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CONTINUATION AND UNIQUENESS FOR GENERALISED EMDEN-FOWLER SYSTEMS

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Abstract

We discuss uniqueness and continuation of solutions to the Cauchy problem for a two dimensional Emden-Fowler differential system.

1. Introduction

In this paper we study continuation and uniqueness of solutions of the generalised Emden-Fowler system

$$x' = a(t)|y|^{\alpha}\operatorname{sgn} y, \qquad y' = -b(t)|x|^{\beta}\operatorname{sgn} x \tag{S}$$

where $a\colon [0,\infty) \to (0,\infty)$ and $b\colon [0,\infty) \to [0,\infty)$ are continuous and $b(t) > 0$ on $(0,\infty)$ and α and β are positive constants. The prototype of system (S) is

$$x'' + c(t)|x|^{\gamma}\operatorname{sgn} x = 0 \tag{E}$$

where $c\colon [0,\infty) \to [0,\infty)$ is continuous and $c(t) > 0$ on $(0,\infty)$. For equation (E), the properties of solutions mentioned above have been considered by Coffman and Ullrich [1], Heidel [4] and Kwong [5]. Some results concerning equations more general than (E) may be found in Hastings [3], Ullrich [9] and Coffman and Wong [2]. Further related results may also be found in Kwong and Wong [6] and in Mirzov [7, 8]. The purpose of this paper is to establish necessary and sufficient conditions in order that all solutions of (S) are continuable to $[0,\infty)$ and for the zero solution of (S) to be unique. Our

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results improve and extend some known theorems in Coffman and Ullrich [1], Heidel [4] and Kwong [5].

We classify (S) into four cases I, II, III and IV according to $\alpha \geq 1$, $\beta \geq 1$; $\alpha \leq 1$, $\beta < 1$; $\alpha > 1$, $\beta < 1$ and $\alpha < 1$, $\beta \geq 1$ respectively. We can prove, similar to [2], that the solution of an arbitrary Cauchy problem for (S) is unique when $\alpha \geq 1$ and $\beta \geq 1$, and that all solutions of (S) are continuable to $[0, \infty)$ when $\alpha \leq 1$ and $\beta < 1$. But for the remaining cases, continuation and uniqueness of solutions of (S) are not guaranteed by the classical theorems.

2. Continuation

In this section we consider continuation of solutions of (S) for the cases I, III and IV. Let $(x(t), y(t))$ be a nontrivial solution of the Cauchy problem (S) with $x(\tau) = x_0$, $y(\tau) = y_0$ on (ω_1, ω_2), where (ω_1, ω_2) is the maximum interval of existence, $\omega_i = \omega_i(\tau, x_0, y_0)$. Clearly $0 \leq \omega_1 < \tau < \omega_2 \leq \infty$. We first establish the following lemma which is similar to the Lemma A in Coffman and Wong [2].

LEMMA 1. *Let $(x(t), y(t))$ be a nontrivial solution of* (S) *on (ω_1, ω_2). Then $\omega_2 < \infty$ $(\omega_1 > 0)$ if and only if $x(t)$ and $y(t)$ have infinitely many zeros in any left (right) neighborhood of $\omega_2(\omega_1)$ and*

$$\limsup_{t\to\omega_2^-} |x(t)| = \limsup_{t\to\omega_2^-} |y(t)| = \infty \qquad \Big(\limsup_{t\to\omega_1^+} |x(t)| = \limsup_{t\to\omega_1^+} |y(t)| = \infty\Big). \tag{1}$$

In fact, the sufficiency is obvious and the proof of the necessity is similar to that of the lemma A.1 in Coffman and Wong [2].

We next introduce the notion of "characteristic sequence on $[T_1, T_2]$" for arbitrary T_1 and T_2, $0 \leq T_1 < \tau < T_2$. We suppose that $\{t_{2k}\}$ and $\{t_{2k+1}\}$ are zeros of $x(t)$ and $y(t)$ on (T_1, T_2) respectively such that

$$0 \leq T_1 < \cdots < t_{-3} < t_{-2} < t_{-1} \leq \tau < t_0 < t_1 < t_2 < \cdots < T_2 < \infty.$$

If the number of the zeros of $x(t)$ is not less than two, then integrating (S) from t_{2k-1} to t_{2k} we have

$$\frac{1}{\alpha+1}|y(t_{2k})|^{\alpha+1} = \frac{1}{\beta+1}\frac{b(\tau_{2k-1})}{a(\tau_{2k-1})}|x(t_{2k-1})|^{\beta+1} \qquad (t_{2k-1} < \tau_{2k-1} < t_{2k}). \tag{2}$$

