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RESEARCH ARTICLE

Negative moments of orthogonal polynomials

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Abstract
If a sequence indexed by nonnegative integers satisfies a linear recurrence without constant terms, one can extend the indices of the sequence to negative integers using the recurrence. Recently, Cigler and Krattenthaler showed that the negative version of the number of bounded Dyck paths is the number of bounded alternating sequences. In this paper, we provide two methods to compute the negative versions of sequences related to moments of orthogonal polynomials. We give a combinatorial model for the negative version of the number of bounded Motzkin paths. We also prove two conjectures of Cigler and Krattenthaler on reciprocity between determinants.

Contents

1. Introduction

Suppose that there is a sequence $(f_n)_{n \in \mathbb{Z}}$ indexed by all integers. If both $|f_n|$ and $|f_{-n}|$ count some combinatorial objects of size $n \geq 1$, such a result is called a *combinatorial reciprocity theorem*, a term first used by Stanley [15]. There are many combinatorial reciprocity theorems; three notable examples are when f_n is the binomial coefficient $\binom{n}{k}$, the chromatic polynomial $\chi_G(n)$ of a graph G and the Ehrhart polynomial $\mathrm{Ehr}_P(n)$ of a lattice polytope P. For more details on combinatorial reciprocity theorems, see the book by Beck and Sanyal [1].

Suppose now that we have a sequence $(f_n)_{n \geq 0}$ indexed by nonnegative integers. If the sequence satisfies a homogeneous linear recurrence relation, then one can extend the indices of this sequence to negative integers $(f_{-n})_{n \geq 1}$ using the recurrence. Recently, Cigler and Krattenthaler [3] showed that,

for a fixed integer k, the negative counterpart of the number of Dyck paths from $(0,0)$ to $(2n,0)$ with bounded height $2k-1$ is the number of alternating sequences $a_1 \le a_2 \ge a_3 \le \cdots \ge a_{2n-1}$ of positive integers at most k. They also showed many other interesting results, including a reciprocity between determinants of these numbers and their connection with orthogonal polynomials.

In this paper, motivated by the work of Cigler and Krattenthaler [3], we find combinatorial reciprocity theorems for more general sequences related to moments of orthogonal polynomials. In particular, we give two methods to study such negative sequences. The first method uses continued fractions and the second one uses matrix inverses. Our first method is new, and the key idea of the second method is due to Hopkins and Zaimi [8]. We also prove two conjectures on reciprocity between determinants proposed by Cigler and Krattenthaler [3, Conjectures 50, 53]. Before stating our results, we first review basic results in orthogonal polynomials and define some notation.

A sequence $(P_n(x))_{n\ge 0}$ of polynomials is called an *orthogonal polynomial sequence*[1] with respect to a linear functional $\mathcal{L}$ if for all $m, n \ge 0$, we have $\deg(P_n(x)) = n$ and

$$\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}K_n, \qquad K_n \ne 0. \tag{1}$$

In this case, we will simply say that $P_n(x)$ are *orthogonal polynomials* (with respect to $\mathcal{L}$).

It is well known [2, Theorem 4.1, p.18] that monic orthogonal polynomials $P_n(x)$ satisfy a three-term recurrence relation:

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \ge 0, \qquad P_{-1}(x) = 0, P_0(x) = 1, \tag{2}$$

for some sequences $\boldsymbol{b} = (b_n)_{n\ge 0}$ and $\boldsymbol{\lambda} = (\lambda_n)_{n\ge 1}$ with $\lambda_n \ne 0$. Conversely, Favard's theorem [2, Theorem 4.4, p.21] states that if monic polynomials $P_n(x)$ satisfy (2) for some sequences $\boldsymbol{b} = (b_n)_{n\ge 0}$ and $\boldsymbol{\lambda} = (\lambda_n)_{n\ge 1}$ with $\lambda_n \ne 0$, then $P_n(x)$ are orthogonal polynomials with respect to a unique linear function $\mathcal{L}$ satisfying (1) and $\mathcal{L}(1) = 1$.

Let $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$ denote the polynomials satisfying (2). Then by Favard's theorem, these are orthogonal polynomials with respect to a unique linear functional $\mathcal{L}$. The *moment* $\mu_n(\boldsymbol{b}, \boldsymbol{\lambda})$ of the orthogonal polynomials $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$ is defined by $\mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) = \mathcal{L}(x^n)$.

Viennot [18] found the following combinatorial interpretation for the moment:

$$\mathcal{L}(x^n) = \mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{p \in \mathrm{Mot}_n} \mathrm{wt}(p; \boldsymbol{b}, \boldsymbol{\lambda}),$$

where Mot_n is the set of Motzkin paths from $(0,0)$ to $(n,0)$ and $\mathrm{wt}(p; \boldsymbol{b}, \boldsymbol{\lambda})$ is a weight of a Motzkin path p depending on the sequences $\boldsymbol{b}$ and $\boldsymbol{\lambda}$. See Section 2 for the precise definitions.

We define the *bounded moments* $\mu_n^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda})$ by

$$\mu_n^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda}) := \sum_{p \in \mathrm{Mot}_n^{\le k}} \mathrm{wt}(p; \boldsymbol{b}, \boldsymbol{\lambda}),$$

where $\mathrm{Mot}_n^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda})$ is the set of Motzkin paths from $(0,0)$ to $(n,0)$ that stay weakly below the line $y = k$. Then the moments are the limits of the bounded moments:

$$\mathcal{L}(x^n) = \mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) = \lim_{k \to \infty} \mu_n^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda}).$$

For certain choices of $\boldsymbol{b}, \boldsymbol{\lambda}$, and k, the sequence $(\mu_n^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda}))_{n\ge 0}$ satisfies a homogeneous linear recurrence relation so that its negative version $(\mu_{-n}^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda}))_{n\ge 1}$ is defined. In this case, we call $\mu_{-n}^{\le k}(\boldsymbol{b}, \boldsymbol{\lambda})$ the *negative (bounded) moments* of the orthogonal polynomials $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$.

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[1]In this paper, we only consider the 'formal' orthogonality in the sense that we do not require the positive-definiteness of the linear functional $\mathcal{L}$, which is often assumed in the literature on orthogonal polynomials.

