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# Fuzzy machines in a category

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"Fuzzy theories" and "distributive laws" are used to define "fuzzy systems" in an arbitrary category. The resulting minimal realization theory provides new insights even in classical cases (so that, for non-deterministic sequential machines, the minimal realization problem is reformulated in terms of the structure of join-irreducibles in finite lattices). The definition of "fuzzy theory" is of independent interest and meshes well with philosophical aspects of fuzzy set theory.

## 1. Introduction

Whereas an ordinary sequential machine has dynamics (state-transition function)

$$Q \times X_0 \to Q ,$$

where $Q$ is the set of states and $X_0$ is the set of inputs; a *nondeterministic sequential machine* has dynamics

(1) $$\delta : Q \times X_0 \to 2^Q ,$$

where we interpret $\delta(q, x) \subset Q$ as the set of *possible* successors to $q$ when acted upon by input $x$ . Again, a *stochastic sequential machine* (of a restricted type) has dynamics

(2) $$Q \times X_0 \to QP ,$$

where $QP$ is the set of probability distributions on $Q$ ; while

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Schützenberger [17] studies what we call *semiring automata* with dynamics

(3) $$Q \times X_0 \to R^{(Q)} = \{f : Q \to R \mid \operatorname{supp}(f) \text{ is finite}\}$$

for some fixed semiring[1] $R$ .

In previous papers [1, 4], we saw that the formation of $Q \times X_0$ from $Q$ could be generalized by considering any category $K$ and any functor $X : K \to K$ ; we formed a category $\mathcal{D}yn(X)$ , with objects $K$-morphisms $\delta : QX \to Q$ , while a morphism $h : (\delta, Q) \to (\delta', Q')$ was a $K$-morphism $h : Q \to Q'$ preserving the dynamics

$$\begin{array}{ccc} QX & \xrightarrow{\delta} & Q \\ {\scriptstyle hX}\downarrow & & \downarrow{\scriptstyle h} \\ Q'X & \xrightarrow[\delta']{} & Q' \end{array}$$

(we call such an $h$ a *dynamorphism*).

We then saw that if the forgetful functor $\mathcal{D}yn(X) \to K$ has a left adjoint (we then call $X$ an *input process*), forming for each $K$-object $Q$ a *free dynamics* $Q\mu_0 : (QX^@)X \to QX^@$ with 'insertion of the generators' $Q\eta : Q \to QX^@$ , we could provide a reachability theory broad enough to encompass sequential machines, linear and group machines, and tree automata. The *reachability map*[2] of an $X$-dynamics $(Q, \delta)$ equipped with *initial state map* $\tau : I \to Q$ is the unique dynamorphic extension $r : IX^@ \to Q$ of $\tau$ :

(4) $$\begin{array}{ccc} I & \xrightarrow{I\eta} & IX^@ \\ & {\scriptstyle\tau}\searrow & \downarrow{\scriptstyle r} \\ & & Q \end{array} \qquad \begin{array}{ccc} IX^@X & \xrightarrow{I\mu_0} & IX^@ \\ {\scriptstyle rX}\downarrow & & \downarrow{\scriptstyle r} \\ QX & \xrightarrow[\delta]{} & Q \end{array}$$

On the other hand, if $\mathcal{D}yn(X) \to K$ has a right adjoint (we call $X$ an *output process*) forming for each $K$-object $Y$ a *cofree dynamics* $YL : (YX_@)X \to YX_@$ with 'evaluation' $Y\Lambda : YX_@ \to Y$ , we could provide an

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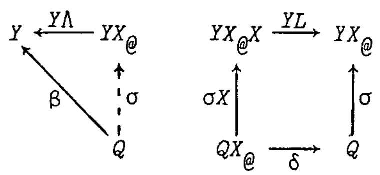
[1] A semiring is unitary and satisfies all the ring axioms save that it need not have additive inverses.

[2] For convenience, we use 'map' and 'morphism' as synonyms in this paper.

observability theory broad enough to emcompass sequential machines, and linear and group machines. The *observability map* of an $X$-dynamics $(Q, \delta)$ equipped with *output map* $\beta : Q \to Y$ is the unique dynamorphic coextension $\sigma : Q \to YX_@$ of $\beta$ :




In particular, we know that if $X$ is an *adjoint process* - that is, $X$ has a right adjoint while $K$ has countable products and coproducts - then it is both an input and output process. Since this includes the case where $K$ is a closed category with countable products and coproducts and $X = - \otimes X_0$ for some fixed object $X_0$ of $K$, we recapture the theory of Goguen [*10*] and Ehrig *et al.* [*8*].

In this paper, we undertake the analogous task for the right hand sides of (1), (2), and (3), asking for conditions on a functor $T$ which will enable $QT$ to serve as an "object of fuzzy states", so that we may give a general theory of nondeterministic automata as systems with dynamics

$$QX \to QT$$

where $X$ is an input or output process, and $T$ is an appropriate functor. In [*8*, Chapter 6], the theory of pseudoclosed categories is advanced as the setting for nondeterministic machines with process $- \otimes X_0$ ; and we shall see in Section 3 that our general theory does indeed include the Ehrig approach.

It would be aesthetically appealing to subsume dynamics $QX \to QT$ in the form $QX \to Q$ of the theory of [*1*, *4*]. To do this, we need a new category $K_T$ with the same objects as $K$, but with a $K_T$-morphism $f : A \rightharpoonup B$ (note the single-headed arrow) being in reality a $K$-morphism $f : A \to BT$, so that a fuzzy dynamics is given by a $K_T$-morphism $QX \rightharpoonup Q$. To be a category, $K_T$ must be equipped with *identities* and *composition*. Now, in each of our examples (1), (2), and (3), we have a map which allows to regard a 'pure' state as a 'fuzzy' state:

$Q \to 2^Q \quad : q \mapsto \{q\}$ ,

$Q \to QP \quad : q \mapsto \varepsilon_q$ where $\varepsilon_q(q') = 1$ if $q' = q$ and is otherwise $0$ ,

$Q \to R^{(Q)} : q \mapsto \varepsilon_q$ where $\varepsilon_q(q') = 1$ if $q' = q$ and is otherwise $0$ .

Each of these is of the form $Qe : Q \rightharpoondown Q$ , and we require, then, that the identity morphisms $Qe$ of $K_T$ generalize this role of letting us interpret pure states as particular examples of fuzzy states. Then for a collection of maps

$$K(A, BT) \times K(B, CT) \to K(A, CT) : \left(A \xrightarrow{\alpha} B,\ B \xrightarrow{\beta} C\right) \mapsto A \xrightarrow{\beta\circ\alpha} C$$

to be composition for $K_T$ it must satisfy the usual rules

$$(\gamma\circ\beta) \circ \alpha = \gamma \circ (\beta\circ\alpha) ,$$

$$\alpha \circ Ae = \alpha = Be \circ \alpha .$$

However, we require one more condition to make $e$ consistent with our interpretation - namely

$$\beta \circ f^{\Delta} = \beta\cdot f \quad \text{for} \quad f : A \to B ,\quad \beta : B \rightharpoondown C ,$$

where we adopt the notation

$$A \xrightarrow{f^{\Delta}} B = A \xrightarrow{f} B \xrightarrow{Be} BT \tag{6}$$

for $f : A \to B$ "viewed as a relation" $A \rightharpoondown B$ .

With this, we may give a formal definition and check it for $QT = 2^Q$ , leaving the other cases to the reader.

DEFINITION 7. Let $T$ be a mapping $\mathrm{Obj}(K) \to \mathrm{Obj}(K)$ , and write $\alpha : A \rightharpoondown B$ for $\alpha : A \to BT$ . Let, then, $\circ$ be an associative "composition of fuzzy relations"; and let $Ae : A \to AT$ for each $A$ in $K$ satisfy

$$\beta \circ f^{\Delta} = \beta\cdot f$$

as well as

$$Ce \circ \beta = \beta$$

for each $f : A \to B$ $\left(\text{where } f^{\Delta} = Be\cdot f\right)$ and each $\beta : B \rightharpoondown C$ . Then we call $T = (T, \circ, e)$ a *fuzzy theory* over $K$. (It is a consequence (Observation 3, Section 2) of these axioms that $T$ can be made into a functor $K \to K$.)

The *Kleisli category* $K_T$ of $T$ has the same objects as $K$, "relations" $A \rightharpoondown B$ as morphisms, $\circ$ for composition, and the $Ae : A \rightharpoondown A$ as identities. That $K_T$ is a category follows from the axioms since

$$\alpha \circ Ae = \alpha \circ (\mathrm{id}_A)^\Delta = \alpha \cdot \mathrm{id}_A = \alpha .$$

(Note: We will use "id" exclusively for $K$ identities since $K_T$ identities are easily denoted with $e$.)

The concept of "monad" or "triple" is well-established in category theory ([13, Chapter V; 2, Chapter 10]) and is entirely coextensive with our notion of fuzzy theory (as is proved in [14, 1.3.18]). If $S$ is the monad corresponding to the fuzzy theory $T$, $K_T$ is the well-known "Kleisli category of $S$" first defined by Kleisli [12]. While a knowledge of monad theory is certainly relevant, it is in no way essential to comprehension of this paper.

We further develop the concept of a fuzzy theory in Section 2. In the remainder of the present section, we develop the theory of nondeterministic sequential machines in such a way as to motivate the general theory of the subsequent sections.

EXAMPLE 8. The theory of nondeterministic sequential machines corresponds to $T = 2^{(-)}$ with

$$Qe : q \mapsto \{q\} ,$$

while for $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, we have $\beta \circ \alpha : A \to 2^C$ defined by

(9) $$\beta \circ \alpha(a) = \cup\{\beta(b) \mid b \in \alpha(a)\} .$$

The reader may check the category axioms. Here we verify $\beta \circ f^\Delta = \beta \cdot f$:

$$\begin{aligned} \beta \circ (Be \cdot f)(a) &= \{\beta(b) \mid b \in Be \cdot f(a)\} \\ &= \beta(f(a)) \quad \text{since } Be \cdot f(a) = \{f(a)\} \\ &= \beta \cdot f(a) . \end{aligned}$$

10. Consider a nondeterministic sequential machine with initial

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"state" $\tau$ in $2^Q$, with dynamics $\delta : Q \times X_0 \to 2^Q$, and with output map $\beta : Q \to \{0, 1\}$. It is usual [15] to simulate such a nondeterministic sequential machine by a *deterministic* sequential machine with state-space $2^Q$, and with

initial state map $\tau : I \to 2^Q$, for some one-element set $I$,

dynamics $\overline{\delta} : 2^Q \times X_0 \to 2^Q$, $(p, x) \mapsto \bigcup_{q \in p} \delta(q, x)$,

and

output map $\beta^{\#} : 2^Q \to \{0, 1\}$, where $\beta^{\#}(p) = 1 \iff \beta(q) = 1$ for some $q \in p$.

We can express this in a more algebraic way:

$$\begin{aligned}\overline{\delta}(p, x) &= \bigcup\{\delta(q, x) \mid q \in p\} \\ &= \bigcup\{\delta(r) \mid r = (q, x) \text{ for some } q \in p\} \\ &= \delta \circ Q\lambda(p, x),\end{aligned}$$

where we recall the form of $\circ$ from (9), and define $Q\lambda : 2^Q \times X_0 \to Q \times X_0$ by

$$Q\lambda : 2^Q \times X_0 \to 2^{Q \times X_0}, \quad (p, x) \mapsto \{(q, x) \mid q \in p\}.$$

11. Thus the passage from the fuzzy $\delta$ to the corresponding deterministic $\overline{\delta}$ may be expressed both in terms of the fuzzy theory composition and the map $Q\lambda$. In Section 3, we shall establish the properties $\lambda$ must satisfy to play this role for a given process $X$ and fuzzy theory $T$ - suitable $\lambda$'s are called *distributive laws*.