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Similarly integrating (S) from t_{2k} to t_{2k+1} we have

$$\frac{1}{\alpha+1}|y(t_{2k})|^{\alpha+1} = \frac{1}{\beta+1}\frac{b(\tau_{2k})}{a(\tau_{2k})}|x(t_{2k+1})|^{\beta+1} \qquad (t_{2k} < \tau_{2k} < t_{2k+1}). \tag{3}$$

We denote

$$A_i = \prod_{k=1}^{i} \frac{b(\tau_{2k-1})/a(\tau_{2k-1})}{b(\tau_{2k-2})/a(\tau_{2k-2})},$$

$$B_i = \prod_{k=1}^{i} \frac{b(\tau_{2k-1})/a(\tau_{2k-1})}{b(\tau_{2k})/a(\tau_{2k})}.$$

If the number of the zeros of $x(t)$ is less than two, A_0 and B_0 are defined by $A_0 = B_0 = 1$. It is easy to show that the following relation holds:

$$A_i = \frac{b(\tau_{2i})a(\tau_0)}{a(\tau_{2i})b(\tau_0)} B_i. \tag{4}$$

The sequences $\{A_i\}$ and $\{B_i\}$ are called characteristic sequences on $[T_1, T_2]$ of the nontrivial solution $(x(t), y(t))$ of (S).

When (S) belongs to any of cases I, III and IV, we have the following result.

THEOREM 1. *All solutions of* (S) *are continuable to* $[0, \infty)$ *if and only if for arbitrary* T_1 *and* T_2, $0 \le T_1 < \tau < T_2$, *the characteristic sequence* $\{A_i\}$ *on* $[T_1, T_2]$ *of every solution of* (S) *is either not defined or is bounded above.*

PROOF. Sufficiency: Let $(x(t), y(t))$ be a nontrivial solution of (S) on (ω_1, ω_2). If $\omega_2 < \infty$, then by Lemma 1 we know that $x(t)$ and $y(t)$ have infinitely many zeros in any left neighbourhood of ω_2 and satisfy (1). On the other hand, from (2) and (3) it follows that

$$|x(t_{2k+1})|^{\beta+1} = B_k |x(t_1)|^{\beta+1}, \tag{5}$$

$$|y(t_{2k})|^{\alpha+1} = A_k |y(t_0)|^{\alpha+1}. \tag{6}$$

From (5) and (6) and by our assumption we have

$$\limsup_{k\to\infty} |x(t_{2k+1})| < \infty, \qquad \limsup_{k\to\infty} |y(t_{2k})| < \infty,$$

which contradicts (1). If $\omega_1 > 0$, then we are led to a contradiction as above. Since the characteristic sequence $\{A_i\}$ on $[0, \tau+1]$ of $(x(t), y(t))$ is bounded above, we know from Lemma 1 that $x(0+)$ and $y(0+)$ exist and are finite. Thus the solution $(x(t), y(t))$ is continuable to $[0, \infty)$.

Necessity: Let $(x(t), y(t))$ be a nontrivial solution of (S) on $[0, \infty)$. By Lemma 1 we know that $x(t)$ and $y(t)$ have finitely many zeros on $[T_1, T_2]$

for arbitrary T_1 and T_2, $0 \leq T_1 < \tau < T_2$. So the sequence $\{A_i\}$ bounded above. This completes the proof of Theorem 1.

In general, it is difficult to verify the conditions of Theorem 1. Therefore we shall introduce some conditions which are easier to apply.

COROLLARY 1. *Let $\alpha \geq 1$ and $\beta \geq 1$ and suppose that $b(0) > 0$ and that $b(t)/a(t)$ is locally of bounded variation on $[0, \infty)$. Then any Cauchy problem of* (S) *has a unique solution on $[0, \infty)$.*

In fact, the uniqueness of solutions is obvious, because the right hand side of (S) satisfies a local Lipschitz condition in x and y. Now we show the extendability of solutions. Let $m = \min_{T_1 \leq t \leq T_2}[b(t)/a(t)]$, where T_1 and T_2 are arbitrary, $0 \leq T_1 < \tau < T_2$. Then the characteristic sequence $\{A_i\}$ on $[T_1, T_2]$ of any nontrivial solution of (S) satisfies

$$\begin{aligned} \ln A_i &\leq \sum_{k=1}^{i} |\ln[b(\tau_{2k-1})/a(\tau_{2k-1})] - \ln[b(\tau_{2k-2})/a(\tau_{2k-2})]| \\ &\leq \frac{1}{m}\sum_{k=1}^{i} |b(\tau_{2k-1})/a(\tau_{2k-1}) - b(\tau_{2k-2})/a(\tau_{2k-2})| \\ &\leq \frac{1}{m}\bigvee_{T_1}^{T_2}\{b/a\} \end{aligned}$$

so that

$$A_i \leq \exp\frac{1}{m}\bigvee_{T_1}^{T_2}\{b/a\} < \infty.$$

Thus the conclusion follows from Theorem 1.