Cigler and Krattenthaler [3] showed the following combinatorial reciprocity theorem for the number $\mu_{2n}^{\leq 2k-1}(\mathbf{0}, \mathbf{1})$ of bounded Dyck paths, where $\mathbf{0} = (0, 0, \dots)$ and $\mathbf{1} = (1, 1, \dots)$.

Theorem 1.1 [3, Corollary 13]. *For positive integers n and k,*

$$\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \mathbf{1}) = |\operatorname{Alt}_{2n-1}^{\leq k}|,$$

where $\operatorname{Alt}_n^{\leq k}$ is the set of alternating sequences $(a_1, \dots, a_n)$ of integers such that $a_1 \leq a_2 \geq a_3 \leq \cdots$ and $1 \leq a_i \leq k$ for all i.

Cigler and Krattenthaler [3] proved Theorem 1.1 using generating functions. We give a new proof of this theorem using continued fractions. To do this, we introduce a notion of ℓ-*peak-valley sequences* in Definition 3.1 and give a simple bijection between alternating sequences and 2-peak-valley sequences.

Using continued fractions, we show in Theorem 3.5 that $\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \lambda)$ is a weight-generating function for 2-peak-valley sequences with some conditions, which is equivalent to [3, Corollary 32]. Our method also applies to Motzkin paths. In Theorems 4.2 and 4.5, we show that if $\mathbf{b}$ and λ satisfy $\lambda_i = b_{i-1}b_i$ for all $i \geq 1$, then $\mu_{-n}^{\leq 3k-1}(\mathbf{b}, \lambda)$ and $\mu_{-n}^{\leq 3k}(\mathbf{b}, \lambda)$ are weight-generating functions for 3-peak-valley sequences with some conditions.

Viennot (see [18, Proposition 17, p. I.15] or [19, (5)]) also showed that the generalized moment $\mu_{n,r,s}(\mathbf{b}, \lambda) := \mathcal{L}(x^n P_r(x; \mathbf{b}, \lambda) P_s(x; \mathbf{b}, \lambda))$ has a similar combinatorial expression

$$\mu_{n,r,s}(\mathbf{b}, \lambda) = \sum_{p \in \operatorname{Mot}_{n,r,s}} \operatorname{wt}(p; \mathbf{b}, \lambda),$$

where $\operatorname{Mot}_{n,r,s}(\mathbf{b}, \lambda)$ is the set of Motzkin paths from $(0, r)$ to (n, s). We define the generalized bounded moments $\mu_{n,r,s}^{\leq k}(\mathbf{b}, \lambda)$ by

$$\mu_{n,r,s}^{\leq k}(\mathbf{b}, \lambda) := \sum_{p \in \operatorname{Mot}_{n,r,s}^{\leq k}} \operatorname{wt}(p; \mathbf{b}, \lambda),$$

where $\operatorname{Mot}_{n,r,s}^{\leq k}(\mathbf{b}, \lambda)$ is the set of Motzkin paths from $(0, r)$ to (n, s) that stay weakly below the line $y = k$.

Cigler and Krattenthaler [3] showed that Theorem 1.1 extends nicely to generalized bounded moments as follows.

Theorem 1.2 [3, Corollary 12]. *For positive integers n, k, r, s with $1 \leq r, s \leq k$, we have*

$$(-1)^{r+s} \mu_{-2n,2r-2,2s-2}^{\leq 2k-1}(\mathbf{0}, \mathbf{1}) = |\operatorname{Alt}_{2n+1,r,s}^{\leq k}|,$$
$$(-1)^{r+s} \mu_{-2n+1,2r-2,2s-1}^{\leq 2k-1}(\mathbf{0}, \mathbf{1}) = |\operatorname{Alt}_{2n,r,s}^{\leq k}|,$$

where $\operatorname{Alt}_{n,r,s}^{\leq k}$ is the set of sequences $(a_1, \dots, a_n)$ of integers such that $a_1 \leq a_2 \geq a_3 \leq \cdots$ and $1 \leq a_i \leq k$ for all i and such that $a_1 = r$ and $a_n = s$.

In Theorems 5.5 and 5.8, we show that if $\mathbf{b}$ and λ satisfy $\lambda_i = b_{i-1}b_i$ for all $i \geq 1$, then $\mu_{-n,r,s}^{\leq 3k-1}(\mathbf{b}, \lambda)$ and $\mu_{-n,r,s}^{\leq 3k}(\mathbf{b}, \lambda)$ are weight-generating functions for 3-peak-valley sequences with some conditions.

Cigler and Krattenthaler [3] showed the following reciprocity theorem relating determinants whose entries are $\mu_n^{\leq 2k-1}(\mathbf{0}, \mathbf{1})$ and their negative versions.

Theorem 1.3 [3, Theorem 15]. *For all nonnegative integers n, k, m, we have*

$$\det\left(\mu_{2n+2i+2j+4m-2}^{\leq 2k+2m-1}(\mathbf{0}, \mathbf{1})\right)_{i,j=0}^{k-1} = \det\left(\mu_{-2n-2i-2j}^{\leq 2k+2m-1}(\mathbf{0}, \mathbf{1})\right)_{i,j=0}^{m-1}.$$

Cigler and Krattenthaler [3] proposed the following two conjectures.

Theorem 1.4 [3, Conjecture 50]. *For all nonnegative integers n, k, m, we have*

$$\det\left(\sum_{s=0}^{2k+2m-1} \mu_{n+i+j+2m-1,0,s}^{\leq 2k+2m-1}(\mathbf{0},\mathbf{1})\right)_{i,j=0}^{k-1} = (-1)^{\left(\binom{k}{2}+\binom{m}{2}\right)(n+1)} \det\left(\left|\mathrm{Alt}_{n+i+j}^{\leq k+m}\right|\right)_{i,j=0}^{m-1}.$$

Theorem 1.5 [3, Conjecture 53]. *For all positive integers n, k, m with $k + m \not\equiv 2 \pmod 3$, we have*

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\mathbf{1},\mathbf{1})\right)_{i,j=0}^{k-1}$$
$$= (-1)^{n\lfloor(k+m)/3\rfloor} \det\left(\mu_{-n-i-j}^{\leq k+m-1}(\mathbf{1},\mathbf{1})\right)_{i,j=0}^{m-1}.$$

In Section 6, we prove a general reciprocity theorem (Theorem 6.1). In Section 7, we prove the above two conjectures using Theorem 6.1. In Section 8, we show that Theorem 6.1 also implies the weighted version of Theorem 1.3 due to Cigler and Krattenthaler [3, Theorem 34]. We then show in Theorem 8.5 that this weighted version gives a bounded and multivariate generalization of the Morales–Pak–Panova ex-conjecture [14] on reverse plane partitions, which has been proved by Hwang et al. [9] and Guo et al. [7], independently.

