZ, ZX, XY;
and their concurrency is established by Steiner's theorem concerning orthologous triangles. See § 6 (1).

Since AH_1, BH_2, CH_3 are respectively perpendicular
to YZ, ZX, XY, they are therefore perpendicular
to H_2H_3, H_3H_1, H_1H_2,
and consequently concurrent at the orthocentre of $H_1H_2H_3$.

(73) *If the homothetic centre of the triangles*

XYZ	*and*	$H_1\ H_2\ H_3$	*be*	T
XYZ	,,	$H_1'\ H_2'\ H_3'$	,,	T_1
XYZ	,,	$H_1''\ H_2''\ H_3''$	,,	T_2
XYZ	,,	$H_1'''H_2'''H_3'''$	,,	T_3 ,

then $T_1T_2T_3$ *is similar and oppositely situated to* ABC,
and T, T_1, T_2, T_3 *form an orthic tetrastigm.*

For T_2 is the mid point of XH_1''
T_3 ,, ,, ,, ,, ,, XH_1''';
therefore T_2T_3 is parallel to $H_1''H_1'''$ and equal to half of it;
therefore T_2T_3 ,, ,, ,, BC ,, ,, ,, ,, ,, ,, .

Again T is the mid point of XH_1
T_1 ,, ,, ,, ,, ,, XH_1';
therefore TT_1 is parallel to H_1H_1' and equal to half of it;
therefore TT_1 is perpendicular to $H_1''H_1'''$ or to T_2T_3,
and T is orthocentre of $T_1T_2T_3$.

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(74) *The point T is the centre of the three parallelograms*

$$YZH_2H_3,\ ZXH_3H_1,\ XYH_1H_2.$$

For YH_2, ZH_3 intersect at T.

Similarly T_1, T_2, T_3 are each the centre of three parallelograms.

(75) *If X', Y', Z' be the mid points of YZ, ZX, XY, then the point of concurrency of*

$H_1\ X'$,	$H_2\ Y'$,	$H_3\ Z'$	*is*	H
$H_1'\ X'$,	$H_2'\ Y'$,	$H_3'\ Z'$	,,	A
$H_1''\ X'$,	$H_2''\ Y'$,	$H_3''\ Z'$	,,	B
$H_1'''X'$,	$H_2'''Y'$,	$H_3'''Z'$	,,	C .

For YH_1 and HZ are parallel, and so are ZH_1 and HY;
therefore HYH_1Z is a parallelogram;
therefore HH_1 and YZ bisect each other,
that is, H_1X' passes through H.

Again, ABC, $H_1'H_1''H_1'''$ are congruent and oppositely situated, and Y in ABC corresponds to Z in $H_1'H_1''H_1'''$;
therefore $AYH_1'Z$ is a parallelogram;
therefore AH_1' and YZ bisect each other,
that is, $H_1'X'$ passes through A.

(76) Let the incircle and excircles of XYZ be denoted by their centres H, A, B, C; then the radical axes of

	H, A;	H, B;	H, C;	B, C;	C, A;	A, B
are	$T_2\ T_3$,	$T_3\ T_1$,	$T_1\ T_2$,	$T_1\ T$,	$T_2\ T$,	$T_3\ T$.

For T_2T_3 is perpendicular to HA, and bisects YZ.

(77) The circles A, B, C; H, C, B; C, H, A; B, A, H
have T, T_1, T_2 T_3
for radical centres.

(78) *X', Y' Z' are the feet of the perpendiculars of $T_1T_2T_3$.*

For in triangle AXH_1' the mid point of XH_1' is T_1,
and T_1T is parallel to AX;
therefore T_1T passes through X', the mid point of AH_1'.

Hence T, T_1, T_2, T_3 are the incentre and the excentres of the triangle $X'Y'Z'$. Compare § 5, (11), (12).