To analyze $\beta^{\#}$ more carefully, we first recall

DEFINITION 12. A *complete semilattice* is a partially ordered set in which every subset has a supremum. A *finite semilattice* is a complete semilattice whose underlying set is finite. A semilattice *homomorphism* is a supremum-preserving map.

We now observe that the output set $\{0, 1\}$ of our machine is a complete semilattice under the supremum operation max, while the state set $2^Q$ of our deterministic simulator is a complete semilattice under the

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supremum operation of union. We now note the commutativity of

$$\begin{array}{ccc} Q & \xrightarrow{Qe} & 2^Q \\ & \searrow^{\beta} & \downarrow{\scriptstyle \beta^\#} \\ & & \{0, 1\} \end{array}$$

In fact, it is easily seen that:

**13.** $\beta^\#$ is the unique homomorphic extension $(2^Q, \cup) \to (\{0, 1\}, \max)$ of $\beta$. In Section 4, we introduce *T-deciders* $(Q, \xi)$ as the generalization for a fuzzy theory $T$ of the complete semilattices associated with $T = 2^{(-)}$, whereas for $T$ as in (2), $(Q, \xi)$ is a 'generalized' convex set. $QT$ is itself always a decider via the canonical map $Qm : QTT \to QT$ defined by

$$Qm = \left(QT \xrightarrow{\mathrm{id}_{QT}} QT\right) \circ \left(QTT \xrightarrow{\mathrm{id}_{QTT}} QTT\right) . \tag{14}$$

For example, for $T$ as in (1), $(2^Q, \cup) = (2^Q, Qm)$ since $Qm : 2^{(2^Q)} \to 2^Q$ assigns to each family its union, as is routinely checked.

This special case of *free* $T$-deciders – $(QT, Qm)$ – (see Theorem 7, Section 4) has been *implicit* in the literature of nondeterministic and stochastic automata theory. It is the contribution of this paper to show, in Section 7, how more general $T$-deciders are used to formulate the fuzzy minimal realization problem.

Returning to our nondeterministic dynamics $\overline{\delta}$, we note on taking $\overline{Q} = 2^Q$ that it provides an example of:

DEFINITION 15. A *complete semilattice with operators indexed by* $X_0$ (we write *FSO* for a finite such structure) is a complete semilattice $(\overline{Q}, \xi)$ together with a function $\overline{\delta} : \overline{Q} \times X_0 \to \overline{Q}$ such that for each $x$ in $X_0$, $\overline{\delta}(-, x) : \overline{Q} \to \overline{Q}$ preserves suprema.

**16.** It is this notion of an object which supports both a $T$-decider structure (semilattice, in this case) and a dynamic structure *in a compatible way* which motivates the theory of *λ-algebras* to be developed in Section 5.

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17. The nondeterministic machine

$$\left(\tau : I \to 2^Q,\ \delta : Q\times X_0 \to 2^Q,\ \beta : Q \to \{0, 1\}\right)$$

will yield the notion of a *λ-machine* in Section 5; while the expanded machine $\left(\tau : I \to 2^Q,\ \overline{\delta} : \overline{Q}\times X_0 \to \overline{Q},\ \overline{\beta} : \overline{Q} \to \{0, 1\}\right)$ - in which $\overline{\delta}$ is a complete semilattice with operators indexed by $X_0$ (that is, a λ-algebra) while $\overline{\beta}$ is a homomorphism - will yield the notion of an *implicit λ-machine*. It is a major observation of this paper that fuzzy machine theory supports two distinct kinds of automaton (which coincide in the deterministic case).

18. In Section 7, we shall give the general theory of *minimal realization* of suitably defined response functions $f$. We shall find that each $f$ has a minimal *implicit* λ-machine which realizes it uniquely up to isomorphism, but that nonisomorphic λ-machines may be minimal and yet realize the same $f$. To illustrate this distinction, we shall give examples from, and an outline of, the theory of minimal realization for nondeterministic sequential machines.

## 2. Fuzzy theories

Much has been written about fuzzy set theory ([18], [9]; see also [11] and the extensive bibliography there). From this point of view, the *fuzzy theories* of Definition 7, Section 1, provide very general categories of fuzzy relations. Our heuristics for the fuzzy category concepts are as follows: $AT$ is the "cloud of fuzzy states over $A$". A morphism $\alpha : A \rightharpoonup B = \alpha : A \to BT$ is "a fuzzy relation from $A$ to $B$". $Ae$ is the "pure state" map.

Now fix a fuzzy theory $\mathbf{T} = (T, e, \circ)$ over $K$. We establish a few general properties, and then provide several examples of fuzzy theories. (For further properties and examples, see [14, 4.3].)

CONTRACTION PRINCIPLE 1. *$\mathrm{id}_{AT} : AT \to AT$ may be regarded as the "relation" $\mathrm{id}_{AT} : AT \rightharpoonup A$.*

OBSERVATION 2. *Each morphism $\alpha : A \to BT$ admits a canonical extension $\alpha^{\#}$,*

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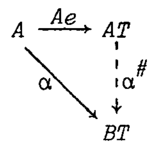


*defined by* $\alpha^{\#} = \alpha \circ \mathrm{id}_{AT}$ (*cf.* 13, Section 1), *such that* $\alpha^{\#} \cdot Ae = \alpha$ .

Proof. By the fuzzy theory axioms we have

$$\alpha^{\#} \cdot Ae = (\alpha \circ \mathrm{id}_{AT}) \circ (Ae)^{\Delta} = \alpha \circ \left[\mathrm{id}_{AT} \circ (Ae)^{\Delta}\right] = \alpha \circ Ae = \alpha . \qquad \square$$

OBSERVATION 3. *$T : K \to K$ becomes a functor when, for $f : A \to B$, we define $fT : AT \to BT$ by $fT = (f^{\Delta})^{\#}$ ; that is,*
$fT = f^{\Delta} \circ \mathrm{id}_{AT} = (Be \cdot f) \circ \mathrm{id}_{AT}$ .

Proof. (i) $\mathrm{id}_A T = Ae \circ \mathrm{id}_{AT} = \mathrm{id}_{AT}$ .

(ii) Given $g : B \to C$ ,

$$(g \cdot f)T = (g \cdot f)^{\Delta} \circ \mathrm{id}_{AT} = (g^{\Delta} \cdot f) \circ \mathrm{id}_{AT} = g^{\Delta} \circ f^{\Delta} \circ \mathrm{id}_{AT} = g^{\Delta} \circ fT =$$
$$= g^{\Delta} \circ (\mathrm{id}_{BT} \cdot fT) = g^{\Delta} \circ \mathrm{id}_{BT} \circ (fT)^{\Delta} = gT \circ (fT)^{\Delta} = gT \cdot fT . \qquad \square$$

OBSERVATION 4. *Given $\alpha : A \rightharpoondown B$ , $\beta : B \rightharpoondown C$ , $\beta \circ \alpha = \beta^{\#} \cdot \alpha$ . In particular, for $\alpha : A \rightharpoondown B$ and $f : B \to C$ , $f^{\Delta} \circ \alpha = fT \cdot \alpha$ .*

Proof. $\beta \circ \alpha = \beta \circ \mathrm{id}_{BT} \circ \alpha^{\Delta} = \beta^{\#} \circ \alpha^{\Delta} = \beta^{\#} \cdot \alpha$ . $\square$

With these sample properties, we now turn to a number of examples, in each of which we let $K$ be the category *Set* of sets and functions. We first summarize our observations in Section 1 on the 'classic' case:

EXAMPLE 5. Let $AT = 2^{A}$ so that $\alpha : A \rightharpoondown B$ is the familiar concept of *relation* via $a \,\alpha\, b \Leftrightarrow b \in \alpha(a)$ . The singleton map $Ae$ , $a \mapsto \{a\}$ , represents the equality relation. $\beta \circ \alpha$ is the usual composition,

$$(\beta \circ \alpha)(a) = \{c \in C \mid \text{there exists } b \text{ in } B \text{ with } b \in \alpha(a) \text{ and } c \in \beta(b)\} .$$

$K_T$ is then the usual category of sets and relations.

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$\mathrm{id}_{AT}$ , as in Contraction Principle 1, is the elementhood relation:

$S\ \mathrm{id}_{AT}\ a \Leftrightarrow a \in S$ .

$\alpha^{\#}(S) = \cup\{\alpha(s) \mid s \in S\}$ .

$fT : AT \to BT$ is the direct image map, $S \mapsto \{f(a) \mid a \in S\}$ .

$Am : ATT \to AT$ (as in 13, Section 1) is the union map.

$\mathcal{S} \mapsto \cup\mathcal{S} = \{a \in A \mid a \in S \text{ for some } S \in \mathcal{S}\}$ .

The same constructions work if $AT$ is restricted to "finite subsets", "non-empty subsets", or "finite, non-empty subsets".

EXAMPLE 6. Let $AT = AP$ as in (2), Section 1. $\alpha : A \to B$ corresponds to an $A$-by-$B$ column-stochastic matrix (we think of $A$ as indexing columns and $B$ as indexing rows). $(Ae)(a)$ is the stochastic column with entry 1 in its $a$th place. $\beta \circ \alpha$ is the usual matrix product.

EXAMPLE 7. Let $AT = R^{(A)}$ as in (3), Section 1. Then $\alpha : A \to B$ is a column-finite matrix of entries from $R$ and a fuzzy theory is constructed in the same way as in Example 6. The "finite subsets" version of Example 5 is recovered by taking $R$ to be a two-element semiring (not a ring!) with the usual mod 2 addition but with conjunction as multiplication.

EXAMPLE 8 ([9, Section 6]). Let $L$ be a complete lattice which is *completely distributive* in the sense that for each $x$ in $L$ , $x \vee (-)$ preserves infima and $x \wedge (-)$ preserves suprema. Set $AT = L^A$ . $Ae$ is the "crisp singleton map"; that is, $(Ae)(a)$ maps $a'$ to the greatest element of $L$ when $a' = a$ and to the least element otherwise. $\alpha : A \to B$ is Goguen's $L$-fuzzy relation. $\beta \circ \alpha$ is defined by

$$[(\beta\circ\alpha)(a)](c) = \vee\{[\beta(b)](c) \wedge [\alpha(a)](b) \mid b \in B\}$$

and the proof that $\circ$ is associative requires complete distributivity. Example 5 (with all subsets) is recovered by taking $L$ to be the two-element lattice. With $L$ the unit interval, $K_T$ is Zadeh's category of fuzzy relations [18].

EXAMPLE 9. Let $M$ be an arbitrary monoid "of credibility values",

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and take $AT = A \times M$ . If $\alpha : A \rightharpoonup B$ , "$\alpha(a) = b$ with credibility $x$ " if $\alpha(a) = (b, x)$ . $(Ae)(a) = (a, 1)$ . If $\alpha(a) = (b, x)$ and $\beta(b) = (c, y)$ then $(\beta \circ \alpha)(a) = (c, xy)$ .

EXAMPLE 10. Let $AT = A^*$ , the free monoid of all strings in $A$ . If $\alpha : A \rightharpoonup B$ , $\alpha(a) = b_1 \ldots b_n$ expresses a "voting preference for choice $a$ ". The empty word represents "abstention". $(Ae)(a) = a$ , the string of length 1 . If $\alpha(a) = b_1 \ldots b_n$ , $(\beta \circ \alpha)(a) = \beta(b_1) \ldots \beta(b_n)$ .