In the final section, Section 9, we consider the negative version of the number of bounded Schröder paths and the negative moments of Laurent biorthogonal polynomials.

2. Preliminaries

In this section, we give some definitions related to negative moments of orthogonal polynomials and prove their basic properties.

We say that a sequence $(f_n)_{n\geq 0}$ *satisfies a homogeneous linear recurrence relation* if there exist a positive integer k and constants $r_1, \ldots, r_k$ with $r_k \neq 0$ such that for all $n \geq k$,

$$f_n = r_1 f_{n-1} + \cdots + r_k f_{n-k}. \tag{3}$$

In this case, we can uniquely extend the sequence f_n to all integers n by requiring that (3) holds for all $n \in \mathbb{Z}$. Therefore, whenever a sequence $(f_n)_{n\geq 0}$ satisfies a homogeneous linear recurrence relation, we can also consider the negatively indexed sequence $(f_{-n})_{n\geq 1}$.

It is not hard to check that the 'negative of negative' of a sequence is itself in the sense that if we write $f = (f_n)_{n\geq 0}$ and $\overline{f} = (f_{-n})_{n\geq 0}$, then $\overline{\overline{f}} = f$.

The following well-known lemma is useful when we study negatively indexed sequences.

Lemma 2.1 [16, Theorem 4.1.1]. *A sequence $(f_n)_{n\geq 0}$ satisfies a homogeneous linear recurrence relation if and only if*

$$\sum_{n\geq 0} f_n x^n = \frac{P(x)}{Q(x)}$$

for some polynomials $P(x)$ and $Q(x)$ with $\deg(P(x)) < \deg(Q(x))$ and $Q(0) \neq 0$. Moreover, in this case, we have

$$\sum_{n\geq 1} f_{-n} x^n = -\frac{P(1/x)}{Q(1/x)},$$

as rational functions.

In this paper, a *lattice path* is a finite sequence $p = (p_0, p_1, \ldots, p_n)$ of points in $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Each $S_i = (x_i - x_{i-1}, y_i - y_{i-1})$, $1 \leq i \leq n$, is called a *step* of p. If the starting point p_0 is fixed, we will often identify the lattice path p with the sequence $(S_1, S_2, \ldots, S_n)$ of its steps.

A *Motzkin path* is a lattice path in which every step is an *up step* $U = (1, 1)$, a *horizontal step* $H = (1, 0)$ or a *down step* $D = (1, -1)$. We denote by $\mathrm{Mot}_{n,r,s}$ the set of Motzkin paths from $(0, r)$ to (n, s). Let $\mathrm{Mot}_{n,r,s}^{\leq k}$ be the set of Motzkin paths in $\mathrm{Mot}_{n,r,s}$ that lie weakly below the line $y = k$. We also define $\mathrm{Mot}_n = \mathrm{Mot}_{n,0,0}$ and $\mathrm{Mot}_n^{\leq k} = \mathrm{Mot}_{n,0,0}^{\leq k}$.

Throughout this paper, we use the following notation:

$$
\begin{aligned}
\boldsymbol{b} &= (b_n)_{n \geq 0} = (b_0, b_1, \dots), \\
\boldsymbol{\lambda} &= (\lambda_n)_{n \geq 1} = (\lambda_1, \lambda_2, \dots), \\
\boldsymbol{b}^2 &= (b_{n-1}b_n)_{n \geq 1} = (b_0 b_1, b_1 b_2, \dots), \\
\boldsymbol{0} &= (0, 0, \dots), \\
\boldsymbol{1} &= (1, 1, \dots).
\end{aligned}
$$

Definition 2.2. The *weight* $\mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{\lambda})$ of a Motzkin path π (with respect to $\boldsymbol{b}$ and $\boldsymbol{\lambda}$) is defined to be the product of b_i for each horizontal step starting at a point with y-coordinate i and λ_i for each down step starting at a point with y-coordinate i. We define

$$
\begin{aligned}
\mu_{n,r,s}(\boldsymbol{b}, \boldsymbol{\lambda}) &= \sum_{\pi \in \mathrm{Mot}_{n,r,s}} \mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{\lambda}), \\
\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) &= \sum_{\pi \in \mathrm{Mot}_{n,r,s}^{\leq k}} \mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{\lambda}), \\
\mu_n(\boldsymbol{b}, \boldsymbol{\lambda}) &= \mu_{n,0,0}(\boldsymbol{b}, \boldsymbol{\lambda}), \\
\mu_n^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) &= \mu_{n,0,0}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}).
\end{aligned}
$$

Recall that $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$, $n \geq 0$, are the orthogonal polynomials defined by the three-term recurrence in (2).

Definition 2.3. The *inverted polynomial* of $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$ is defined by $P_n^*(x; \boldsymbol{b}, \boldsymbol{\lambda}) = x^n P_n(1/x; \boldsymbol{b}, \boldsymbol{\lambda})$. We also define

$$
\begin{aligned}
\delta P_n(x; \boldsymbol{b}, \boldsymbol{\lambda}) &= P_n(x; \delta\boldsymbol{b}, \delta\boldsymbol{\lambda}), \\
\delta P_n^*(x; \boldsymbol{b}, \boldsymbol{\lambda}) &= P_n^*(x; \delta\boldsymbol{b}, \delta\boldsymbol{\lambda}),
\end{aligned}
$$

where, for a sequence $\boldsymbol{s} = (s_n)_{n \geq 0}$, we denote $\delta\boldsymbol{s} = (s_{n+1})_{n \geq 0}$.

The main focus of this paper is to study the negative versions of $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$.

Definition 2.4. Let k, r, s be fixed integers. If the sequence $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ for $n = 0, 1, \dots$ satisfies a homogeneous linear recurrence relation, then we define $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ for $n = 1, 2, \dots$ in the unique way so that the sequence $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ for all $n \in \mathbb{Z}$ satisfies the recurrence. We call $\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) := \mu_{-n,0,0}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ the *negative moments* of the orthogonal polynomials $P_n(x; \boldsymbol{b}, \boldsymbol{\lambda})$.