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(79) The homothetic centre of the triangles

$$\begin{array}{llll} T_1T_2T_3 & \text{and} & H_1'H_1''H_1''' & \text{is } X \\ T_1T_2T_3 & ,, & H_2'H_2''H_2''' & ,, \ Y \\ T_1T_2T_3 & ,, & H_3'H_3''H_3''' & ,, \ Z. \end{array}$$

For T_2, T_3 are mid points of XH_1'', XH_1'''.

(80) Since $X'Y'Z'$ is the complementary triangle of XYZ, and T, T_1, T_2, T_3 are the incentre and excentres of $X'Y'Z'$, and H, A, B, C ,, ,, ,, ,, ,, ,, XYZ; therefore HT, AT_1, BT_2, CT_3 all pass through the centroid of XYZ. See § 2.

If G′ denote this centroid *

then $$HG' : TG' = AG' : T_1G' = BG' : T_2G' = CG' : T_3G'$$
$$= 2 : 1.$$

(81) Since H is the incentre, G′ the centroid, of XYZ, and T the incentre of $X'Y'Z'$, if HG′T be produced to J′ so that TJ′ = HT, then J′ will be the incentre of $X_1Y_1Z_1$.

Similarly J_1', J_2', J_3', situated on AT_1, BT_2, CT_3, so that $T_1J_1' = AT_1$ and so on, will be the first, second, and third excentres of $X_1Y_1Z_1$.

These statements follow from the first few corollaries of § 2.

(82) The tetrads of points

H, G′, T, J′; A, G′, T_1, J_1'; B, G′, T_2, J_2'; C, G′, T_3, J_3'

form harmonic ranges.

(83) Since triangles $I_1I_2I_3$, ABC stand to each other in the same relation as ABC, XYZ, the second being the orthic triangle of the first, it may be convenient to state in another form some of the results already established.

The means of transliteration from the one form to the other will be afforded by the following lists of corresponding points.

* G′ would naturally denote the centroid of triangle A′B′C′, but G is the centroid both of ABC and A′B′C′.

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$$A,\ B,\ C,\ H,\ X,\ Y,\ Z,\ H_1,\ H_2,\ H_3$$

correspond to

$$I_1,\ I_2,\ I_3,\ I,\ A,\ B,\ C,\ H_a,\ H_b,\ H_c$$

and

$$O,\ O_a,\ O_b,\ O_c,\ G',\ T,\ T_1,\ T_2,\ T_3$$

correspond to

$$O_0,\ O_1,\ O_2,\ O_3,\ G,\ L,\ L_1,\ L_2,\ L_3$$

and

$$X',\ Y',\ Z',\ X_1,\ Y_1,\ Z_1,\ J',\ J_1',\ J_2',\ J_3'$$

correspond to

$$A',\ B',\ C',\ A_1,\ B_1,\ C_1,\ J,\ J_1,\ J_2,\ J_3.$$

Hence the following results* are obtained :

(*a*) The orthocentres of the triangles BCI_1, CAI_2, ABI_3 form the vertices of a triangle $H_aH_bH_c$ which is congruent to the fundamental triangle ABC, and has its sides parallel to the corresponding sides of ABC.

(*b*) AH_a, BH_b, CH_c are concurrent at the radical centre of I_1, I_2, I_3.

(*c*) The radical centre bisects AH_a, BH_b, CH_c.

(*d*) The radical axes of the I circles bisect the sides of $H_aH_bH_c$.

(*e*) O_0 is the orthocentre of $H_aH_bH_c$.

(*f*) AD_1, BE_2, CF_3 are concurrent at J the incentre of $H_aH_bH_c$. The points J, I, L are collinear.

(*g*) H_a, A′, I ; H_b, B′, I ; H_c, C′, I are collinear.

(*h*) A′, B′, C′ bisect H_aI, H_bI, H_cI.

FIGURE 59.

In triangle ABC, the points H, X, Y, Z are the orthocentre and feet of the perpendiculars ;
the various I, D, E, F points are the centres and points of contact of the incircle and the excircles.