## 3. Distributive laws

We motivated the construction of the Kleisli category $K_T$ of Definition 7, Section 1, by suggesting that nondeterministic dynamics $QX \to QT$ in the category $K$ be viewed as normal dynamics

$$QX \rightharpoonup Q$$

in $K_T$ . Now while the functor $X : K \to K$ gives us an object function $Q \mapsto QX$ in $K_T$ , it does *not* give us a map on morphisms in $K_T$ , since $(A \to BT)X$ is not of the form $AX \rightharpoonup BX = AX \to BXT$ . Our task, then, is to find a "lifting" $\overline{X}$ of $X$:

$$A \xrightarrow{g} B \longmapsto AX \xrightarrow{g\overline{X}} BX .$$

OBSERVATION 1. *$K$ may be viewed as a subcategory of $K_T$ via "a morphism is a relation"; that is,* $f \mapsto f^\Delta$ .

Proof. We have $(\mathrm{id}_A)^\Delta = Ae$ and, given $f : A \to B$ and $g : B \to C$ , we further have $(g \cdot f)^\Delta = g^\Delta \cdot f = g^\Delta \circ f^\Delta$ . □

DEFINITION 2. By a *lifting* of a functor $X : K \to K$ to $K_T$ we mean a functor $\overline{X} : K_T \to K_T$ such that

$$\begin{array}{ccc} K_T & \overset{\overline{X}}{\dashrightarrow} & K_T \\ {\scriptstyle (\ )^\Delta}\uparrow & & \uparrow{\scriptstyle (\ )^\Delta} \\ K & \underset{X}{\longrightarrow} & K \end{array}$$

is a commutative square of functors.

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Note now that if $\overline{X}$ is such a lifting, we have

$$\begin{aligned}\alpha\overline{X} &= (\mathrm{id}_{BT}\cdot\alpha)\overline{X}\\ &= \left(\mathrm{id}_{BT}\circ\alpha^{\Delta}\right)\overline{X}\\ &= \mathrm{id}_{BT}\overline{X}\circ\alpha^{\Delta}\overline{X} \quad \text{since } \overline{X} \text{ is a functor}\\ &= \mathrm{id}_{BT}\overline{X}\circ(\alpha X)^{\Delta} \quad \text{by Definition 2}\\ &= \mathrm{id}_{BT}\overline{X}\cdot\alpha X\ .\end{aligned}$$

Summarizing, then:

**LEMMA 3.** *If* $X$ *is as in Definition* 2, *then for all* $\alpha : A \to B$ , $\alpha\overline{X} : AX \to BX$ *is given by*

$$\alpha\overline{X} = AX \xrightarrow{\alpha X} BTX \xrightarrow{B\lambda} BXT\ ,$$

*where*

$$B\lambda : BTX \to BX = \mathrm{id}_{BT}\overline{X}\ . \qquad \Box$$

NOTE. $B(TX)$ means $(BT)X$ , not $(BX)T$ , in accordance with our notation for the action of functors.

The lemma raises the question: "Under what conditions on $\lambda$ does $\alpha\overline{X} = B\lambda\cdot\alpha X$ define a lifting of $X$ ?" The appropriate definition and proposition are:

**DEFINITION 4.** A *distributive law* of $X$ over $T$ is an assignment to each object $A$ of $K$ of a morphism $A\lambda : ATX \to AXT$ such that the following two diagrams commute for all $A$ and $\alpha : A \to B$ .

$$\begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT\\ & {\scriptstyle AeX}\nwarrow \quad \nearrow{\scriptstyle AXe} & \\ & AX & \end{array} \qquad \begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT\\ {\scriptstyle \alpha^{\#}X}\downarrow & & \downarrow{\scriptstyle (B\lambda\cdot\alpha X)^{\#}}\\ BTX & \xrightarrow[B\lambda]{} & BXT \end{array}$$

This definition can be shown to be coextensive with the distributive laws between monads of [6] – where the terminology is motivated by the distributive law of ring theory.

**PROPOSITION 5.** *The correspondence, Lemma* 3, *establishes a bijection between liftings* $\overline{X}$ *as in Definition* 2 *and distributive laws as in*

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*Definition* 4.

Proof. If $\overline{X}$ is a lifting and $\lambda$ is defined by $B\lambda = \mathrm{id}_{BT}\overline{X}$, then

$$A\lambda \cdot AeX = \mathrm{id}_{AT}\overline{X} \cdot AeX = \mathrm{id}_{AT}\overline{X} \circ (AeX)^{\Delta} = \mathrm{id}_{AT}\overline{X} \circ (Ae)^{\Delta}\overline{X}$$

$$= \left(\mathrm{id}_{AT} \circ (Ae)^{\Delta}\right)\overline{X} = (Ae)\overline{X} = AXe$$

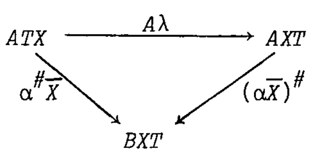
since $\overline{X}$ preserves identities. Moreover,

$(B\lambda \cdot \alpha X)^{\#} \cdot A\lambda = \left(\alpha\overline{X} \circ \mathrm{id}_{AXT}\right) \cdot \left(\mathrm{id}_{AT}\overline{X}\right)$ by the argument preceding Lemma 3 and Observation 2, Section 2

$$= \alpha\overline{X} \circ \mathrm{id}_{AXT} \circ \left(\mathrm{id}_{AT}\overline{X}\right)^{\Delta} = \alpha\overline{X} \circ \left(\mathrm{id}_{AT}\overline{X}\right) = \left(\alpha \circ \mathrm{id}_{AT}\right)\overline{X}$$

$$= \alpha^{\#}\overline{X} = B\lambda \cdot \alpha^{\#}X .$$

Now let $\lambda$ be a distributive law and define $\overline{X}$ by $\alpha\overline{X} = B\lambda \cdot \alpha X$. That $Ae\overline{X} = A\lambda \cdot AeX = AXe$ is immediate. The second diagram in Definition 4 reduces to




Since $\beta \circ \alpha = \beta^{\#} \cdot \alpha$ by Observation 2, Section 2, we have

$$(\beta \circ \alpha)\overline{X} = C\lambda \cdot \left(\beta^{\#} \cdot \alpha\right)X = C\lambda \cdot \beta^{\#}X \cdot \alpha X = \beta^{\#}\overline{X} \cdot \alpha X = (\beta\overline{X})^{\#} \cdot A\lambda \cdot \alpha X$$

$$= (\beta\overline{X})^{\#} \cdot (\alpha\overline{X}) = (\beta\overline{X}) \circ (\alpha\overline{X}) .$$

Thus $\overline{X}$ is indeed a functor. For $\overline{X}$ thus defined from $\lambda$ we have

$$\left(\mathrm{id}_{AT}\right)\overline{X} = A\lambda \cdot \left(\mathrm{id}_{AT}\right)X = A\lambda ,$$

which does indeed recover $\lambda$.

To see that $\overline{X}$ is a lifting of $X$, we note that

$$\alpha^{\Delta}\overline{X} = BT\lambda \cdot \alpha^{\Delta}X = BT\lambda \cdot BTeX \cdot \alpha X = BTXe \cdot \alpha X = (\alpha X)^{\Delta} . \qquad \square$$

In the future we shall write $X_{\lambda}$ (rather than $\overline{X}$) for the lifting corresponding to $\lambda$.

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The following proposition is sensibly stated now although its proof relies on results to be independently established below.

PROPOSITION 6. *Let* $T$ *be a fuzzy theory over* **Set** *and let* $X = - \times X_0 : \mathbf{Set} \to \mathbf{Set}$ . *Then*

$$QT \times X_0 \xrightarrow{Q\lambda} (Q \times X_0)T$$

$$(p,\ x) \mapsto (\mathrm{in}_x T)(p)$$

(*where* $\mathrm{in}_x : Q \to Q \times X_0$ , $q \mapsto (q,\ x)$ ) *is a distributive law of* $X$ *over* $T$ .

Proof. The proof that $AXe = A\lambda \cdot AeX$ is clear since $e : \mathrm{id} \to T$ is a natural transformation (see (2), Section 6 below). To prove the second distributive law axiom it suffices to prove that (II) commutes for every $x$ in $X_0$ ;

$$\begin{array}{ccccc}
A & \xrightarrow{\quad Ae \quad} & AT & \xrightarrow{\quad \alpha^{\#} \quad} & BT \\
\Big\downarrow{\scriptstyle \mathrm{in}_x} & \mathrm{I} & \Big\downarrow{\scriptstyle \mathrm{in}_x T} & \mathrm{II} & \Big\downarrow{\scriptstyle \mathrm{in}_x T} \\
A \times X_0 & \xrightarrow[(A \times X_0)e]{} & (A \times X_0)T & \xrightarrow[(B\lambda \cdot (\alpha \times X_0))^{\#}]{} & (B \times X_0)T
\end{array} .$$

I always commutes ((2), Section 6). Using Observation 2, Section 2, and the definition of $\lambda$ , the outer rectangle commutes; that is, (II) commutes preceded by $Ae$ . It follows from Theorem 7, Section 4, that (II) commutes. □

The value of distributive laws for machine theory is given by the following:

PROPOSITION 7. *Let* $\lambda$ *be a distributive law and let* $\left(AX^{@},\ A\mu_0;\ A\eta\right)$ *be a free* $X$*-dynamics over* $A$ . *Then* $\left(AX^{@},\ (A\mu_0)^{\Delta};\ (A\eta)^{\Delta}\right)$ *is a free* $X_\lambda$*-dynamics over* $A$ . *Thus if* $X$ *is an input process, so is each* $X_\lambda$ .

Proof. Let $(Q,\ \delta : QX_\lambda \to Q)$ be an $X_\lambda$-dynamics and let $\alpha : A \to Q$ . Not surprisingly, we use the associated $X$-dynamics as in 10, Section 1 - recall the formula $\overline{\delta} = \delta \circ Q\lambda = \delta^{\#} \cdot Q\lambda : QTX \to QT$ . Specifically, for

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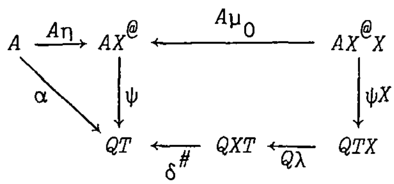
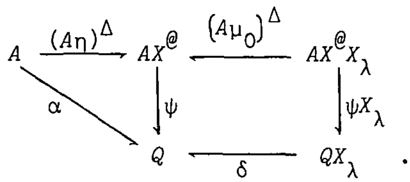
$\psi : AX^{@} \to Q$ compare

(8)




with

(9)




Since

$$\psi \circ (A\eta)^{\Delta} = \psi \cdot A\eta ,$$

$$\psi \circ (A\mu_0)^{\Delta} = \psi \cdot A\mu_0 ,$$

and

$$\delta \circ \psi X_{\lambda} = \delta \circ (Q\lambda \cdot \psi X) = \delta^{\#} \cdot Q\lambda \cdot \psi X ,$$

we have "(8) commutes, for unique $\psi$, in $K$" if and only if "(9) commutes, for unique $\psi$, in $K_T$". □

EXAMPLE 10. As a special case of Proposition 6, $\lambda$ as in 10, Section 1, is a distributive law.

EXAMPLE 11. Let $\Omega$ be the operator domain with one binary operation and let $AX = A \times A$ be the corresponding tree automaton input process in $\mathcal{Set}$ (see [1]). Let $T$ be as in Example 5, Section 2. Then

$$2^A \times 2^A \xrightarrow{A\lambda} 2^{A \times A}$$

$$(S_1, S_2) \mapsto S_1 \times S_2$$

is a distributive law. This example generalizes easily to arbitrary operator domains.