Now we prove some basic properties of the (generalized) negative moments $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$.

Viennot [18, Ch. V, (27)] found the following generating function for $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$.

Lemma 2.5. *Let r, s and k be integers with $0 \leq r, s \leq k$. If $r \leq s$, then*

$$
\sum_{n \geq 0} \mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) x^n = \frac{x^{s-r} P_r^*(x; \boldsymbol{b}, \boldsymbol{\lambda}) \delta^{s+1} P_{k-s}^*(x; \boldsymbol{b}, \boldsymbol{\lambda})}{P_{k+1}^*(x; \boldsymbol{b}, \boldsymbol{\lambda})}. \tag{4}
$$

If $r > s$, then

$$\sum_{n \geq 0} \mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \lambda) x^n = \frac{P_s^*(x; \boldsymbol{b}, \lambda) \delta^{r+1} P_{k-r}^*(x; \boldsymbol{b}, \lambda)}{P_{k+1}^*(x; \boldsymbol{b}, \lambda)} \prod_{i=s+1}^{r} \lambda_i. \tag{5}$$

By Lemmas 2.1 and 2.5, we can also find the generating function for $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$.

Proposition 2.6. *Let r, s and k be integers with $0 \leq r, s \leq k$. Suppose that $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ is well-defined for $n \geq 1$. If $r \leq s$, then*

$$\sum_{n \geq 1} \mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda) x^n = -\frac{x P_r(x; \boldsymbol{b}, \lambda) \delta^{s+1} P_{k-s}(x; \boldsymbol{b}, \lambda)}{P_{k+1}(x; \boldsymbol{b}, \lambda)}. \tag{6}$$

If $r > s$, then

$$\sum_{n \geq 1} \mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda) x^n = -\frac{x^{r-s+1} P_s(x; \boldsymbol{b}, \lambda) \delta^{r+1} P_{k-r}(x; \boldsymbol{b}, \lambda)}{P_{k+1}(x; \boldsymbol{b}, \lambda)} \prod_{i=s+1}^{r} \lambda_i. \tag{7}$$

Using Flajolet's combinatorial theory of continued fractions [5], Viennot [18] showed that

$$\sum_{n \geq 0} \mu_n^{\leq k}(\boldsymbol{b}, \lambda) x^n = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{1 - b_2 x - \cfrac{\ddots}{\ddots - \cfrac{\lambda_k x^2}{1 - b_k x}}}}}. \tag{8}$$

There is a similar continued fraction expression for the generating function for $\mu_{-n}^{\leq k}(\boldsymbol{b}, \lambda)$.

Proposition 2.7. *If $(\mu_{-n}^{\leq k}(\boldsymbol{b}, \lambda))_{n \geq 1}$ is defined, we have*

$$\sum_{n \geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \lambda) x^n = \cfrac{-x}{x - b_0 - \cfrac{\lambda_1}{x - b_1 - \cfrac{\lambda_2}{x - b_2 - \cfrac{\ddots}{\ddots - \cfrac{\lambda_k}{x - b_k}}}}}.$$

Proof. By Lemma 2.1 and (8),

$$\sum_{n \geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \lambda) x^n = \cfrac{-1}{1 - b_0 x^{-1} - \cfrac{\lambda_1 x^{-2}}{1 - b_1 x^{-1} - \cfrac{\lambda_2 x^{-2}}{1 - b_2 x^{-1} - \cfrac{\ddots}{\ddots - \cfrac{\lambda_k x^{-2}}{1 - b_k x^{-1}}}}}}.$$

Multiplying x to the numerator and the denominator for each fraction, we obtain the desired formula. $\quad\square$

For the rest of this paper, we mainly consider the bounded moments $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ and their negatives $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ when $\boldsymbol{b} = \boldsymbol{0}$ or $\lambda = \boldsymbol{b}^2$. The choice of λ satisfying $\lambda = \boldsymbol{b}^2$ becomes more natural if we define

the weight of a Motzkin path using 'points' instead of 'steps' as follows: the *point-weight* $\text{pwt}(\pi; \boldsymbol{b})$ of a Motzkin path $\pi \in \text{Mot}_{n,r,s}$ is defined by

$$\text{pwt}(\pi; \boldsymbol{b}) = \prod_{(i,j) \in \pi} b_j.$$

If $\lambda = \boldsymbol{b}^2$, there is a simple relation between the usual weight $\text{wt}(\pi; \boldsymbol{b}, \boldsymbol{b}^2)$ and the point-weight $\text{pwt}(\pi; \boldsymbol{b})$.

Lemma 2.8. *For $\pi \in \text{Mot}_{n,r,s}$, we have*

$$\text{wt}(\pi; \boldsymbol{b}, \boldsymbol{b}^2) = \frac{b_0 \cdots b_{r-1}}{b_0 \cdots b_s} \, \text{pwt}(\pi; \boldsymbol{b}).$$

Proof. We first show this for $r = s = 0$. Suppose $\tau \in \text{Mot}_{n,0,0}$. Since each down step of τ corresponds to a unique up step, we can redistribute the weight $b_{i-1} b_i$ attached to a down step ending at height $i - 1$ in such a way that the weight of the down step is b_{i-1} and the weight of its corresponding up step ending at height i is b_i. Therefore $\text{wt}(\tau; \boldsymbol{b}, \boldsymbol{b}^2)$ is equal to the product of the new weights of the steps, where the weight of each step ending at height i is given by b_i. This is equivalent to assigning the weight b_j for each lattice point (i, j) in τ except the starting point $(0, 0)$. Thus, $\text{wt}(\tau; \boldsymbol{b}, \boldsymbol{b}^2) = b_0^{-1} \text{pwt}(\tau; \boldsymbol{b})$, which shows the lemma for $r = s = 0$.