The rest of the notation will be explained as it is wanted.

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* See Professor Johann Döttl's *Neue merkwürdige Punkte des Dreiecks*, pp. 40-46 (no date). The proofs given in this noteworthy pamphlet are analytical.

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(84) The perpendicular AX contains the intersection of

D_2E_2 ,	D_3F_3	namely	X_0
D_3E_3 ,	D_2F_2	,,	X_1
D E ,	D_1F_1	,,	X_2
D_1E_1 ,	D F	,,	X_3 .

The perpendicular BY contains the intersection of

E_3F_3 ,	E_1D_1	namely	Y_0
E_2F_2 ,	E D	,,	Y_1
E_1F_1 ,	E_3D_3	,,	Y_2
E F ,	E_2D_2	,,	Y_3 .

The perpendicular CZ contains the intersection of

F_1D_1 ,	F_2E_2	namely	Z_0
F D ,	F_3E_3	,,	Z_1
F_3D_3 ,	F E	,,	Z_2
F_2D_2 ,	F_1E_1	,,	Z_3 .

FIGURE 60.

Through A draw a parallel to BC;
let D_2E_2 meet AX at X_0 and the parallel at S.

Then triangles CD_2E_2, ASE_2 are similar;

and because $CD_2 = CE_2$

therefore $AS = AE_2 = s_3$.

Now triangles AX_0S, DIC have their sides respectively parallel to each other; therefore they are similar.

But $AS = s_3 = DC$;

therefore $AX_0 = DI = r$.

Again if D_3F_3 meet the parallel through A at T, and AX at X_0', it may be proved that

$$AT = AF_3 = s_2 = DB$$

and that triangles $AX_0'T$, DIB are congruent;

therefore $AX_0' = DI = r$,

and X_0, X_0' are the same point.

The other properties are proved in a manner exactly analogous.

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(85) FIGURE 59.

$$AX_0 = BY_0 = CZ_0 = r$$
$$AX_1 = BY_1 = CZ_1 = r_1$$
$$AX_2 = BY_2 = CZ_2 = r_2$$
$$AX_3 = BY_3 = CZ_3 = r_3.$$

Attention may be directed here and later on to the way in which the various suffixes occur.

The triads of lines				the triangles
E_1F_1,	F_2D_2,	D_3E_3	determine	$X_1Y_2Z_3$
EF,	F_3D_3,	D_2E_2		$X_0Y_3Z_2$
E_3F_3,	FD,	D_1E_1		$X_3Y_0Z_1$
E_2F_2,	F_1D_1,	DE		$X_2Y_1Z_0$

(86) The four triangles

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are respectively similar and oppositely situated to

$$I_1\ I_2\ I_3, \quad I\ I_3\ I_2, \quad I_3\ I\ I_1, \quad I_2\ I_1\ I$$

and H, the orthocentre of ABC, is the circumcentre of the four.

Since Y_2Z_3 is perpendicular to AI_1,
therefore Y_2Z_3 is parallel to I_2I_3.
Similarly for Z_3X_1 and X_1Y_2.

Again $\angle HY_2Z_3 = \angle CAI_1$
because the sides of the one are perpendicular to those of the other:
and $\angle HZ_3Y_2 = \angle BAI_1$, for a similar reason:
therefore $\angle HY_2Z_3 = \angle HZ_3Y_2$:
therefore $HY_2 = HZ_3$.
Similarly $HZ_3 = HX_1$;
therefore H is the circumcentre of $X_1Y_2Z_3$.