We can now sketch the approach to fuzzy machine theory of [8]. (The reader unfamiliar with closed and monoidal categories may consult [2].) Ehrig *et al.* work only in a closed category $K$ with countable products and coproducts, and use only the adjoint process $- \otimes X_0$ . Then, rather than construct a Kleisli category $K_T$ as we did in (2), Section 1, they consider any category $L$ which is *pseudoclosed* with respect to $K$ ; that is:

(1) $L$ has the same objects as the closed category $K$ and is structured with a pair of functors

$$K \underset{R}{\overset{(\ )^\Delta}{\rightleftarrows}} L ,$$

where $R$ is right adjoint to $(\ )^\Delta$ ;

(2) $L$ is a monoidal category and $(\ )^\Delta$ is a monoidal functor; that is

$$(f \otimes g)^\Delta = f^\Delta \otimes' g^\Delta .$$

The condition (1) is equivalent to our conditions defining a fuzzy theory $T$ . Given (1), define $KT = K^\Delta R$ and let $e$ and $\circ$ be induced by the identity and composition of $L$ ; $L$ may then be identified with $K_T$ . Conversely, given $T$ , set $B^\Delta R = BT$ ; the desired adjointness is

$$\frac{A^\Delta \to B}{A \to BT} .$$

The point of (2) is to ensure that $- \otimes X_0$ lifts to a functor on $L$ . This, then, corresponds to our distributive laws, in view of our Proposition 5. In short:

OBSERVATION 12. *The Ehrig theory* [8, Chapter 6] *is a special case of ours, applying when* $X : K \to K$ *has the special form* $- \otimes X_0$ *in a closed category.*

## 4. Deciders

For the balance of this paper, $T$ denotes a fuzzy theory in an

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arbitrary category $K$ .

Our task now is to define a $T$-decider as the generalization of the complete semilattices which, in 13, Section 1, we associated with the 'classical' fuzzy theory $T$ , with $T = 2^{(-)}$ , of Example 5, Section 2. Clearly, the supremum $\xi(\{q\})$ of a one-element subset of a complete semilattice $Q$ is simply $q$ - and we may rewrite this equality as $\xi \cdot Qe = \mathrm{id}_Q$ . More subtly, recall that in Example 5, Section 2, we had $\alpha^{\#}(S) = \cup\{\alpha(s) \mid s \in S\}$ . The crucial property of suprema that "$\xi(S_a) = \xi(\overline{S}_a)$ for each $a$ in $A$ implies that $\xi(\cup S_a) = \xi(\cup\overline{S}_a)$ " may then be re-expressed by stating that if $\alpha, \beta : A \to 2^Q$ satisfy $\xi\cdot\alpha = \xi\cdot\beta$ , then we have $\xi\cdot\alpha^{\#} = \xi\cdot\beta^{\#}$ .

This yields the general definition:

**DEFINITION 1.** A *T-decider* is a pair $(Q, \xi)$ with $\xi : QT \to Q$ a $K$-morphism subject to the conditions

(i) $\xi\cdot Qe = \mathrm{id}_Q$ ; and

(ii) whenever $\alpha, \beta : A \to QT$ are such that $\xi\cdot\alpha = \xi\cdot\beta$ , then $\xi\cdot\alpha^{\#} = \xi\cdot\beta^{\#}$ .

The heuristic meaning of the first axiom is clear. The second axiom asserts that the structure of $T$ imposes some 'deterministic' ways of building fuzzy states from other fuzzy states which must be respected by $\xi$ ; further intuition may be inferred from the following examples:

**EXAMPLE 2.** Let $T$ be as in Example 10, Section 2. If $M$ is a monoid, the map $\xi : M^* \to M$ which realizes formal multiplication is a decider and, conversely, every decider is a monoid with multiplication $(x, y) \mapsto \xi(xy)$ ; the two concepts are the same. Axiom (ii) may be formulated as "whenever $(w_1, \ldots, w_n)$, $(\overline{w}_1, \ldots, \overline{w}_n)$ are $n$-tuples of elements of $Q^*$ such that $\xi(w_i) = \xi(\overline{w}_i)$ , then $\xi(w_1 \ldots w_n) = \xi(\overline{w}_1 \ldots \overline{w}_n)$ ."

**EXAMPLE 3.** For $T$ as in Example 9, Section 2, (ii) asserts that "if $\xi(q, x) = \xi(\overline{q}, \overline{y})$ then $\xi(q, xz) = \xi(\overline{q}, \overline{y}z)$ for all $z$ ." A $T$-decider

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is an $M$-set: the more standard equivalent axioms being (i) and "$\xi(q, xy) = \xi(\xi(q, x), y)$ ."

EXAMPLE 4. Let $T$ be as in Example 6, Section 2. Every convex subset of a real linear space is a decider, where $\xi$ converts elements of $QT$ (thought of as formal convex combinations) into actual convex combinations. The arbitrary stochastic decider is not of this form since

$$\xi : \{x, y\}T \to \{x, y\} , \quad \lambda x + \mu y \longmapsto \begin{cases} x & \text{if } \lambda \neq 0 , \\ y & \text{if } \lambda = 0 , \end{cases}$$

satisfies the axioms. Axiom (ii) asserts that

$$\xi(\lambda_1 p_1 + \ldots + \lambda_n p_n) = \xi(\lambda_1 \overline{p}_1 + \ldots + \lambda_n \overline{p}_n)$$

for any $(\lambda_1, \ldots, \lambda_n)$ (all nonnegative and summing to $1$ ), so long as $\xi(p_i) = \xi(\overline{p}_i)$ .

In the language of monads, deciders are the well-known algebras over a monad. Deciders, then, are coextensive with universal algebra (see [2, Chapter 10], [13, Chapter 6], and [14]).

DEFINITION 5. If $(Q, \xi)$, $(R, \theta)$ are $T$-deciders, a morphism $f : Q \to R$ is a $T$-*homomorphism* $f : (Q, \xi) \to (R, \theta)$ just in case we have

$$\begin{array}{ccc} QT & \xrightarrow{\xi} & Q \\ {\scriptstyle fT}\downarrow & & \downarrow{\scriptstyle f} \\ RT & \xrightarrow[\theta]{} & R \end{array} \quad . \tag{6}$$

Since $T$ is a functor, $\mathrm{id}_Q : (Q, \xi) \to (Q, \xi)$ is a $T$-homomorphism and the composition of $T$-homomorphisms is again a $T$-homomorphism. Let $K^T$ denote the category of $T$-deciders and $T$-homomorphisms.

The following fundamental result establishes a canonical free decider over any object. Recall ((14), Section 1) that

$$Am = \left(AT \xrightarrow{\mathrm{id}_{AT}} A\right) \circ \left(ATT \xrightarrow{\mathrm{id}_{ATT}} AT\right)$$

(which equals $(\mathrm{id}_{AT})^{\#}$ in the sense of Observation, Section 2).

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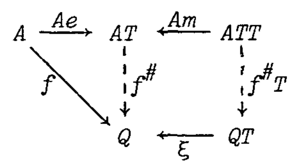
**THEOREM 7.** *$(AT, Am)$ is a decider. Moreover, given any decider $(Q, \xi)$ and morphism $f : A \to Q$ there exists a unique $T$-homomorphism $f^{\#} : (AT, Am) \to (Q, \xi)$ ,*




*such that $f^{\#} \cdot Ae = f$ . The formula for $f^{\#}$ is*

(8) $$f^{\#} = AT \xrightarrow{fT} QT \xrightarrow{\xi} Q .$$

The notation is not ambiguous since, given $(BT, Bm)$ and any $\alpha : A \to BT$ , $\alpha^{\#} = \alpha \circ \text{id}_{AT}$ , as in Observation 2, Section 2, is equal to $Bm \cdot \alpha T$ .

Proof. First, to check consistency of the $\#$ notation, we have for $\alpha : A \to BT$ ,

$$\alpha \circ \text{id}_{AT} = \text{id}_{BT} \circ \alpha^{\Delta} \circ \text{id}_{AT} = \text{id}_{BT} \circ \alpha T = (\text{id}_{BT})^{\#} \cdot \alpha T = Bm \cdot \alpha T .$$

We next show that $(QT, Qm)$ is a decider. For axiom (i), we have

$$\begin{aligned} Qm \cdot QTe &= (\text{id}_{QT})^{\#} \cdot QTe \\ &= \text{id}_{QT} \circ QTe = \text{id}_{QT} . \end{aligned}$$

For (ii), if $Qm \cdot \alpha = Qm \cdot \beta$ ,

$$\begin{aligned} Qm \cdot \alpha^{\#} &= (\text{id}_{QT})^{\#} \cdot \alpha^{\#} = \text{id}_{QT} \circ \alpha^{\#} = \text{id}_{QT} \circ \alpha \circ \text{id}_{AT} = (Qm \cdot \alpha) \circ \text{id}_{QT} \\ &= (Qm \cdot \beta) \circ \text{id}_{AT} = Qm \cdot \beta^{\#} . \end{aligned}$$

Next, we must show that $f^{\#}$ as in (8) is a $T$-homomorphism. Since $\xi \cdot \text{id}_{QT} = \xi = \text{id}_Q \cdot \xi = \xi \cdot \xi^{\Delta}$ by Definition 1 (i), we may use Definition 1 (ii) to deduce that

$$\xi \cdot (\text{id}_{QT})^{\#} = \xi \cdot (\xi^{\Delta})^{\#} = \xi \cdot \xi T .$$

Therefore,

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$$\begin{aligned} f^{\#}\cdot Am &= \xi\cdot fT\cdot Am = \xi\cdot(f^{\Delta}\circ Am) \quad \text{(recall Observation 4, Section 2)} \\ &= \xi\cdot\left[f^{\Delta}\circ \mathrm{id}_{AT}\circ \mathrm{id}_{ATT}\right] = \xi\cdot(fT\circ \mathrm{id}_{ATT}) = \xi\cdot\left[\mathrm{id}_{QT}\circ(fT)^{\Delta}\circ \mathrm{id}_{ATT}\right] \\ &= \xi\cdot\left[\mathrm{id}_{QT}\circ(fT)^{\Delta\#}\right] = \xi\cdot(\mathrm{id}_{QT})^{\#}\cdot(fT)^{\Delta\#} \\ &= \xi\cdot\xi T\cdot fTT = \xi\cdot f^{\#}T \ . \end{aligned}$$

Moreover, $f^{\#}\cdot Ae = \xi\cdot fT\cdot Ae = \xi\cdot(f^{\Delta})^{\#}\cdot Ae = \xi\cdot f^{\Delta} = \xi\cdot Qe\cdot f = f$ .

It only remains to verify uniqueness. Suppose $g : (AT, Am) \to (Q, \xi)$ is a $T$-homomorphism – $g\cdot Am = \xi\cdot gT$ – such that $g\cdot Ae = f$ . Then

$$\begin{aligned} g &= g\cdot(Ae\circ \mathrm{id}_{AT}) = g\cdot\left[\mathrm{id}_{AT}\circ(Ae)^{\Delta}\circ \mathrm{id}_{AT}\right] = g\cdot(\mathrm{id}_{AT}\circ AeT) \\ &= g\cdot(\mathrm{id}_{AT})^{\#}\cdot AeT = g\cdot Am\cdot AeT = \xi\cdot gT\cdot AeT \\ &= \xi\cdot(g\cdot Ae)T = \xi\cdot fT = f^{\#} \ . \qquad \square \end{aligned}$$

By Theorem 7, $K_T$ , the Kleisli category of "fuzzy relations" is isomorphic to the full subcategory of $K^T$ comprising all free deciders $(AT, Am)$ , for

$$(Ae)^{\#} = \mathrm{id}_{AT} \quad \text{and} \quad (\beta\circ\alpha)^{\#} = \beta^{\#}\cdot\alpha^{\#}$$

(since both $T$-homomorphisms are $\beta \circ \alpha$ when preceded by $Ae$ ). As will be clarified in the remainder of the paper, many automata-theoretic constructions operating on objects in $K_T$ cannot be defined in $K_T$ but can be defined in the larger – and much better behaved – universe $K^T$ .

