# COMPRESSIBLE CONVECTION

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## 1. INTRODUCTION

Stellar convection zones often extend over several pressure scale heights and convective velocities can be comparable to the local sound speed. Neither laboratory convection experiments nor analytic solution of the non-linear equations are feasible in such regimes. In order to gain insight into the details of stellar convection we are obliged to use numerical simulations. At the present time, even this approach cannot be applied to the parameter range typical of stellar interiors; however solutions can be obtained which extend over many scale heights and have non-negligible Mach numbers. Under these conditions it is necessary to employ the full compressible equations rather than the anelastic approximation (Gough [1]) or the Boussinesq approximation (Spiegel & Veronis [2] ).

## 2. THE PROBLEM

Rather than attempting to model a complete star, we will employ a simplified geometry. In this way we can facilitate the numerical calculation, while avoiding the complexities of treating the transition between the convective zone and the optically thin region.

As a standard problem, we consider a gas confined in a rectangular box with slippery walls. The upper and lower faces are maintained at fixed but different temperatures, $T_u$ and $T_l$. The side walls are thermally insulating. A constant gravitational field is imposed which has sufficient magnitude to produce a significant density variation with height.

The equation governing the problem are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 ,$$

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - g_i \rho = 0$$

and

$$\rho T \left( \frac{\partial S}{\partial t} + u_i \frac{\partial S}{\partial x_i} \right) - \tau_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left( K \frac{\partial T}{\partial x_i} \right) = 0 ,$$

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where $$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell}\right),$$

S is the specific entropy, $\eta$ is the coefficient of viscosity, K is the thermal conductivity and all the other symbols have their usual meaning.

Even if we prescribe the equation of state, the functional forms of the conductivity and the viscosity and the aspect ratios of the box, we still have five degrees of freedom in setting up the problem (see Graham [3]). In addition we can choose our initial velocity, density and temperature distribution.

Numerical solutions for the two-dimensional problem have been presented by Graham [3]. The most important parameters are found to be the Rayleigh number, R, the Prandtl number, $\sigma$ , and the layer depth parameter, Z, given by

$$R = \frac{(g/T_u)\, d^4 \left[ (T_\ell - T_u)/d - g/c_p \right]}{(K/\rho_u c_p)(\eta/\rho_u)} ,$$

$$\sigma = c_p \eta / K , \qquad Z = (T_\ell - T_u)/T_u .$$

It is sometimes useful to have solutions of the linear equations for the onset of convection. These equations have been treated by Spiegel [4], Gough et al [5] and by Graham and Moore [6]. A relative Rayleigh number, $\lambda$ , can be defined by scaling R by the critical Rayleigh number of the linear problem.

## 3. NUMERICAL METHODS

A variety of numerical methods have been used for compressible convection. The first solutions were obtained for two-dimensional motions using a modified Lax-Wendroff finite difference scheme. This method is described by Graham [3]. The scheme has been generalised to three-dimensional flows. An alternative to the finite difference method is the pseudo-spectral or collocation method. This has been successfully used for compressible convection equations by Graham (unpublished). Each dependent variable is approximated by a truncated Chebyshev series in two or three space dimensions. The series are substituted into the differential equations. The time derivatives of the coefficients of the series are determined by requiring that the differential equations be satisfied at selected collocation points. Because Chebyshev transforms can be calculated using fast Fourier techniques, the method is economical. Both the Chebyshev scheme and the Lax-Wendroff scheme suffer from numerical stability problems for low Prandtl numbers and large values of the Rayleigh number. Solutions have been obtained with $\lambda$ =100, $\sigma$ =0.1 and Z=10. Current work is directed at developing an alternating direction implicit finite difference scheme.

## 4. THE RESULTS

Because of the computational labour involved in obtaining three-dimensional solutions, most of the calculations are restricted to two dimensional flows. The calculations reported by Graham [3] relate to a perfect gas law and constant K and $\eta$ . A number of general results were found.