(87) The radii of the circumcircles of

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are $\quad 2R + r, \quad 2R - r_1, \quad 2R - r_2, \quad 2R - r_3.$

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For

$$\begin{aligned} & HX_1 + HY_2 + HZ_3 \\ &= AX_1 + BY_2 + CZ_3 + HA + HB + HC \\ &= r_1 + r_2 + r_3 + 2(k_1 + k_2 + k_3) \\ &= 4R + r + 2r + 2R \\ &= 6R + 3r. \end{aligned}$$

(88)

$$\begin{array}{llll} X_0D = AI, & X_1D_1 = AI_1, & X_2D_2 = AI_2, & X_3D_3 = AI_3 \\ Y_0E = BI, & Y_1E_1 = BI_1, & Y_2E_2 = BI_2, & Y_3E_3 = BI_3 \\ Z_0F = CI, & Z_1F_1 = CI_1, & Z_2F_2 = CI_2, & Z_3F_3 = CI_3 \end{array}$$

Because AX_0 is equal and parallel to ID,
therefore $AIDX_0$ is a parallelogram;
therefore $X_0D = AI$.

(89) In the four pairs of triangles

$$\begin{array}{llll} X_1Y_2Z_3, & X_0Y_3Z_2, & X_3Y_0Z_1, & X_2Y_1Z_0 \\ I_1I_2I_3, & I\,I_3I_2, & I_3I\,I_1, & I_2I_1I \end{array}$$

consider the intersections of the sides.*

Y_2Z_3	intersects	I_1I_2,	I_1I_3	at	V_1, W_1
Z_3X_1	,,	I_2I_3,	I_2I_1	,,	W_2, U_2
X_1Y_2	,,	I_3I_1,	I_3I_2	,,	U_3, V_3
Y_3Z_2	,,	$I\,I_3$,	$I\,I_2$	,,	V', W'
Z_2X_0	,,	I_3I_2,	I_3I	,,	W_2, U''
X_0Y_3	,,	I_2I,	I_2I_3	,,	U''', V_3
Y_0Z_1	,,	I_3I,	I_3I_1	,,	V', W_1
Z_1X_3	,,	$I\,I_1$,	$I\,I_3$	,,	W'', U''
X_3Y_0	,,	I_1I_3,	I_1I	,,	U_3, V'''
Y_1Z_0	,,	I_2I_1,	I_2I	,,	V_1, W'
Z_0X_2	,,	I_1I,	I_1I_2	,,	W'', U_2
X_2Y_1	,,	$I\,I_2$,	$I\,I_1$	,,	U''', V'''.

It will be seen that several theorems are embedded in the preceding notation.

* The notation here is somewhat complicated, but it could not well be otherwise. I have made various attempts to simplify it, but with little success; what is gained in one respect is lost in another.

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(90) The sides of the four triangles

$$DEF, \quad D_1E_1F_1, \quad D_2E_2F_2, \quad D_3E_3F_3$$

contain each four other points of the diagram.

EF	contains	Y_3	Z_2	V'	W'
FD	,,	Z_1	X_3	W''	U''
DE	,,	X_2	Y_1	U'''	V'''
E_1F_1	,,	Y_2	Z_3	V_1	W_1
F_1D_1	,,	Z_0	X_2	W'''	U_2
D_1E_1	,,	X_3	Y_0	U_3	V'''
E_2F_2	,,	Y_1	Z_0	V_1	W'
F_2D_2	,,	Z_3	X_1	W_2	U_2
D_2E_2	,,	X_0	Y_3	U'''	V_3
E_3F_3	,,	Y_0	Z_1	V'	W_1
F_3D_3	,,	Z_2	X_0	W_2	U''
D_3E_3	,,	X_1	Y_2	U_1	V_1

(91) The twelve EF, FD, DE lines determine, by their intersections with the six lines of the orthic tetrastigm $II_1I_2I_3$, pairs of feet of the perpendiculars of the triangles

$$\begin{array}{lll} I_1BC, & AI_2C, & ABI_3 \\ IBC, & AI_3C, & ABI_2 \\ I_3BC, & AIC, & ABI_1 \\ I_2BC, & AI_1C, & ABI. \end{array}$$

The other twelve feet are the various D, E, F points.