## 5. Fuzzy machines

For the rest of the paper, fix a functor $X : K \to K$ and a distributive law $\lambda : TX \to XT$ of $X$ over $T$ . We now fulfill the promise of 16, Section 1.

DEFINITION 1. A *λ-algebra* is a triple $(Q, \delta, \xi)$ with $(Q, \delta)$ an $X$-dynamics and $(Q, \xi)$ a $T$-decider in such a way that $\xi$ is an $X$-dynamorphism $(QT, \delta T\cdot Q\lambda) \to (Q, \delta)$ ; that is, in such a way that we have

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$$\begin{array}{ccccc} QTX & \xrightarrow{Q\lambda} & QXT & \xrightarrow{\delta T} & QT \\ \downarrow{\scriptstyle \xi X} & & & & \downarrow{\scriptstyle \xi} \\ QX & & \xrightarrow[\delta]{} & & Q \end{array} .$$

By a *λ-homomorphism* $f : (Q, \delta, \xi) \to (Q', \delta', \xi')$ between λ-algebras we mean a simultaneous dynamorphism and $T$-homomorphism.

EXAMPLE 2. Set $AT = A$ , $Ae = \mathrm{id}_A$ , and $\beta \circ \alpha = \beta \cdot \alpha$ . Then $T$ is a fuzzy theory in $K$ , the *identity theory*. Clearly $K_T = K$ . Moreover, if $(Q, \xi)$ is a $T$-decider for this theory, then axiom (i) of Definition 1, Section 4, stipulates that $\xi = \mathrm{id}_Q$ . Thus we also have $K = K^T$ . Finally, $A\lambda = \mathrm{id}_{AX} : ATX \to AXT$ is a distributive law of $X$ over $T$ , and the category of λ-algebras is just $\mathcal{D}yn(X)$ .

If $K$ has an initial object $0$ (for example, the empty set in $\mathcal{S}et$ ) then for arbitrary $T$ and $X$ defined by $AX = 0$ , $fX = \mathrm{id}_0$ , the $\lambda$ defined by $A\lambda = \mathrm{id}_0$ is a distributive law of $X$ over $T$ and the category of λ-algebras may be identified with $K^T$ .

EXAMPLE 3. When $X = - \times X_0 : \mathcal{S}et \to \mathcal{S}et$ and $\lambda$ is as in 10, Section 1, $(\overline{Q}, \overline{\delta}, \xi)$ satisfies Definition 1 if and only if $\overline{\delta}_x = \overline{\delta}(-, x) : (\overline{Q}, \xi) \to (\overline{Q}, \xi)$ is a $T$-homomorphism for every $x$ in $X_0$ . Thus the semilattices with operators discussed in Definition 15, Section 1, are λ-algebras.

For the distributive law $Q\lambda : (p, x) \to \mathrm{in}_x T \cdot p$ of Proposition 6, Section 3, the diagram of Definition 1 says that

$$\begin{aligned} \overline{\delta}_x \xi(p) &= \xi \cdot \overline{\delta} T \cdot \mathrm{in}_x T(p) \\ &= \xi \cdot \left(\overline{\delta} \cdot \mathrm{in}_x\right) T(p) \\ &= \xi \cdot \overline{\delta}_x T(p) \ ; \end{aligned}$$

that is, each $\overline{\delta}_x$ is a $T$-homomorphism $(\overline{Q}, \xi) \to (\overline{Q}, \xi)$ .

LEMMA 4. *Let* $K = \mathcal{S}et$ . *Then the image of a* λ*-homomorphism is a* λ*-subalgebra. More precisely, let* $f : (Q, \delta, \xi) \to (Q', \delta', \xi')$ *be a*

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*λ-homomorphism with image* $I = \{f(q) \mid q \in Q\}$ *and with* $i : I \to Q'$ *the inclusion. Then if* $p : Q \to I$ *has* $p(q) = f(q)$ *, there exist unique* $\delta'' : IX \to I$ *,* $\xi'' : IT \to I$ *such that* $p : (X, \delta, \xi) \to (I, \delta'', \xi'')$ *and* $i : (I, \delta'', \xi'') \to (Q', \delta', \xi')$ *are λ-homomorphisms. Moreover, given any other diagram of λ-homomorphisms*

$$(Q, \delta, \xi) \xrightarrow{p_1} (I_1, \delta_1, \xi_1) \xrightarrow{i_1} (Q', \delta', \xi'), \qquad f : (Q, \delta, \xi) \to (Q', \delta', \xi')$$

*with* $p_1$ *surjective,* $i_1$ *injective, there exists a unique λ-isomorphism* $\Gamma$ *as shown below:*

$$\begin{array}{ccccc} & \overset{p}{\nearrow} & (I, \delta'', \xi'') & \overset{i}{\searrow} & \\ (Q, \delta, \xi) & & \downarrow \Gamma & & (Q', \delta', \xi') . \\ & \underset{p_1}{\searrow} & (I_1, \delta_1, \xi_1) & \underset{i_1}{\nearrow} & \end{array}$$

Proof. As $p$ is onto, there exists $d : I \to Q$ with $pd = \mathrm{id}_I$ . As $T$ is a functor, $pT\, dT = \mathrm{id}_{IT}$ ,

$$\begin{array}{ccccccc} & & QT & \xrightarrow{pT} & IT & \xrightarrow{iT} & Q'T \\ & & \downarrow \xi & & \downarrow \xi'' & & \downarrow \xi' \\ I & \xrightarrow[d]{} & Q & \xrightarrow[p]{} & I & \xrightarrow[i]{} & Q' \end{array}$$

and $pT$ is onto. We leave the remainder of the proof to the reader with the following three hints: $\xi''$ as above exists uniquely because $pT$ is onto, $i$ is injective and the perimeter commutes; the two decider axioms and the λ-law of Definition 1 follow from the principle that to prove $f, g : A \to I$ are equal it suffices to prove $i \cdot f = i \cdot g$ ; $\Gamma$ is defined by $\Gamma(p(q)) = p_1(q)$ . □

With the motivation provided by 17, Section 1, we may immediately give the definitions of λ-machine and implicit λ-machine:

**DEFINITION 5.** A *λ-machine* is a 7-tuple $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ where $(Y, \theta)$ is a $T$-decider and

$$I \xrightarrow{\tau} QT \ , \quad QX \xrightarrow{\delta} QT \ , \quad Q \xrightarrow{\beta} Y$$

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are $K$-morphisms (the *initial state*, *dynamics*, and *output map*, respectively).

While this definition is independent of which $\lambda$ relates $X$ and $T$, the way such machines "run" - as described in Sections 6 and 7 - is not. A similar definition was presented by Burroni in [7].

DEFINITION 6. An *implicit* $\lambda$-*machine* is an 8-tuple $\overline{M} = (\overline{Q}, \overline{\delta}, \overline{\xi}, I, \overline{\tau}, Y, \theta, \overline{\beta})$ where

$(\overline{Q}, \overline{\delta}, \overline{\xi})$ is a $\lambda$-algebra,

$\overline{\tau} : I \to \overline{Q}$ is a $K$-morphism, and

$\overline{\beta} : (\overline{Q}, \overline{\xi}) \to (Y, \theta)$ is a $T$-homomorphism.

Observe that both definitions collapse to the usual one in the deterministic case where $T$ is the identity theory as in Example 2. The difficulties encountered in fuzzy minimal realization theory are a direct consequence of the fact that the algebraic machinery producing unique reachable-observable realizations depends on Definition 6, whereas what we really want are "states" which require Definition 5. The proof that these two types of machine can simulate each other is given in Section 7.

Recall that in the classic case, 10-13, Section 1, we passed from a nondeterministic sequential machine (that is, $\lambda$-machine)

$$\left(\tau \in 2^Q;\ \delta : Q\times X_0 \to 2^Q;\ \beta : Q \to \{0, 1\}\right)$$

to a deterministic sequential machine which is an implicit $\lambda$-machine

$$\left(\tau : I \to 2^Q;\ \overline{\delta} = \delta\circ Q\lambda : 2^Q\times X_0 \to 2^Q;\ \beta^{\#} : 2^Q \to \{0, 1\}\right) .$$

We now show that $(QT,\ \delta^{\#}\cdot Q\lambda = \delta\circ Q\lambda,\ Qm)$ is a $\lambda$-algebra for general $X$, $T$, $\lambda$, and any $X$-dynamics $(Q, \lambda)$. We know (Theorem 7, Section 4) that we can always pass from an output map $\beta : Q \to Y$ with a $T$-decider structure $(Y, \theta)$ on $Y$ to the unique $T$-homomorphism $\beta^{\#} = \theta\cdot\beta T$ which extends $\beta$.

LEMMA 7. *For any* $QX \xrightarrow{\delta} QT$, $(QT,\ \delta^{\#}\cdot Q\lambda,\ Qm)$ *is a* $\lambda$-*algebra.*

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Proof.

$$\begin{array}{ccc}
QTTX & \xrightarrow{QmX \,=\, (\mathrm{id}_{QT})^{\#}X} & QTX \\
{\scriptstyle QT\lambda}\downarrow & & \\
QTXT & & \downarrow{\scriptstyle Q\lambda} \\
{\scriptstyle Q\lambda T}\downarrow & & \\
 & \nearrow{\scriptstyle QXm} & QXT \\
QXTT & & \\
{\scriptstyle \delta^{\#}T}\downarrow & & \downarrow{\scriptstyle \delta^{\#}} \\
QTT & \xrightarrow[Qm]{\qquad} & QT
\end{array} \quad .$$

The top piece is an instance of the square of 3.4 with $\alpha = \mathrm{id}_{QT}$ since $Qm = (\mathrm{id}_{QT})^{\#}$ and $(Q\lambda)^{\#} = QXm \cdot Q\lambda T$ .

The bottom piece just says that $\delta^{\#}$ is a $T$-homomorphism. □

Our next result generalizes the observation that any sequential machine dynamics $\delta : Q \times X_0 \to Q_0$ gives rise to a nondeterministic dynamics

$$2^Q \times X_0 \xrightarrow{Q\lambda} 2^{Q \times X_0} \xrightarrow{2^{\delta}} 2^Q : (p, x) \to \{\delta(q, x) \mid q \in p\} .$$

Moreover, this construction lifts dynamorphisms $\beta$ to $\lambda$-homomorphism $\beta^{\#}$ :

**THEOREM 8.** *For each $X$-dynamics $(Q, \delta)$ , we have that $(QT, \delta T \cdot Q\lambda, Qm)$ is a $\lambda$-algebra. Moreover, given any $\lambda$-algebra $(Y, \gamma, \theta)$ and dynamorphism $\beta : (Q, \delta) \to (Y, \gamma)$ , we have that $\beta^{\#} = \theta \cdot \beta T$ is a $\lambda$-homomorphism $(QT, \delta T \cdot Q\lambda, Qm) \to (Y, \gamma, \theta)$ .*

Proof. The first claim is just Lemma 7, with $\delta^{\Delta}$ replacing $\delta$ .