1. Two-dimensional solutions evolve to steady state flows. The time taken to reach a steady state increases with increasing horizontal box dimension and decreasing $\sigma$. This suggests that for more extreme configurations, there may be no steady solution.

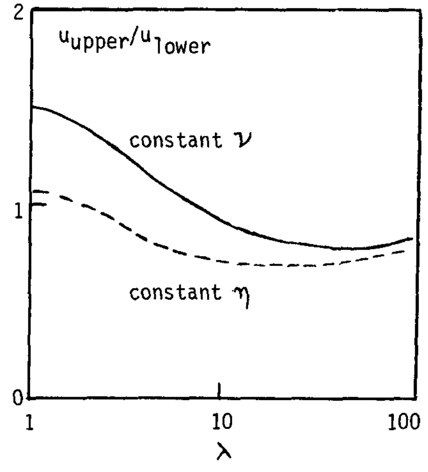


Figure 1a

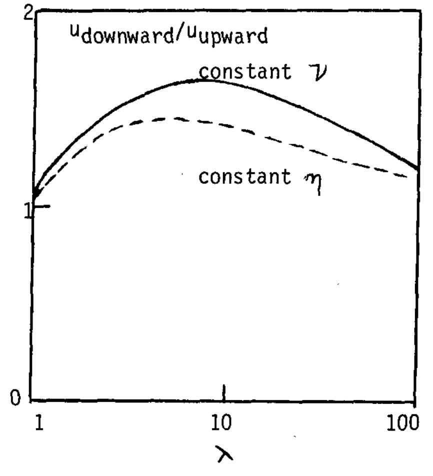


Figure 1b

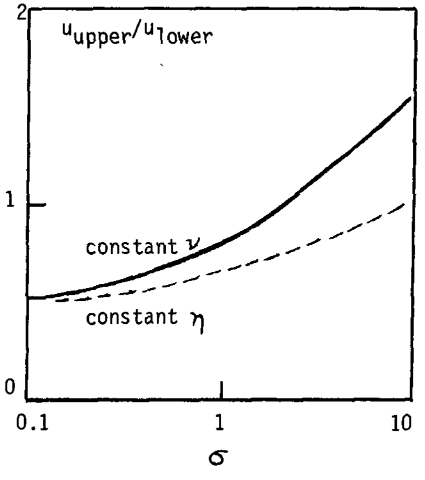


Figure 1c

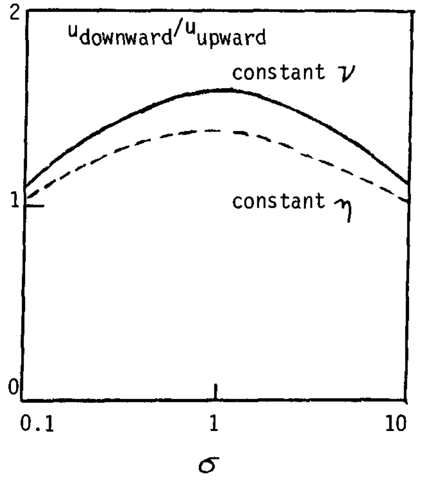


Figure 1d

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2. There is an asymmetry between upward and downward velocities, downward velocities usually being larger. Horizontal velocities are similar at the upper and lower surfaces, with the lower velocity often being slightly larger. This is a surprising result, particularly for large values of Z, because continuity arguments have been proposed to suggest that convective velocities are larger in low density regions.
3. When the horizontal box dimension is large enough to permit several convective rolls, the horizontal wavelength differs significantly from that which would maximise heat flux.
4. Convective cells extend over several pressure scale heights in the vertical direction. No cases were found where the flow breaks up into several rolls in the vertical.