REMARK 1. Letting $\alpha = 1$, $\beta > 1$ and $a(t) \equiv 1$ in (S), Corollary 1 reduces to a theorem of Coffman and Ullrich [1].

3. Uniqueness

In this section we consider uniqueness of solutions of (S) for the cases II, III and IV. For this we first prove the following seven lemmas.

LEMMA 2. *Let $x_0 \neq 0$ and $y_0 \neq 0$. Then the Cauchy problem* (S) *with $x(\tau) = x_0$, $y(\tau) = y_0$, $\tau \in [0, \infty)$ has a locally unique solution.*

This is clear since the right hand sides of (S) are continuous and satisfy a Lipschitz condition in a neighborhood of (x_0, y_0).

Lemma 3. *Let* $x_0 = 0$, $y_0 \neq 0$. *Then the Cauchy problem* (S) *with* $x(\tau) = x_0$, $y(\tau) = y_0$, $\tau \in [0, \infty)$ *has a locally unique solution* $(x(t), y(t))$ *such that* $x(t) \neq 0$, $y(t) \neq 0$ *on* $(\tau, \tau + \varepsilon]$, $\varepsilon > 0$ *sufficiently small.*

Proof. The existence is obvious. We now prove the uniqueness. Let $(x(t), y(t))$ be a solution of the Cauchy problem such that $x(t) \neq 0, y(t) \neq 0$ on $(\tau, \tau + \varepsilon]$, and set

$$u(t) = x(t)/(t - \tau), \quad v(t) = (y(t) - y_0)/(t - \tau).$$

It is easy to see that $u(t)$ and $v(t)$ are continuous on $[\tau, \tau + \varepsilon]$ and satisfy the integral system

$$\begin{aligned} u(t) &= \int_\tau^t a(s)\frac{(s-\tau)^\alpha}{t-\tau}\left[v(s) + \frac{y_0}{s-\tau}\right]^\alpha ds \\ v(t) &= -\int_\tau^t b(s)\frac{(s-\tau)^\beta}{t-\tau} u^\beta(s)\,ds \end{aligned} \tag{7}$$

for $t \in (\tau, \tau + \varepsilon)$; furthermore the local uniqueness for solutions of the Cauchy problem is equivalent to that of the integral system (7). Let $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ be two solutions of (7). Using (7) we have

$$\begin{aligned} u_2(t) - u_1(t) &= \int_\tau^t a(s)\frac{(s-\tau)^\alpha}{t-\tau}\left\{\left[v_2(s) + \frac{y_0}{s-\tau}\right]^\alpha - \left[v_1(s) + \frac{y_0}{s-\tau}\right]^\alpha\right\} ds, \\ v_2(t) - v_1(t) &= -\int_\tau^t b(s)\frac{(s-\tau)^\beta}{t-\tau}\left[u_2^\beta(s) - u_1^\beta(s)\right] ds. \end{aligned} \tag{8}$$

Computing the two factors of the integrands in (8), we have

$$\begin{aligned} &\left[v_2(s) + \frac{y_0}{s-\tau}\right]^\alpha - \left[v_1(s) + \frac{y_0}{s-\tau}\right]^\alpha \\ &\qquad = \alpha\left\{v_1(s) + \frac{y_0}{s-\tau} + \theta_1\left[v_2(s) - v_1(s)\right]\right\}^{\alpha-1} \\ &\qquad\quad \times [v_2(s) - v_1(s)] \\ &\qquad = f(s)[v_2(s) - v_1(s)], \qquad (0 < \theta_1 < 1) \end{aligned} \tag{9}$$

$$\begin{aligned} u_2^\beta(s) - u_1^\beta(s) &= \beta\{u_1(s) + \theta_2[u_2(s) - u_1(s)]\}^{\beta-1} \times [u_2(s) - u_1(s)] \\ &= g(s)[u_2(s) - u_1(s)], \qquad (0 < \theta_2 < 1). \end{aligned} \tag{10}$$

Clearly $f(s)$ and $g(s)$ are integrable on $[\tau, \tau+\varepsilon]$ for $\alpha > 0$ and $0 < \beta < 1$. From (8), (9) and (10) we get

$$\begin{aligned} &|u_2(t) - u_1(t)| + |v_2(t) - v_1(t)| \\ &\quad \leq \int_\tau^t \left[a(s)\frac{(s-\tau)^\alpha}{t-\tau}|A(s)| + b(s)\frac{(s-\tau)^\beta}{t-\tau}\right] \\ &\qquad \times [|u_2(s) - u_1(s)| + |v_2(s) - v_1(s)|]\,ds. \end{aligned}$$

From which it follows that $u_1(t) \equiv u_2(t)$, $v_1(t) \equiv v_2(t)$. This completes the proof of Lemma 3.