It may be useful to remember that these four triads of triangles are similar to

$$I_1I_2I_3, \quad II_3I_2, \quad I_3II_1, \quad I_2I_1I.$$

The following proof of one of these properties may be sufficient:

Because $\qquad CD_1 = CE_1$

therefore triangles CD_1V_1, CE_1V_1 are congruent

and $\qquad \angle CD_1V_1 = \angle CE_1V_1 = \frac{1}{2}(B + C)$

$$= \angle I_3AB.$$

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Now triangle I_1BC is similar to $I_1I_2I_3$,
the sides I_1B, I_1C, BC being homologous to
,, ,, I_1I_2, I_1I_3, I_2I_3
and I_1D_1 being homologous to I_1A,
so that D_1 and A are homologous points;
therefore V_1 ,, B ,, ,, ,, ,
since $\angle CD_1V_1 = \angle I_3AB$.
But B is the foot of the perpendicular on I_1I_3 from I_2;
therefore V_1 ,, ,, ,, ,, ,, ,, ,, I_1C ,, B.

(92) The following quartets of points form orthic tetrastigms:

$$X_0X_1D_2D_3\ ;\quad Y_0Y_2E_3E_1\ ;\quad Z_0Z_3F_1F_2$$
$$X_2X_3D\,D_1\ ;\quad Y_3Y_1E\,E_2\ ;\quad Z_1Z_2F\,F_3\,.$$

(93) Through the mid point of

BC	pass	X_0I,	X_1I_1,	X_2I_2,	X_3I_3
CA	,,	Y_0I,	Y_1I_1,	Y_2I_2,	Y_3I_3
AB	,,	Z_0I,	Z_1I_1,	Z_2I_2,	Z_3I_3.

Take* for example X_1I_1.

Triangles $X_1D_2D_3$, I_1BC are similar and oppositely situated; therefore X_1I_1 passes through their centre of similitude.
But D_3C ,, ,, ,, ,, ,, ,, ;
therefore if X_1I_1 cut D_3C at A', the centre of similitude is A'.

Hence
$$\frac{A'B}{A'C} = \frac{A'D_2}{A'D_3}$$
$$= \frac{A'B + A'D_2}{A'C + A'D_3}$$
$$= \frac{BD_2}{CD_3} = \frac{s}{s};$$

therefore A' is the mid point of BC.

A shorter demonstration of this would be obtained if (95) were proved before (93).

* This method of proof is due to Professor Neuberg.

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For $X_1U_3I_1U_2$ is a parallelogram;

therefore X_1I_1 bisects U_3U_2;

therefore ,, ,, BC.

(94) The four centres of homology of the four pairs of triangles $X_1Y_2Z_3$, $I_1I_2I_3$, and so on, are the symmedian points of these pairs of triangles.

For I_1X_1 bisects BC,

and BC is antiparallel to I_2I_3 with respect to $\angle I_1$;

therefore I_1X_1 is a symmedian of $I_1I_2I_3$.

Since X_1I_1 bisects BC, it must also bisect D_2D_3.

Now D_2D_3 is antiparallel to Y_2Z_3 with respect to $\angle X_1$;

therefore X_1I_1 is a symmedian of $X_1Y_2Z_3$.

(95) All the U points are on a line parallel to BC

,, ,, V ,, ,, ,, ,, ,, ,, ,, CA

,, ,, W ,, ,, ,, ,, ,, ,, ,, AB.

(96)

$$U_2U_3 = V_3V_1 = W_1W_2 = s$$
$$U''U''' = V_3V' = W'W_2 = s_1$$
$$U''U_3 = V'''V' = W_1W'' = s_2$$
$$U_2U''' = V'''V_1 = W'W'' = s_3.$$

Because D_2U_2 is parallel to BU_3,

and CU_2 ,, ,, ,, D_3U_3,

and $CD_2 = s_1 = BD_3$;

therefore $D_2U_2 = BU_3$;

therefore $U_2U_3BD_2$ is a parallelogram;

therefore U_2U_3 is parallel and equal to BD_2, that is to s.