Now consider the general case $\pi \in \text{Mot}_{n,r,s}$. Let τ be the Motzkin path obtained from π by adding r up steps at the beginning and s down steps at the end. Then

$$\text{wt}(\pi; \boldsymbol{b}, \boldsymbol{b}^2) = \frac{\text{wt}(\tau; \boldsymbol{b})}{b_0 b_1^2 \cdots b_{s-1}^2 b_s}, \qquad \text{pwt}(\pi; \boldsymbol{b}) = \frac{\text{pwt}(\tau; \boldsymbol{b})}{b_0 \cdots b_{r-1} b_0 \cdots b_{s-1}}.$$

Since $\tau \in \text{Mot}_{n+r+s,0,0}$, we have $\text{wt}(\tau; \boldsymbol{b}, \boldsymbol{b}^2) = b_0^{-1} \text{pwt}(\tau; \boldsymbol{b})$, which together with the equations above implies the desired identity. $\qquad \square$

Lemma 2.8 immediately implies the following proposition, which shows that $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2)$ is a natural point-weight generating function for Motzkin paths.

Proposition 2.9. *We have*

$$\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2) = \frac{b_0 \cdots b_{r-1}}{b_0 \cdots b_s} \sum_{\pi \in \text{Mot}_{n,r,s}^{\leq k}} \text{pwt}(\pi; \boldsymbol{b}).$$

We finish this section by giving sufficient conditions for $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ to be well-defined.

Proposition 2.10. *If $P_{k+1}(0; \boldsymbol{b}, \lambda) \neq 0$, then $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ is well-defined for $n \geq 1$.*

Proof. Since $P_{k+1}(x; \boldsymbol{b}, \lambda)$ has the nonzero constant term $P_{k+1}(0; \boldsymbol{b}, \lambda)$, its inverted polynomial $P_{k+1}^*(x; \boldsymbol{b}, \lambda)$ has degree $k + 1$. Moreover, $P_{k+1}^*(0; \boldsymbol{b}, \lambda) = 1$ because it is the leading coefficient of the monic polynomial $P_{k+1}(x; \boldsymbol{b}, \lambda)$.

Now we consider the generating function for $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ in Lemma 2.5. If $r \leq s$,

$$\deg(x^{s-r} P_r^*(x; \boldsymbol{b}, \lambda) \delta^{s+1} P_{k-s}^*(x; \boldsymbol{b}, \lambda)) \leq k < \deg(P_{k+1}^*(x; \boldsymbol{b}, \lambda)).$$

If $r > s$,

$$\deg(P_s^*(x; \boldsymbol{b}, \lambda) \delta^{r+1} P_{k-r}^*(x; \boldsymbol{b}, \lambda)) \leq s + k - r < k < \deg(P_{k+1}^*(x; \boldsymbol{b}, \lambda)).$$

Therefore, by Lemma 2.1, $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \lambda)$ is well-defined in either case. $\qquad \square$

Proposition 2.11. *The sequence* $(\mu_{-n,r,s}^{\leq k}(\mathbf{0}, \lambda))_{n \geq 1}$ *is well-defined if and only if k is odd. The sequence* $(\mu_{-n,r,s}^{\leq k}(\mathbf{b}, \mathbf{b}^2))_{n \geq 1}$ *is well-defined if and only if $k \not\equiv 1 \pmod 3$.*

Proof. Substituting $x = 0$ in (2) gives a recurrence for $P_n(0; \mathbf{b}, \lambda)$. Therefore, by induction, one can easily show that

$$P_{2k}(0; \mathbf{0}, \lambda) = (-1)^k \prod_{i=1}^{k} \lambda_{2i-1},$$

$$P_{2k+1}(0; \mathbf{0}, \lambda) = 0,$$

$$P_{3k}(0; \mathbf{b}, \mathbf{b}^2) = b_0 \cdots b_{3k-1},$$

$$P_{3k+1}(0; \mathbf{b}, \mathbf{b}^2) = -b_0 \cdots b_{3k},$$

$$P_{3k+2}(0; \mathbf{b}, \mathbf{b}^2) = 0.$$

Then the proof follows from Proposition 2.10. $\qquad\square$

3. Reciprocity for bounded Dyck paths

In this section, we introduce a method to compute negative moments using continued fractions. Using this method, we give a combinatorial model for $\mu_{-n}^{\leq k}(\mathbf{0}, \lambda)$, which is equivalent to Cigler and Krattenthaler's result stated in Theorem 1.1.

We begin with the following definitions.

Definition 3.1. An *ℓ-peak-valley sequence* (*ℓ-PV sequence* for short) is a sequence $(a_1, \ldots, a_n)$ of nonnegative integers such that for $i = 1, \ldots, n$,

○ if $a_i \equiv 0 \pmod \ell$, then a_i is a valley; that is, $a_{i-1} > a_i < a_{i+1}$,
○ if $a_i \equiv -1 \pmod \ell$, then a_i is a peak; that is, $a_{i-1} < a_i > a_{i+1}$,

where we set $a_0 = a_{n+1} = 0$. Let $\mathrm{PV}_n^{\ell,k}$ denote the set of all ℓ-PV sequences $(a_1, \ldots, a_n)$ with bound k (i.e., $0 \leq a_i \leq k$ for all $i = 1, \ldots, n$).

Definition 3.2. We define the *weight* $\mathrm{wt}(\pi)$ of a sequence $\pi = (a_1, \ldots, a_n)$ of nonnegative integers by $\mathrm{wt}(\pi) = V_{a_1} \cdots V_{a_n}$, where V_i's are indeterminates.

For convenience, we define $\mathrm{PV}_0^{\ell,k} = \{\emptyset\}$, where $\emptyset$ is the empty sequence with $\mathrm{wt}(\emptyset) = 1$.

In this paper, we only need to consider ℓ-PV sequences for $\ell = 2, 3$. In this section, (resp. Section 4) we will show that if $\mathbf{b} = \mathbf{0}$ (resp. $\lambda = \mathbf{b}^2$), the negative moment $\mu_{-n}^{\leq k}(\mathbf{b}, \lambda)$ is a generating function for certain 2-PV sequences (resp. 3-PV sequences).

Note that a sequence is a 2-PV sequence if and only if every even integer is a valley and every odd integer is a peak assuming a zero is padded at both ends. For example, $(3, 2, 7, 0, 1)$ is a 2-PV sequence because the odd integers $3, 7, 1$ are peaks and the even integers $2, 0$ are valleys. Equivalently, a sequence $(a_1, \ldots, a_n)$ is a 2-PV sequence if and only if n is odd, $a_1 > a_2 < a_3 > \cdots$, each a_{2i-1} is odd and a_{2i} is even.