LEMMA 4. *Let* $\alpha \geq 1$, $0 < \beta < 1$ *and let* $x_0 \neq 0$, $y_0 = 0$. *Then the Cauchy problem* (S) *with* $x(\tau) = x_0$, $y(\tau) = y_0$, $\tau \in [0, \infty)$ *has a locally unique solution* $(x(t), y(t))$ *such that* $x(t) \neq 0$, $y(t) \neq 0$ *on* $(\tau, \tau+\varepsilon]$, $\varepsilon > 0$ *small enough.*

The proof of Lemma 4 and the next lemma are similar to that of Lemma 3 and so we omit them.

LEMMA 5. *Let* $b(0) > 0$, $0 < \alpha < 1$ *and* $\beta > 0$, *and let* $x_0 \neq 0$, $y_0 = 0$. *Then the Cauchy problem* (S) *with* $x(\tau) = x_0$, $y(\tau) = y_0$, $\tau \in [0, \infty)$ *has locally unique solution* $(x(t), y(t))$ *such that* $x(t) \neq 0$, $y(t) \neq 0$ *on* $(\tau, \tau+\varepsilon]$, $\varepsilon > 0$ *small enough.*

According to Lemmas 2, 3, 4 and 5 we know that the uniqueness of solutions of (S) for the cases $\alpha \geq 1$ and $\beta < 1$ or $0 < \alpha < 1$, $\beta > 0$ and $b(0) > 0$ reduces to that of the zero solution of (S). We now consider the uniqueness of solutions of (S). We first give a generalisation of the lemma in Heidel [4].

LEMMA 7. *Let* $\alpha \geq 1$, $0 < \beta < 1$. *Then a nontrivial solution* $(x(t), y(t))$ *of* (S) *is singular if and only if* $x(t)$ *and* $y(t)$ *have infinitely many zeros on a finite interval.*

PROOF. Sufficiency. Suppose that $x(t)$ and $y(t)$ have infinitely many zeros on some interval $[T_1, T_2]$. Then the exists a limit point of zeros, say $T \in [T_1, T_2]$. By continuity of solutions we get $x(T) = y(T) = 0$. So $(x(t), y(t))$ is singular.

Necessity. Suppose that $x^2(\tau) + y^2(\tau) \neq 0$, $x(T) = y(T) = 0$ and $\tau < T$ and that $x(t) \neq 0$ on (τ, T). Without loss of generality we can assume that $x(t) > 0$ on (τ, T). Then $y'(t) \leq 0$ on (τ, T), from which it follows that $y(t) \leq y(T) = 0$ on (τ, T). We thus obtain $x(t) \leq x(T) = 0$, which contradicts our assumption. This completes the proof of Lemma 7.

LEMMA 8. *Let* $0 < \alpha < 1$, $\beta > 0$ *and* $b(0) > 0$. *Then the conclusion of Lemma 7 holds.*

The proof of Lemma 8 is similar to that of Lemma 7 and so is omitted.

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Theorem 2. *Let $\alpha \geq 1$, $0 < \beta < 1$. Then the zero solution of* (S) *is unique if and only if for arbitrary T_1 and T_2, $0 \leq T_1 < \tau < T_2$, the characteristic sequence $\{A_i\}$ on $[T_1, T_2]$ of every nontrivial solution of* (S) *has a positive lower bound.*

Proof. Sufficiency: Let $(x(t), y(t))$ be a singular solution of the Cauchy problem (S) with $x(\tau) = x_0$, $y(\tau) = y_0$ and let $x(T) = y(T) = 0$. By Lemma 7 we can choose T_1 and T_2 such that $T_1 < T_2$ and $x(t)$ and $y(t)$ have infinitely many zeros on $[T_1, T_2]$. Therefore, the characteristic sequences $\{A_i\}$ and $\{B_i\}$ and the amplitudes of $x(t)$ and $y(t)$ satisfy (5) and (6). From (6) it follows that

$$\begin{aligned}\lim_{k\to\infty} A_k &= \lim_{k\to\infty} |y(t_{2k})|^{\alpha+1}/|y(t_0)|^{\alpha+1} \\ &= |y(T)|^{\alpha+1}/|y(t_0)|^{\alpha+1} \\ &= 0,\end{aligned}$$

which is a contradiction.