To establish the second statement, we next observe that $\lambda : TX \to XT$ is a natural transformation; that is,

$$(9) \qquad \begin{array}{ccc}
ATX & \xrightarrow{A\lambda} & AXT \\
{\scriptstyle fTX}\downarrow & & \downarrow{\scriptstyle fXT} \\
BTX & \xrightarrow[B\lambda]{} & BXT
\end{array}$$

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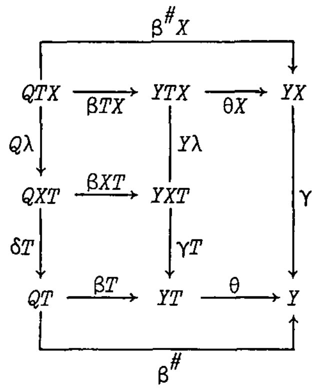
commutes for all $f : A \to B$ . This follows from Definition 4, Section 3, with $\alpha = f^{\Delta}$ since $\left(f^{\Delta}X_{\lambda}\right)^{\#} = (fX)^{\Delta\#} = fXT$ . Now use the diagram



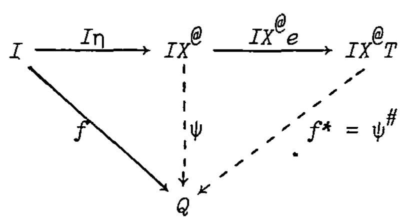

. □

## 6. Reachability and observability

To define reachability for $\lambda$-machines we need to use a free $\lambda$-dynamics; that is, a free $\lambda$-algebra. Given the free $X$-dynamics $\left(IX^{@}, I\mu_0\right)$ , we may apply the construction of Theorem 8, Section 5, to obtain the $\lambda$-algebra $\left(IX^{@}T, I\mu_0T\cdot IX^{@}\lambda, IX^{@}m\right)$ . We now show that it, too, is free.

THEOREM 1. *If* $X$ *is an input process, then the* $\lambda$-*algebra* $\left(IX^{@}T, I\mu_0T\cdot IX^{@}\lambda, IX^{@}m\right)$ *is the free* $\lambda$-*algebra over* $I$ *with "inclusion of the generators"* $IX^{@}e\cdot I\eta$ .

Proof. We must show that if $(Q, \delta, \xi)$ is any $\lambda$-algebra and $f : I \to Q$ is any $K$-morphism,




,

then there exists a unique $\lambda$-homomorphism $f^{*}$ with $f^{*}\cdot IX^{@}e\cdot I\eta = f$ . We

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will show that $f^* = \psi^\#$ where $\psi$ is the unique dynamorphism from $(IX^@, I\mu_0)$ to $(Q, \delta)$ such that $\psi \cdot I\eta = f$. $\psi^\#$ is a $\lambda$-homomorphism by Theorem 8, Section 5.

To prove that $f^*$ is unique it suffices to observe that $IX^@e$ is a dynamorphism, and this is seen at once from the diagram

$$\begin{array}{ccc}
IX^@X & \xrightarrow{IX^@eX} & IX^@TX \\
\Big\downarrow I\mu_0 & \searrow^{IX^@Xe} & \Big\downarrow IX^@\lambda \\
 & & IX^@XT \\
 & & \Big\downarrow I\mu_0 T \\
IX^@ & \xrightarrow[IX^@e]{} & IX^@T
\end{array}$$

as soon as we note that $e : \mathrm{id} \to T$ is a natural transformation; that is

(2)
$$\begin{array}{ccc}
A & \xrightarrow{Ae} & AT \\
f\Big\downarrow & & \Big\downarrow fT \\
B & \xrightarrow[Be]{} & BT
\end{array}$$

holds for all $f : A \to B$; for $fT \cdot Ae = (f^\Delta)^\# \cdot Ae = f^\Delta = Be \cdot f$. □

With this machinery, we may now define the reachability map for a $\lambda$-machine.

DEFINITION 3. The *reachability map* of the $\lambda$-machine $M$ of Definition 5, Section 5, is the $\lambda$-morphism

$$\tau^* : \left(IX^@T,\ I\mu_0 T \cdot IX^@\lambda,\ IX^@m\right) \to \left(QT,\ \delta^\# \cdot Q\lambda,\ Qm\right) .$$

Notice that the same map arises as the "deterministic" $X_\lambda$-reachability map in $K_T$ :

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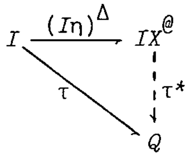


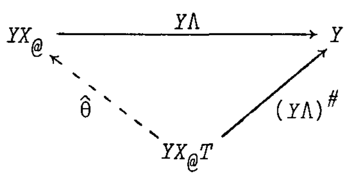

.

More generally, the *reachability map* of the implicit $\lambda$-machine $\overline{M}$ of Definition 6, Section 5, is the $\lambda$-homomorphism

$$\tau^* : \left(IX^@T,\ I\mu_0T\cdot IX^@\lambda,\ IX^@m\right) \to (\overline{Q},\ \overline{\delta},\ \xi) .$$

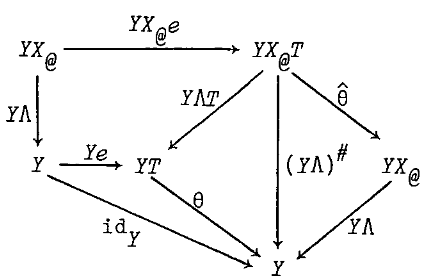
We now construct the machinery which lets us build the observability map.

THEOREM 4. *If $X$ is an output process then, for each decider $(Y, \theta)$, $\left(YX_@,\ YL,\ \hat{\theta}\right)$ is a $\lambda$-algebra where $\hat{\theta}$ is the unique dynamorphic*




*coextension of $(Y\Lambda)^\#$, (where $\left(YX_@T,\ YLT\cdot YX_@\lambda,\ YX_@m\right)$ is a $\lambda$-algebra as in Theorem 8, Section 5). Moreover, for each $\lambda$-algebra $(Q, \delta, \xi)$ and $T$-homomorphism $f : (Q, \xi) \to (Y, \theta)$, the unique dynamorphic coextension $\psi : (Q, \delta) \to \left(YX_@,\ YL\right)$ is a $\lambda$-homomorphism.*

*Proof.* We must first show that $\left(YX_@,\ \hat{\theta}\right)$ is a decider. Since we proved that $YX_@e$ is a dynamorphism in proving Theorem 1, the diagram

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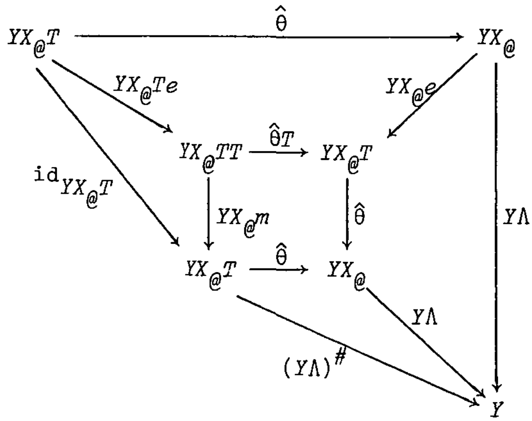
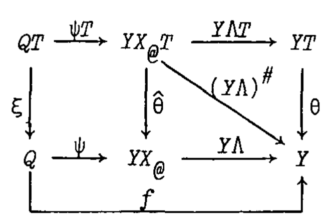
shows that the dynamorphisms $\hat{\theta}\cdot YX_{@}e$ and $\mathrm{id}_{YX_{@}}$ are equal followed by $Y\Lambda$, and hence equal. In a similar vein, from the diagram




we see first that the $T$-homomorphisms $(Y\Lambda)^{\#}\cdot YX_{@}m$ and $(Y\Lambda)^{\#}\cdot\hat{\theta}T$ are equal preceded by $YX_{@}Te$ and hence are equal. We next observe that $\hat{\theta}T$, $YX^{@}m$ are dynamorphisms; for use Theorem 8, Section 5, with $\beta = (\hat{\theta})^{\Delta}$, $\mathrm{id}_{YX_{@}T}$ respectively. Finally, the dynamorphisms $\hat{\theta}\cdot\hat{\theta}T$, $\hat{\theta}\cdot YX_{@}m$ are equal followed by $Y\Lambda$, and so are equal. Axiom 1, Section 5, is clear since $\hat{\theta}$ is a dynamorphism.

For the second statement, consult the diagram




using Theorem 8, Section 5, to see that $\psi T$ is a dynamorphism. □

DEFINITION 5. The *observability map* of the $\lambda$-machine $M$ of Definition 5, Section 5, is the $\lambda$-homomorphism

$$\sigma : (QT,\ \delta^{\#}\cdot Q\lambda,\ Qm) \to (YX_{@},\ YL,\ \hat{\theta})$$

which is the unique dynamorphic coextension of the $T$-homomorphism

$\beta^{\#} : (QT, Qm) \to (Y, \theta)$ .

More generally, the *observability map* of the implicit $\lambda$-machine $\overline{M}$ of Definition 6, Section 5, is the $\lambda$-homomorphism

$$\sigma : (\overline{Q}, \overline{\delta}, \xi) \to \left(YX_{@}, YL, \hat{\theta}\right)$$

which is the unique dynamorphic coextension of the $T$-homomorphism $\overline{\beta} : (\overline{Q}, \xi) \to (Y, \theta)$ .

The *total response map* of $M$ (or $\overline{M}$ ) is the $\lambda$-homomorphism

$$\left(IX^{@}T, I\mu_0 T\cdot IX^{@}\lambda, IX^{@}m\right) \xrightarrow{\sigma\cdot\tau^*} \left(YX_{@}, YL, \hat{\theta}\right) .$$

## 7. Minimal realization

In this section we will always assume $X$ is an input process. A morphism of the form $f : IX^{@} \to Y$ may be the response of either a $\lambda$-machine or an implicit $\lambda$-machine. Indeed, given either machine, we shall see (Theorem 10) that there exists a suitable machine of the other sort so that both have the same response. We shall see that while the familiar algebraic construction of the minimal deterministic realization generalizes easily to implicit $\lambda$-machines (Proposition 4), the associated $\lambda$-machine need not have the "minimal number of states".

Let us first dispatch the generalities. In the minimal realization theory for state-behavior machines [4], one factorizes $\mathcal{D}yn(X)$-morphisms

$$IF \to YG ,$$

where the left and right adjoints $F$ and $G$ of the forgetful functor $U : \mathcal{D}yn(X) \to K$ yield $IF = \left(IX^{@}, I\mu_0\right)$ and $YG = \left(YX_{@}, YL\right)$ .

In the present study, our total response maps (Definition 5, Section 6) have the form of $K^{\lambda}$-morphisms

$$IF \to YG ,$$

where it is the left and right adjoints $F$ and $G$ of the forgetful functor $U : K^{\lambda} \to K$ that yield $IF = \left(IX^{@}T, I\mu_0 T\cdot IX^{@}\lambda, IX^{@}m\right)$ (by Theorem 1, Section 6) and $GY = \left(YX_{@}, YL, \hat{\theta}\right)$ (by Theorem 4, Section 6).

In both cases, then, we have a minimal realization theory for A-morphisms

$$IF \to YG ,$$

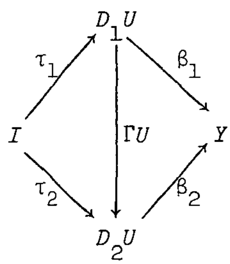
where $U : A \to B$ is a functor with left adjoint $F$ and right adjoint $G$. This then suggests a general axiomatic treatment of minimal realization (which is in fact implicit in the work of Bainbridge [5] - see [3, p. 58] for further references).

DEFINITION 1. Let $A$ be a category equipped with a functor $U : A \to B$. Then an $A$-*machine* is $(D, \tau, \beta)$, where

$\tau : I \to DU$ is a $B$-morphism called the *input map*; and

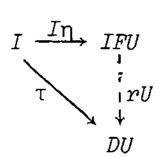
$\beta : DU \to Y$ is a $B$-morphism called the *output map*.