Recall that $\mathrm{Alt}_n^{\leq k}$ is the set of all sequences $(b_1, \ldots, b_n)$ such that $b_1 \leq b_2 \geq b_3 \leq \cdots$ and $1 \leq b_i \leq k$. There is a close connection between alternating sequences and 2-PV sequences as follows.

Proposition 3.3. *The map from $\mathrm{PV}_{2n+1}^{2,2k-1}$ to $\mathrm{Alt}_{2n+1}^{\leq k}$ defined by*

$$(a_1, \ldots, a_{2n+1}) \mapsto (k - \lfloor a_1/2 \rfloor, \ldots, k - \lfloor a_{2n+1}/2 \rfloor)$$

is a bijection.

Proof. One can easily see that the map from $\mathrm{Alt}_{2n+1}^{\leq k}$ to $\mathrm{PV}_{2n+1}^{2,2k-1}$ defined by $(b_1, \ldots, b_{2n+1}) \mapsto (c_1, \ldots, c_{2n+1})$, where $c_{2i+1} = 2(k - b_{2i+1}) + 1$ and $c_{2i} = 2(k - b_{2i})$, is the inverse of the given map. $\quad\square$

There is a simple continued fraction expression for the generating function for 2-PV sequences.

Proposition 3.4. *For an integer $k \geq 1$, we have*

$$\sum_{n \geq 0} \sum_{\pi \in \mathrm{PV}_{2n+1}^{2,2k-1}} \mathrm{wt}(\pi)x^{2n+1} = \cfrac{1}{-V_0 x - \cfrac{1}{-V_1 x - \cfrac{}{\ddots - \cfrac{1}{-V_{2k-1}x}}}}.$$

Proof. The proof is by induction on k. If $k = 1$, since there is only one 2-PV sequence $(1, 0, 1, \ldots, 0, 1)$ of length $2n + 1$ with bound 1, we have

$$\sum_{n \geq 0} \sum_{\pi \in \mathrm{PV}_{2n+1}^{2,1}} \mathrm{wt}(\pi)x^{2n+1} = \sum_{n \geq 0} V_1 x (V_1 V_0 x^2)^n = \frac{V_1 x}{1 - V_1 V_0 x^2}.$$

Hence, it is true for $k = 1$.

Suppose $k > 1$, and let $\overline{\mathrm{PV}}_{2n+1}^{2,2k-1}$ be the set of sequences $(a_1, \ldots, a_{2n+1})$ in $\mathrm{PV}_{2n+1}^{2,2k-1}$ such that $a_i \geq 2$ for all $i = 1, \ldots, 2n + 1$. For convenience, let

$$\mathcal{A} = \bigcup_{n \geq 0} \mathrm{PV}_{2n+1}^{2,2k-1} \quad \text{and} \quad \overline{\mathcal{A}} = \bigcup_{n \geq 0} \overline{\mathrm{PV}}_{2n+1}^{2,2k-1}.$$

Then, by induction with indices shifted properly, it is enough to show that

$$\sum_{\pi \in \mathcal{A}} \mathrm{wt}(\pi)x^{|\pi|} = \cfrac{1}{-V_0 x - \cfrac{1}{-V_1 x - \sum_{\pi \in \overline{\mathcal{A}}} \mathrm{wt}(\pi)x^{|\pi|}}}, \tag{9}$$

where if $\pi = (a_1, \ldots, a_n)$, we denote $|\pi| = n$.

Let $A = (a_1, \ldots, a_{2n+1}) \in \mathcal{A}$. We divide A into subsequences using the locations of 0's as follows. Let $i_1, \ldots, i_m$ be the indices j such that $a_j = 0$, where $i_1 < \cdots < i_m$. Let $A_0 = (a_1, \ldots, a_{i_1-1})$ and $A_j = (a_{i_j}, \ldots, a_{i_{j+1}-1})$ for $j = 1, \ldots, m$, where $i_{m+1} - 1 = 2n + 1$, so that A is the concatenation of $A_0, A_1, \ldots, A_m$. For example, if $A = (3, 2, 7, 0, 1, 0, 5, 2, 3, 0, 7, 6, 7)$, then $A_0 = (3, 2, 7)$, $A_1 = (0, 1)$, $A_2 = (0, 5, 2, 3)$ and $A_3 = (0, 7, 6, 7)$.

Since every even integer is a valley and every odd integer is a peak in A, one can easily check the following:

○ A_0 is either (1) or an element in $\overline{\mathcal{A}}$,
○ for each $1 \leq j \leq m$, A_j is either $(0, 1)$ or $(0, y)$ for some $y \in \overline{\mathcal{A}}$.

Conversely, any choice of $A_0, A_1, \ldots, A_m$ satisfying the above conditions gives an element in $\mathcal{A}$. Therefore, if we set $S = \sum_{\pi \in \overline{\mathcal{A}}} \mathrm{wt}(\pi)x^{|\pi|}$, then

$$\sum_{\pi \in \mathcal{A}} \mathrm{wt}(\pi)x^{|\pi|} = \sum_{m \geq 0} (V_1 x + S)(V_0 V_1 x^2 + V_0 x S)^m = \frac{V_1 x + S}{1 - V_0 V_1 x^2 - V_0 x S}.$$

Dividing the numerator and the denominator by $V_1 x + S$, we obtain (9), and the proof follows by induction. □

We now give a combinatorial interpretation for $\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \lambda)$ using 2-PV sequences.