Necessity: By Lemma 7 any nontrivial solution of (S) has finitely many zeros on $[T_1, T_2]$ for arbitrary T_1 and T_2, $0 \leq T_1 < \tau < T_2$. Then the characteristic sequence $\{A_i\}$ is finite and positive, and so $\{A_i\}$ has a positive lower bound. This completes the proof of Theorem 2.

Theorem 3. *Let $0 < \alpha < 1$, $\beta > 0$ and $b(0) > 0$. Then the conclusion of Theorem 2 holds.*

The proof of Theorem 3 is similar to that of Theorem 2 and so is omitted. The following result is an analogue of Corollary 1.

Corollary 2. *Let $b(0) > 0$ and suppose that $b(t)/a(t)$ is locally of bounded variation on $[0, \infty)$. Then the zero solution of* (S) *is unique.*

In fact, for arbitrary T_1 and T_2, $0 \leq T_1 < \tau < T_2$, the characteristic sequence $\{A_i\}$ on $[T_1, T_2]$ of any nontrivial solution of (S) satisfies

$$\begin{aligned}\ln A_i &= \sum_{k=1}^{i}\{\ln[b(\tau_{2k-1})/a(\tau_{2k-1})] - \ln[b(\tau_{2k-2})/a(\tau_{2k-2})]\} \\ &= \sum_{k=1}^{i}\int_{\tau_{2k-2}}^{\tau_{2k-1}} \frac{d[b(t)/a(t)]}{b(t)/a(t)}\,dt.\end{aligned}$$

Then we get

$$A_i \geq \exp\left\{-\int_{T_1}^{T_2} \frac{|d[b(t)/a(t)]|}{b(t)/a(t)}\,dt\right\}.$$

Therefore by Theorems 2 and 3, the result follows.

COROLLARY 3. *Let* $b(0) > 0$ *and suppose that* $\ln[b(t)/a(t)]$ *has finite lower variation on any finite interval. Then the zero solution of* (S) *is unique.*

In fact, the characteristic sequence $\{A_i\}$ on $[T_1, T_2]$ satisfies

$$\begin{aligned}\ln A_i &= \ln \prod_{k=1}^{i} \frac{b(\tau_{2k-1})/a(\tau_{2k-1})}{b(\tau_{2k-2})/a(\tau_{2k-2})} \\ &\geq -\{\text{lower variation of } \ln[b(t)/a(t)] \text{ on } [T_1, T_2]\} = -L.\end{aligned}$$

Thus $A_i \geq \exp(-L)$. From this, and by Theorems 2 and 3, we get the conclusion of Corollary 3.

REMARK 2. Letting $\alpha = 1$, $0 < \beta < 1$ and $a(t) \equiv 1$ in (S), then Corollaries 2 and 3 reduce to Theorem 1 of Heidel [4] and Corollary A.5 of Coffman and Wong [2], respectively.

4. Global continuation and uniqueness

In this section we shall establish necessary and sufficient conditions for global continuation and uniqueness of solutions of (S). From the results of Sections 2 and 3, it is easy to prove the following results.

THEOREM 4. *Let* $\alpha \geq 1$ *and* $\beta > 0$. *Then all solutions of* (S) *are unique and continuable to* $[0, \infty)$ *if and only if for arbitrary* T_1 *and* T_2, $0 \leq T_1 < \tau < T_2$, *the characteristic sequence* $\{A_i\}$ *on* $[T_1, T_2]$ *of every nontrivial solution of* (S) *has positive lower and upper bounds.*

THEOREM 5. *Let* $0 < \alpha < 1$, $\beta > 0$ *and let* $b(0) > 0$. *Then all solutions of* (S) *are unique and continuable to* $[0, \infty)$ *if and only if for arbitrary* T_1 *and* T_2, $0 < T_1 < \tau < T_2$, *the characteristic sequence* $\{A_i\}$ *on* $[T_1, T_2]$ *of every nontrivial solution of* (S) *has positive lower and upper bounds.*

The final corollary improves and extends results of [1] and is related to results of [5].

COROLLARY 4. *Let* $\alpha > 0$, $\beta > 0$ *and suppose that* $b(0) > 0$ *and* $b(t)/a(t)$ *is locally of bounded variation on* $[0, \infty)$. *Then all solutions of* (S) *are unique and continuable to* $[0, \infty)$.

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