A *homomorphism* of A-machines, $(D_1, \tau_1, \beta_1) \to (D_2, \tau_2, \beta_2)$ with the same $I$ and $Y$ is an A-morphism $\Gamma : D_1 \to D_2$ such that




With this definition, such machines form a category $Mach(A, I, Y)$.

DEFINITION 2. If $U : A \to B$ has left adjoint $F$, we define the *reachability map* of $M = (D, \tau, \beta)$ to be the unique A-morphism $r : IF \to D$ satisfying




If $U$ has right adjoint $G$, we define the *observability map* of $M$ to be the unique A-morphism $\sigma : D \to YG$ satisfying

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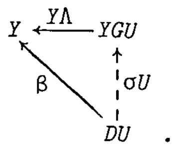



If $U$ has both left and right adjoints, we call $\phi = \sigma \cdot r : IF \to YG$ the *total response map* of $M$ . We call $M$ a *realization* of $\phi$ .

Given a factorization system[3] $(E, M)$ for $B$ , we say $M$ is *reachable* if $rU$ is in $E$ , *observable* if $\sigma U$ is in $M$ .

We immediately have the:

AXIOMATIC MINIMAL REALIZATION THEOREM 3. *Let* $U : A \to B$ *be a functor with left adjoint* $F$ *, and right adjoint* $G$ *. Let* $(E, M)$ *be a factorization system for* $B$ *with the property*

IF *Given any* A*-morphism* $f : A_1 \to A_2$ *, and an* $E - M$ *factorization*

$$A_1 U \xrightarrow{p} B \xrightarrow{\sigma} A_2 U$$

*of* $fU$ *, there exists a unique* A*-object* $A$ *and unique* A*-morphisms* $\overline{p} : A_1 \to A$ *and* $\overline{\sigma} : A \to A_2$ *such that* $\overline{p}U = p$ *and* $\overline{\sigma}U = \sigma$ *;*

THEN *for every* A*-morphism*

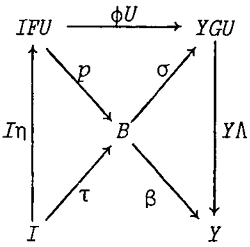
$$\phi : IF \to YG$$

*there exists an* A*-machine* $\overline{M}_\phi$ *which is a reachable and observable realization of* $\phi$ *; moreover* $\overline{M}_\phi$ *is unique up to isomorphism in* $\mathrm{Mach}(A, I, Y)$ .

Proof. We simply use the diagram

---

[3] At this level of axiomatic treatment, we do *not* need an image factorization system - all we require is that each map factorizes into an $E$ followed by an $M$ , and that this factorization is unique up to isomorphism.




to define $\tau$ and $\beta$ in terms of the $(E, M)$ factorization of $\phi U$ . Let then $\overline{p} : IF \to A$ and $\overline{\sigma} : A \to YG$ satisfy $\overline{p}U = p$ and $\overline{\sigma}U = \sigma$ . Then $\overline{p}$ and $\overline{\sigma}$ are the reachability and observability maps of $(A, \tau, \beta)$ , which is thus a reachable and observable realization of $\phi$ . Uniqueness up to isomorphism is immediate. □

Stripped to its essentials like this, minimal realization seems rather dull!

If we specialize all this to $K^\lambda \to K^T$ , with $K = \mathcal{S}et$ , $X = - \times X_0 : \mathcal{S}et \to \mathcal{S}et$ , $\lambda$ as in Proposition 6, Section 3, $(Y, \theta)$ a $T$-decider and $I$ a one-element set so that $IX^@ = X_0^*$ , we use Lemma 4, Section 5 to immediately obtain

**PROPOSITION 4.** *For every* $\lambda$*-homomorphism* $\phi : X_0^*T \to Y^{X_0^*}$ *, there exists an implicit* $\lambda$*-machine* $\overline{M}_\phi$ *which is a reachable and observable realization of* $\phi$ *; moreover,* $\overline{M}_\phi$ *is unique up to isomorphism.* □

We now turn to the relationship between $\lambda$-machines and implicit $\lambda$-machines. For this we only require that $X$ be an input process, and so study the response map $f_M : IX^@ \to Y$ rather than the total response map $\phi_M : IX^@T \to YX_@$ .

**DEFINITION 5.** The *response* $f_M$ of the $\lambda$-machine $M$ of Definition 5, Section 5 is the response of the deterministic $X$-machine

$$I \xrightarrow{\tau} QT \ , \quad QTX \xrightarrow{Q\lambda} QXT \xrightarrow{\delta^\#} QT \ , \quad QT \xrightarrow{\beta^\#} Y \ ;$$

that is, $f_M$ is the $K$-morphism

$$IX^{@} \xrightarrow{r} QT \xrightarrow{\beta^{\#}} Y ,$$

where $r$ is the reachability map of (4), Section 1; that is, the unique dynamorphic extension of $\tau$ .

Since $\tau^{*}\cdot IX^{@}e$ is a dynamorphism and $\left(\tau^{*}\cdot IX^{@}e\right)\cdot I\eta = \tau$ , $\tau^{*}\cdot IX^{@}e = r$ . We then have the following relationships (delete $Y\Lambda\cdot\sigma$ if $X$ is not output):

(6)
$$\begin{array}{ccccc} IX^{@}T & \xrightarrow{\tau^{*}} & QT & \xrightarrow{\sigma} & YX_{@} \\ {\scriptstyle IX^{@}e}\uparrow & \nearrow{\scriptstyle r} & & \searrow{\scriptstyle \beta^{\#}} & \downarrow{\scriptstyle Y\Lambda} \\ IX^{@} & & \xrightarrow[f_M]{} & & Y \end{array} .$$

Thus the total response map $\sigma\cdot\tau^{*}$ is the unique $T$-homomorphic extension of the dynamorphic coextension $\sigma\cdot r$ of $f_M = \beta^{\#}\cdot r$ .

DEFINITION 7. The *response* $f_{\overline{M}}$ of the implicit $\lambda$-machine of Definition 6, Section 5, is the composition

$$IX^{@} \xrightarrow{r} \overline{Q} \xrightarrow{\overline{\beta}} Y$$

where $r$ is the unique dynamorphic extension of $\tau$ .

It is clear from the definitions that if $M$ is a $\lambda$-machine, then its associated deterministic machine $\left(QT, \delta^{\#}\cdot Q\lambda, Qm, I, \tau, Y, \beta^{\#}\right)$ is an implicit $\lambda$-machine whose response is $f_M$ . We now set out to establish an appropriate converse:

DEFINITION 8. If $(\overline{Q}, \overline{\xi})$ is a $T$-decider, a *scoop* of $(\overline{Q}, \overline{\xi})$ is a triple $(Q, i, c)$ where

$$Q \xrightarrow{i} \overline{Q} \xrightarrow{c} QT$$

are such that $i^{\#}\cdot c = \mathrm{id}_{\overline{Q}}$ $\left(\text{where } i^{\#} = \overline{\xi}\cdot iT\right)$.

The terminology is due to Ehrig *et al.* [8, 10.3] and our definition is a variation of theirs. Notice, however, that $(\overline{Q}, \mathrm{id}, \overline{Q}e)$ is always a scoop.

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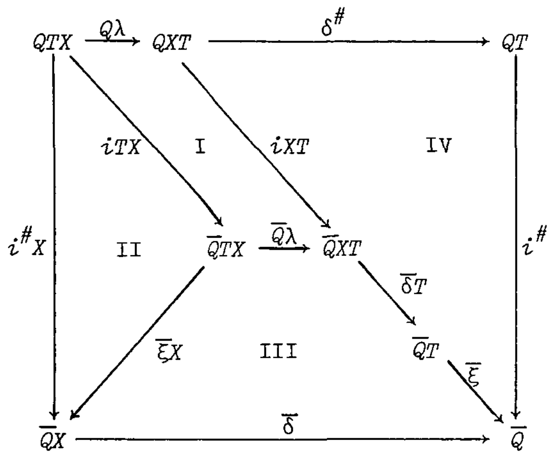
**PROPOSITION 9.** *Given an implicit* $\lambda$*-machine* $\overline{M}$ *then any scoop* $(Q, i, c)$ *of* $(\overline{Q}, \overline{\xi})$ *provides the state object* $Q$ *for a* $\lambda$*-machine* $M$ *with* $f_M = f_{\overline{M}}$ *;* $M$ *is defined by*

$$\delta = QX \xrightarrow{iX} \overline{Q}X \xrightarrow{\overline{\delta}} \overline{Q} \xrightarrow{c} QT ,$$

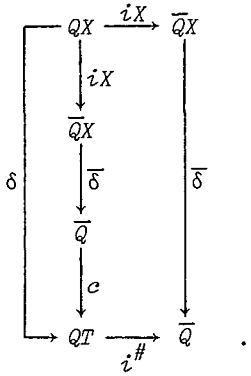
$$\tau = I \xrightarrow{\overline{\tau}} \overline{Q} \xrightarrow{c} QT ,$$

$$\beta = Q \xrightarrow{i} \overline{Q} \xrightarrow{\overline{\beta}} Y .$$

**Proof.** We prove that the diagram




commutes. I commutes since $\lambda$ is a natural transformation. II commutes since $i^{\#} = \overline{\xi} \cdot iT$ . III just says that $(\overline{Q}, \overline{\delta}, \overline{\xi})$ is a $\lambda$-algebra. Since $i^{\#} \cdot c = \mathrm{id}_{\overline{Q}}$ by the definition of 'scoop', we have




.

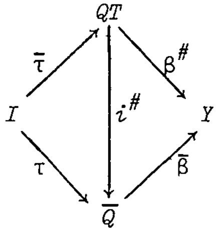
Noting that the two paths of IV yield $T$-homomorphisms, we recall Theorem 7, Section 4, to see that they are equal since, when preceded by $QXe$ , both equal $\overline{\delta}\cdot iX$ :

$$i^{\#}\cdot\delta^{\#}\cdot QXe = i^{\#}\cdot\delta = \overline{\delta}\cdot iX$$

by the last diagram while

$$\begin{aligned}\overline{\xi}\cdot\overline{\delta}T\cdot iXT\cdot QXe &= \overline{\xi}\cdot\left((\overline{\delta}\cdot iX)T^{\Delta}\circ QXe\right)\\ &= \overline{\xi}\cdot(\overline{\delta}\cdot iX)T^{\Delta}\\ &= \overline{\xi}\cdot Qe\cdot(\overline{\delta}\cdot iX)\\ &= \overline{\delta}\cdot iX\ .\end{aligned}$$

Since $\overline{\beta}$ is a $T$-homomorphism we also have




.

Thus $i^{\#}\ :\ \left(QT,\ \delta^{\#}\cdot Q\lambda,\ I,\ \tau,\ Y,\ \beta^{\#}\right) \to (\overline{Q},\ \overline{\delta},\ I,\ \overline{\tau},\ Y,\ \overline{\beta})$ is a homomorphism of $X$-machines and, in particular, $f_M = f_{\overline{M}}$ . □

We immediately have

THEOREM 10. *Let there be given* $f : IX^{@} \to Y$ , *a* $T$*-decider structure* $(Y,\ \theta)$ *and an object* $Q$ *of* $K$ . *Then a* $\lambda$*-machine* $M$ *with state object* $Q$ *and response* $f_M$ *equal to* $f$ *exists iff there exists an implicit* $\lambda$*-machine* $M$ *with* $f_{\overline{M}} = f$ *and whose decider* $(\overline{Q},\ \overline{\xi})$ *admits a scoop* $(Q',\ i,\ c)$ *with* $Q$ *and* $Q'$ *isomorphic in* $K$ .