Further calculations have been performed with a constant kinematic viscosity, $\nu$, rather than a constant dynamic viscosity, $\eta$. It had been conjectured that the increase of $\nu$ near the surface reduced the upper horizontal velocity. Figure 1a shows the ratio of upper to lower velocity as a function of $\lambda$ with Z=10. We see that it is only for small values of $\lambda$ that the upper velocity is enhanced. Figure 1b shows the corresponding behaviour of the ratio of downward to upward velocities. The variations with Prandtl number is shown in figures 1c and 1d. The general conclusion is that the solutions are insensitive to the form of the viscosity law in the cases of large R and small $\sigma$, which is the regime found in stellar convection zones.

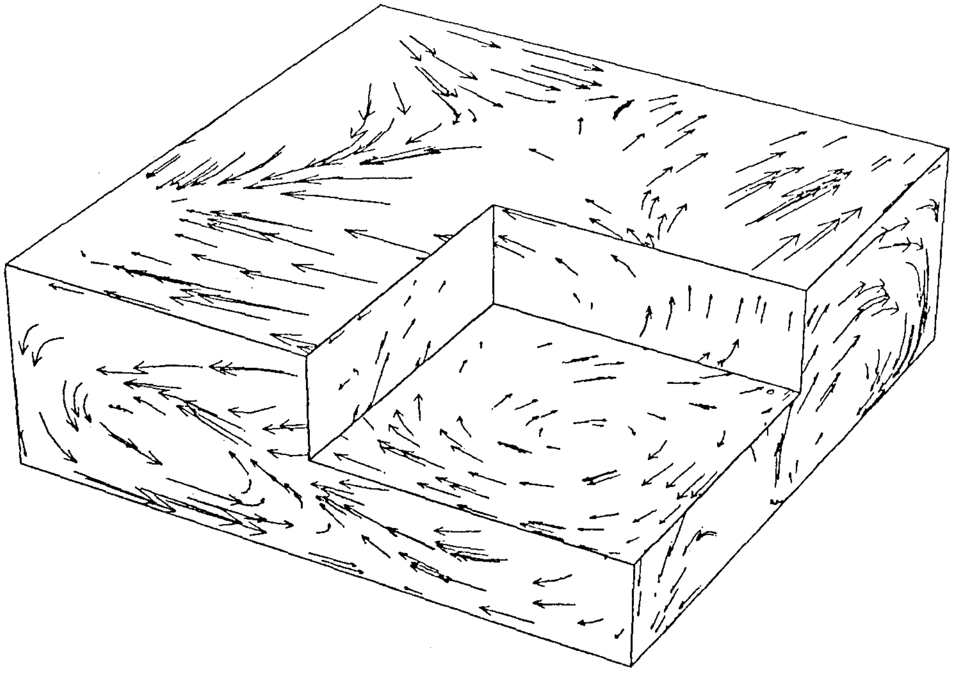

Figure 2

Relatively few three-dimensional calculations have been performed. If the horizontal box size is comparable to the vertical size, two-dimensional flow patterns are found. As the horizontal size is increased, the flow pattern becomes time dependent even for modest values of $\lambda$ . Figure 2 shows a velocity field for Z=1 and $\lambda \approx 10$. In this perspective picture, a rectangular bite has been removed from one corner of the box to reveal the interior. The arrows represent velocity components parallel to the faces. The arrows are distributed at random with a probability proportional to the density. The cut away portion shows that the fluid has significant vertical vorticity. Such regions are observed to be short lived, being dissipated and then reforming in a new position.

## 5. CONCLUSIONS

Numerical simulation of compressible convection provides a way of obtaining a detailed picture of stellar convection. At the present time, solutions are still far from the parameter range found in stellar interiors. However the solutions are well removed from the Boussinesq limit of laboratory convection experiments. It is to be hoped that future developments in the form of more efficient algorithms for computational fluid dynamics, turbulence theories for handling the fine scale features of the flow and increases in the available computing resources will all help in attempts to construct more realistic models of stellar convection zones.

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