Theorem 3.5. *Suppose that the sequence $\lambda = (\lambda_i)_{i \geq 1}$ is given by $\lambda_i = V_{i-1}^{-1} V_i^{-1}$ for $i \geq 1$. Then we have*

$$\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \lambda) = V_0 \sum_{\pi \in PV_{2n-1}^{2,2k-1}} \mathrm{wt}(\pi).$$

Proof. Let $\lambda_0 = V_0^{-1}$. Observe that for each integer $m \geq 0$, $\lambda_m^{-1} \lambda_{m-1} \cdots \lambda_0^{(-1)^{m+1}} = V_m$. By Proposition 2.7,

$$\sum_{n \geq 1} \mu_{-n}^{\leq 2k-1}(\mathbf{0}, \lambda) x^n = \cfrac{-x}{x - \cfrac{\lambda_1}{x - \cfrac{\lambda_2}{x - \cfrac{\ddots}{ - \cfrac{\lambda_{2k-1}}{x}}}}}$$

$$= \cfrac{\lambda_0^{-1} x}{-\lambda_0^{-1} x - \cfrac{1}{-\lambda_1^{-1} \lambda_0 x - \cfrac{1}{-\lambda_2^{-1} \lambda_1 \lambda_0^{-1} x - \cfrac{\ddots}{ - \cfrac{1}{-\lambda_{2k-1}^{-1} \lambda_{2k-2} \cdots \lambda_0^{(-1)^{2k}} x}}}}}$$

$$= \cfrac{V_0 x}{-V_0 x - \cfrac{1}{-V_1 x - \cfrac{\ddots}{ - \cfrac{1}{-V_{2k-1} x}}}}.$$

Then the proof follows from Proposition 3.4.   □

By the above theorem with $\lambda = \mathbf{1}$, we get the following corollary.

Corollary 3.6. *We have*

$$\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \mathbf{1}) = \left| PV_{2n-1}^{2,2k-1} \right|.$$

By Proposition 3.3, Corollary 3.6 is equivalent to Theorem 1.1 due to Cigler and Krattenthaler. Moreover, using Proposition 3.3, one can easily check that Theorem 3.5 is equivalent to the following proposition, which is a weighted version of Theorem 1.1.

Proposition 3.7 [3, Corollary 32]. *Suppose that $\lambda_{2i-1} = V_i^{-1} A_i^{-1}$ and $\lambda_{2i} = A_i^{-1} V_{i+1}^{-1}$ for all $i \geq 1$, and let $\lambda_0 = V_1^{-1}$. Then we have*

$$\mu_{-2n}^{\leq 2k-1}(\mathbf{0}, \lambda) = V_1 R_{AV}^{(k)} \left(\sum_{\pi \in \mathrm{Alt}_{2n-1}^{\leq k}} \mathrm{wt}_{AV}(\pi) \right),$$

where the operator $R_{AV}^{(k)}$ replaces A_i by V_{k+1-i} and V_i by A_{k+1-i}, and

$$\mathrm{wt}_{AV}(\pi) = V_{a_1} V_{a_3} \cdots V_{a_{2n-1}} A_{a_2} A_{a_4} \cdots A_{a_{2n-2}}. \tag{10}$$

4. Reciprocity for bounded Motzkin paths

In this section, we find a combinatorial interpretation for $\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2)$. We only need to consider the case $k \not\equiv 1 \pmod 3$ because, otherwise, $\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2)$ is not defined by Proposition 2.11. We show that $\mu_{-n}^{\leq 3k-1}(\boldsymbol{b}, \boldsymbol{b}^2)$ is a generating function for 3-PV sequences (Theorem 4.2) and $\mu_{-n}^{\leq 3k}(\boldsymbol{b}, \boldsymbol{b}^2)$ is a generating function for modified 3-PV sequences (Theorem 4.5).

Recall that a sequence $(a_1, \ldots, a_n)$ is a 3-PV sequence if each a_i is a valley when $a_i \equiv 0 \pmod 3$, and a_i is a peak when $a_i \equiv 2 \pmod 3$.

Using arguments similar to those in the previous section, we find a continued fraction expression for the generating function for 3-PV sequences.

Proposition 4.1. *For an integer $k \geq 1$, we have*

$$\sum_{n \geq 1} \sum_{\pi \in \mathrm{PV}_n^{3,3k-1}} \mathrm{wt}(\pi) x^n = \cfrac{1}{-V_0 x - 1 - \cfrac{1}{-V_1 x - 1 - \cfrac{1}{\ddots - \cfrac{1}{-V_{3k-1} x - 1}}}}.$$

Proof. The proof is similar to (but slightly more complicated than) that of Proposition 3.4. Let $\overline{\mathrm{PV}}_n^{3,3k-1}$ be the set of sequences $(a_1, \ldots, a_n)$ in $\mathrm{PV}_n^{3,3k-1}$ such that $a_i \geq 3$ for all $i = 1, \ldots, n$, and let

$$\mathcal{A} = \bigcup_{n \geq 0} \mathrm{PV}_n^{3,3k-1} \quad \text{and} \quad \overline{\mathcal{A}} = \bigcup_{n \geq 0} \overline{\mathrm{PV}}_n^{3,3k-1}.$$

We first claim that

$$\sum_{\pi \in \mathcal{A}} \mathrm{wt}(\pi) x^{|\pi|} = \cfrac{1}{-V_0 x - 1 - \cfrac{1}{-V_1 x - 1 - \cfrac{1}{-V_2 x - 1 - \sum_{\pi \in \overline{\mathcal{A}}} \mathrm{wt}(\pi) x^{|\pi|}}}}. \tag{11}$$

It is easy to see that the proposition follows from the claim by induction on k. Therefore, it suffices to prove the claim (11).

For a sequence $\pi = (a_1, \ldots, a_n) \in \mathcal{A}$, let $i_1, \ldots, i_m$ be the indices j such that $a_j = 0$ or $a_{j-1} \geq a_j = 1$, where $i_1 < \cdots < i_m$. Let $A_0 = (a_1, \ldots, a_{i_1-1})$ and $A_j = (a_{i_j}, \ldots, a_{i_{j+1}-1})$ for $j = 1, \ldots, m$, where $i_{m+1} - 1 = n$, so that π is the concatenation of $A_0, A_1, \ldots, A_m$.

Observe that the possible sequences for A_0 are $(1), (1, 2), (2), (1, y)$ and (y) where $y \in \overline{\mathcal{A}}$. For $1 \leq j \leq m$, the first entry of A_j is 0 or 1. If the first entry is 0, the possible sequences for A_j are $(0, 1), (0, 1, 2), (0, 2), (0, 1, y), (0, y)$, where $y \in \overline{\mathcal{A}}$, and if the first entry is 1, the possible sequences for A_j are $(1), (1, 2), (1, y)$ where $y \in \overline{\mathcal{A}}$. Hence, if we set $S = \sum_{\pi \in \overline{\mathcal{A}}} \mathrm{wt}(\pi) x^{|\pi|}$, then we have

$$\sum_{\pi \in \mathcal{A}} \mathrm{wt}(\pi) x^{|\pi|} = \sum_{m \geq 0} (V_1 x + V_1 V_2 x^2 + V_2 x + V_1 x S + S)$$

$$\times (V_0 x (V_1 x + V_1 V_2 x^2 + V_2 x + V_1 x S + S) + V_1 x (1 + V_2 x + S))^m$$

$$= \frac{V_1 x + V_1 V_2 x^2 + V_2 x + V_1 x S + S}{1 - V_0 x (V_1 x + V_1 V_2 x^2 + V_2 x + V_1 x S + S) - V_1 x (1 + V_2 x + S)},$$

which is easily seen to be equal to the right-hand side of (11). This completes the proof. $\square$

Using Proposition 4.1, we can find a combinatorial interpretation for $\mu_{-n}^{\leq 3k-1}(\boldsymbol{b}, \boldsymbol{b}^2)$.