Similarly the other UU lines are parallel to BC;

therefore the U points are collinear.

The U points lie on the line B′C′.

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(97) The following sets of six points are concyclic

$$U_2 , \quad U_3 , \quad V_3 , \quad V_1 , \quad W_1 , \quad W_2$$
$$U'' , \quad U''' , \quad V_3 , \quad V' , \quad W' , \quad W_2$$
$$U'' , \quad U_3 , \quad V''' , \quad V' , \quad W_1 , \quad W''$$
$$U_2 , \quad U''' , \quad V''' , \quad V_1 , \quad W' , \quad W''.$$

Because D_2V_3, D_3W_2 are two of the perpendiculars of $X_1D_2D_3$;
therefore W_2V_3 is antiparallel to D_2D_3 with respect to $\angle X_1$;
therefore W_2V_3 is antiparallel to U_2U_3 ;
therefore W_2, V_3, U_3, U_2 are concyclic.
Similarly U_3, W_1, V_1, V_3 ,, ,,
and V_1, U_2, W_2, W_1 ,, ,,
Hence all the six points are concyclic.

The four circles are the Taylor circles of the orthic tetrastigm $II_1I_2I_3$.

(98) If the centres of these circles be denoted by

$$O_0 , \quad O_1 , \quad O_2 , \quad O_3$$

then these four points form an orthic tetrastigm.

They are the incentre and the excentres of the complementary triangle A′B′C′.

(99) The six II lines of the orthic tetrastigm $II_1I_2I_3$ are the radical axes of the circles O_0, O_1, O_2, O_3 taken in pairs ; and the four I points of the same tetrastigm are the radical centres of the circles O_0, O_1, O_2, O_3 taken in threes.

(100) The following are symmetrical trapeziums:

$$W_2V_3W_1V_1 ; \quad U_3W_1U_2W_2 ; \quad V_1U_2V_3U_3 ;$$
$$W_2V_3W'V' ; \quad U'''W'U''W_2 ; \quad V'U''V_3U''' ;$$
$$W''V'''W_1V' ; \quad U_3W_1U''W'' ; \quad V'U''V'''U_3 ;$$
$$W''V'''W'V_1 ; \quad U'''W'U_2W'' ; \quad V_1U_2V'''U'''.$$

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Professor Fuhrmann gives the following property, but his proof is too long for insertion here :

The axis of homology of the triangles

$$ABC \text{ and } X_1Y_2Z_3$$

is perpendicular to HI.

Of the last seventeen properties, (84), (85), (91) are given by W. H. Levy of Shalbourne in the *Lady's and Gentleman's Diary* for 1857, pp. 50-1, in his answer to a question proposed by him the previous year.

At the *Concours d'agrégation des sciences mathématiques* (Paris, 1873) the following question was proposed :

The points of contact of the excircles of a triangle ABC which are situated on the sides produced are joined, and a new triangle A'B'C' is formed. (1) *Find the angles of A'B'C'.* (2) *Prove that AA', BB', CC' are the altitudes of ABC.* (3) *Determine the centre and the radius of the circumcircle of A'B'C'.*

In the *Nouvelle Correspondance Mathématique*, I. 50-3 (1874), Professor Neuberg gives a geometrical solution of the question, in which (confining himself to triangle $X_1Y_2Z_3$) he proves (85), (86), (87), (91), (92), (93), (94) and one or two other properties. Professor Neuberg in the 6th edition of Casey's *Sequel to Euclid*, p. 278 (1892) and Professor Fuhrmann in his *Synthetische Beweise planimetrischer Sätze*, p. 89 (1890), give the first part of (97).

The seventeen properties were communicated to the Edinburgh Mathematical Society in 1889.

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