Proof. $\left(Q,\ Qe,\ \mathrm{id}_{QT}\right)$ is a scoop of $(QT,\ Qm)$ . □

In the next section, we characterize scoops in terms of extremal elements.

## 8. Examples

We now relate the scoops of Definition 8, Section 7 to extremal

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elements; and then give examples of realization by nondeterministic automata. Throughout this section we restrict $K$ to be *Set* .

If $(\overline{Q}, \xi)$ is a $T$-decider and $Q$ is a subset of $\overline{Q}$ , let $\langle Q \rangle$ denote the image of $i^{\#}$ in $Q$ where $i : Q \to \overline{Q}$ is the inclusion function. In the context of Examples 2, 3, 4 of Section 4, $\langle Q \rangle$ is, respectively, the submonoid generated by $Q$ , the union of all $M$-orbits intersecting $Q$ , and the set of all 'convex combinations' of elements in $Q$ . For $T$ as in (5), Section 2, $\langle Q \rangle$ is the set of all suprema of elements of $Q$ .

$Q$ *generates* $(\overline{Q}, \xi)$ if $\langle Q \rangle = \overline{Q}$ ; that is, $(Q, i, c)$ is a scoop for some $c$ . $\overline{q}$ in $\overline{Q}$ is an *isolated element* of $(\overline{Q}, \xi)$ if $q \notin \langle \overline{Q} \backslash \{q\} \rangle$ . $(\overline{Q}, \xi)$ is *extremal* if $(\overline{Q}, \xi)$ is generated by its isolated elements (in which case the isolated elements are contained in every set of generators and so constitute the unique minimal set of generators), and $T$ is *extremal* if every finitely-generated $T$-decider is extremal.

The $T$ of Example 10, Section 2, is not extremal (the 3-element group is generated by either of its non-units) even though free monoids are.

Isolated elements of convex sets are the well-known *extreme points* and thus finitely-generated convex sets are extremal. We do not know if all finitely-generated stochastic deciders are extremal, but this is true for all doubly-generated stochastic deciders (classified in [14, 4.3, Exercise 1]) which, incidentally, would appear to be an intriguing generalization of Zadeh's unit interval as a universe for "assigning weights to two truth values consistent with the laws of probability theory".

With respect to $T$ as in (5), Section 2, an extremal element of a complete semi-lattice is more conventionally called a *join-irreducible*. This $T$ is extremal:

PROPOSITION 1. *Every finite (equals finitely-generated complete) semilattice* $\overline{Q}$ *is extremal.*

Proof. Let $Q_1$ be the set of minimal elements of $\overline{Q} \backslash \{0\}$ (0 denotes the empty supremum; that is, the least element) and let $Q_{n+1}$ be the set of minimal elements of $\overline{Q} \backslash \langle Q_1 \cup \dots \cup Q_n \rangle$ (where $\langle A \rangle$ denotes the set of

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suprema of elements of $A$ ), continuing until $\overline{Q} = \langle Q_1 \cup \ldots \cup Q_m \rangle$ . It suffices to prove that each $Q_n$ consists of join-irreducibles. This is clear for $n = 1$ . Suppose $q \in Q_{n+1}$ and $q = q_1 \vee \ldots \vee q_k$ with no $q_i = q$ . Since $q \notin \langle Q_1 \cup \ldots \cup Q_n \rangle$ , $k \neq 0$ . Since each $q_i < q$ , we must have $q_i \in \langle Q_1 \cup \ldots \cup Q_n \rangle$ , the desired contradiction. □

Recalling the conventions stated just before Definition 7, Section 7, we have

DEFINITION 2. A *finite-state minimal realization* of $f : X_0^* \to Y$ is an implicit $\lambda$-machine $\overline{M} = (\overline{Q}, \overline{\delta}, \overline{\xi}, \overline{\tau}, \overline{\beta})$ and a scoop $(Q, i, c)$ of $(\overline{Q}, \overline{\xi})$ satisfying

(i) $f_{\overline{M}} = f$ ,

(ii) $Q$ is finite,

(iii) subject to (i), (ii) the number of elements of $Q$ is as small as possible.

In this context, Theorem 10, Section 7 asserts

THEOREM 3. *To find a nondeterministic minimal realization of $f$ it is necessary and sufficient to find an implicit nondeterministic realization of $f$ having fewest possible join-irreducibles.* □

In the deterministic case, $T = \text{id}$ , $Q = \overline{Q}$ is the only scoop and the notion is the usual one and coincides with the unique reachable and observable realization. The following example shows, however, that 'minimal' need not even imply 'reachable'.

EXAMPLE 4. We now present an example due to Ehrig *et al.* [8, 10.6] of a response whose nondeterministic minimal realization has fewer join-irreducibles than the realization of Proposition 4, Section 7. Let $X_0 = \{x\}$ have one element, let $Y$ be the semilattice of subsets of $\{a, b, c, d\}$ and let $f : X_0^* \to Y$ be the sequence

$$f = a/b/c/d//ab/ac/ad//$$

(that is, $f(x^0) = a$ , $f(x^4) = f(x^7) = \{a, b\}$ ; // denotes cycling).

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As is well known, the state space $Q_f$ of the deterministic minimal realization of $f$ may be constructed as the closure under the left-shift of $f$ in $Y^{X_0^*}$. Writing $fw$ for $fL_w$, we readily compute

$$fx = b/c/d//ab/ac/ad// ,$$

$$fx^2 = c/d//ab/ac/ad// ,$$

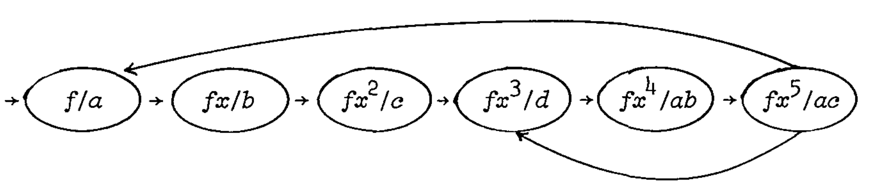
$$fx^3 = d//ab/ac/ad// ,$$

$$fx^{4+3k} = //ab/ac/ad// ,$$

$$fx^{5+3k} = //ac/ad/ab// ,$$

$$fx^{6+3k} = //ad/ab/ac// ,$$

so that $Q_f$ has the seven states: $Q_f = \{f, \ldots, fx^6\}$. Set $\overline{Q}$ to be the subset of $Y^{X_0^*}$ of all suprema (that is, pointwise unions) of elements of $Q_f$. Then $\overline{Q}$ is closed under left shift and is easily seen to be an *FSO*. Moreover, $\overline{Q}$ becomes an implicit nondeterministic realization of $f$ with $\overline{\tau} = f$ and $\overline{\beta}(g) = g(\Lambda)$ where $\Lambda$ is the empty word in $X_0^*$. It is clear from the earlier discussion of isolated elements that the join-irreducibles of $\overline{Q}$ are just those elements of $Q_f$ which cannot be expressed as suprema of other elements of $Q_f$. In this case, the only such relation is $fx^6 = f \vee fx^3$, so that $\overline{Q}$ has six join-irreducibles. The state-graph of the corresponding nondeterministic sequential machine as in the proof of Proposition 9, Section 7 (notice that we only use the values of $c$ on elements of form $\overline{\tau}$ or $\overline{\delta}(q, x)$ and that on these values $c$ is unique in this particular example) is

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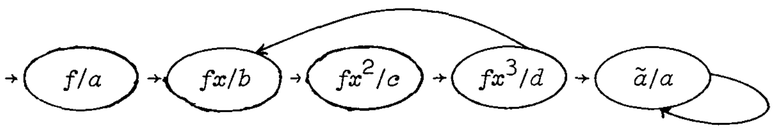
Let $\tilde{a} \in Y^{X_0^*}$ be constantly $a$. Then $\overline{R} = \overline{Q} \cup \{\overline{a}\}$ is an $FSO$ containing $\overline{Q}$ as a sub-$RSO$. However, $\overline{R}$ has only the five join-irreducibles $R = \{f, fx, fx^2, fx^3, \tilde{a}\}$ owing to the relationships $fx \vee \tilde{a} = fx^4$, $fx^2 \vee \tilde{a} = fx^5$, $fx^6 = f \vee fx^3 \vee \tilde{a}$. The corresponding nondeterministic sequential machine is




It is obvious that no nondeterministic realization of $f$ can have only four states, so this realization is minimal. The image of the reachability map $\tau^* : 2^{\{x\}^*} \to \overline{R}$ (as in Theorem 3, Section 7) does not contain $\tilde{a}$.

EXAMPLE 5. We present a simple example of a response possessing two minimal realizations corresponding to non-isomorphic implicit realizations (only one is reachable and observable). Let $X_0 = \{x\}$ have one element, let $Y$ be the semilattice of subsets of $\{a, b\}$ and let $f : X_0^* \to Y$ be the sequence

$$q_0 = f : a/b/a//ab//$$

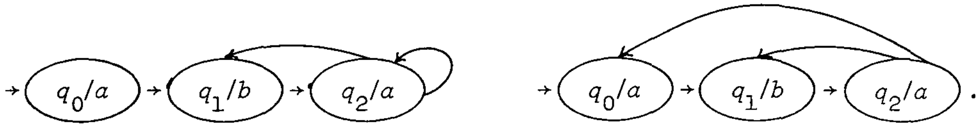
(for notations see Example 4). Then

$$q_1 = fx = b/a//ab// ,$$

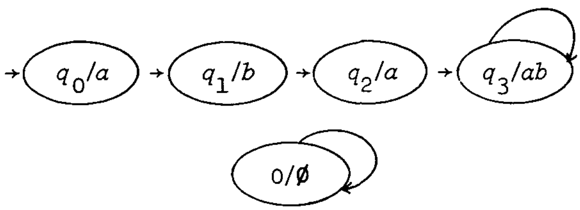
$$q_2 = fx^2 = a//ab// ,$$

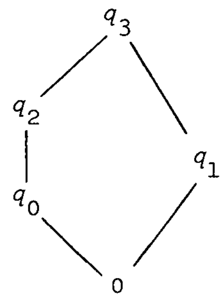
$$q_3 = fx^{3+k} = //ab// .$$

Then $q_3 = q_2 \vee q_1 = q_0 \vee q_1$ giving rise to two possible scoops and the following two $\lambda$-machines



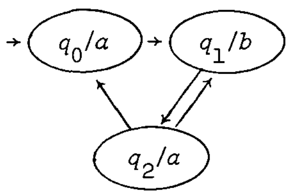

The $\lambda$-algebra structure in both cases is



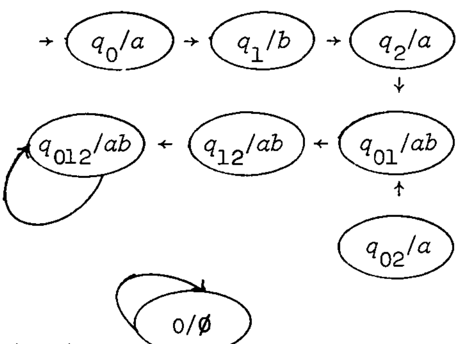



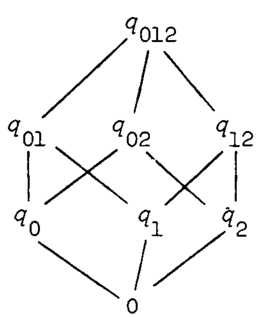

Dynamic Structure  Semilattice Structure

It is easy to check that $f$ cannot be realized using only two states, so this implicit realization is minimal. Now consider the following three-state $\lambda$-machine which realizes $f$ .




The corresponding implicit $\lambda$-machine is shown below:




Dynamic structure  Semilattice structure

Both implicit realizations are reachable but only the first is observable.

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