Theorem 4.2. *Let $\boldsymbol{b} = (b_i)_{i \geq 0}$ and $\boldsymbol{\lambda} = (\lambda_i)_{i \geq 1}$ be the sequences given by $b_i = -V_i^{-1}$ and $\lambda_i = V_i^{-1}V_{i-1}^{-1}$ for all i. Then we have*

$$\mu_{-n}^{\leq 3k-1}(\boldsymbol{b}, \boldsymbol{\lambda}) = V_0 \sum_{\pi \in \text{PV}_{n-1}^{3,3k-1}} \text{wt}(\pi).$$

Proof. By Proposition 2.7, we have

$$\sum_{n \geq 1} \mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})x^n = \cfrac{-x}{x - b_0 - \cfrac{\lambda_1}{x - b_1 - \cfrac{\lambda_2}{x - b_2 - \ddots - \cfrac{\lambda_k}{x - b_k}}}}$$

$$= \cfrac{b_0^{-1}x}{1 - b_0^{-1}x - \cfrac{b_0^{-1}b_1^{-1}\lambda_1}{1 - b_1^{-1}x - \cfrac{b_1^{-1}b_2^{-1}\lambda_2}{1 - b_2^{-1}x - \ddots - \cfrac{b_{k-1}^{-1}b_k^{-1}\lambda_k}{1 - b_k^{-1}x}}}}$$

$$= \cfrac{V_0 x}{-V_0 x - 1 - \cfrac{1}{-V_1 x - 1 - \cfrac{1}{-V_2 x - 1 - \ddots - \cfrac{1}{-V_k x - 1}}}}.$$

The proof follows from Proposition 4.1. □

Now we find a combinatorial interpretation for $\mu_{-n}^{\leq 3k}(\boldsymbol{b}, \boldsymbol{b}^2)$. To this end, we need the following definition.

Definition 4.3. A *modified 3-PV sequence* is a sequence $(a_1, \ldots, a_n)$ of nonnegative integers such that for $i = 1, \ldots, n$,

- if $a_i \equiv 1 \pmod 3$, then a_i is a valley; that is, $a_{i-1} > a_i < a_{i+1}$,
- if $a_i \equiv 2 \pmod 3$, then a_i is a peak; that is, $a_{i-1} < a_i > a_{i+1}$,

where we set $a_0 = a_{n+1} = 0$. Let $\widetilde{\text{PV}}_n^{3,k}$ denote the set of all modified 3-PV sequences of length n with bound k (i.e., $0 \leq a_i \leq k$ for all i).

Similar to Proposition 4.1, there is a continued fraction expression for the generating function for $\widetilde{\text{PV}}_n^{3,3k}$ (see the proposition below). We note, however, that the proof of Proposition 4.4 is different from that of Proposition 4.1 due to the fact that in Proposition 4.4, the sum is over $n \geq 0$ whereas in Proposition 4.1 the sum is over $n \geq 1$.

Proposition 4.4. *For an integer $k \geq 1$, we have*

$$\sum_{n \geq 0}(-1)^{n+1} \sum_{\pi \in \widetilde{\text{PV}}_n^{3,3k}} \text{wt}(\pi)x^n = \cfrac{1}{-V_0 x - 1 - \cfrac{1}{-V_1 x - 1 - \ddots - \cfrac{1}{-V_{3k} x - 1}}}.$$

Proof. Let $\mathcal{A} = \cup_{n \geq 0} \widetilde{\mathrm{PV}}_n^{3,3k}$, and let $\mathcal{B}$ be the set of sequences $\beta = (b_1, \ldots, b_m)$ for $m \geq 0$ such that b_i is a valley if $b_i \equiv 1 \pmod 3$, b_i is a peak if $b_i \equiv 0 \pmod 3$ and $1 \leq b_i \leq 3k$ for all i, where we set $b_0 = b_{m+1} = 0$. Here $(b_1, \ldots, b_m)$ means the empty sequence $\emptyset$ if $m = 0$. By Proposition 4.1, we have

$$\sum_{\beta \in \mathcal{B}} \mathrm{wt}(\beta) x^{|\beta|} = 1 + \cfrac{1}{-V_1 x - 1 - \cfrac{1}{-V_2 x - 1 - \cfrac{\ddots}{\ddots - \cfrac{1}{-V_{3k} x - 1}}}}.$$

We claim that

$$\sum_{\alpha \in \mathcal{A}} \mathrm{wt}(\alpha)(-x)^{|\alpha|} \left(V_0 x + \sum_{\beta \in \mathcal{B}} \mathrm{wt}(\beta) x^{|\beta|} \right) = 1. \tag{12}$$

For $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B} \cup \{(0)\}$, define the weight $\overline{\mathrm{wt}}(\alpha, \beta)$ of the pair (α, β) to be $(-1)^{|\alpha|} \mathrm{wt}(\alpha) \mathrm{wt}(\beta)$. To prove the claim, it suffices to find a sign-reversing involution φ from $\mathcal{A} \times (\mathcal{B} \cup \{(0)\})$ to itself with unique fixed point $(\emptyset, \emptyset)$, where $\emptyset$ is the empty sequence.

For a nonempty sequence $\alpha = (a_1, \ldots, a_m) \in \mathcal{A}$, define $I(\alpha)$ to be the largest integer i such that $1 \leq i \leq m - 1$ and $a_i, a_{i+1} \not\equiv 1 \pmod 3$. If there is no such i, we define $I(\alpha) = 0$. Similarly, for a nonempty sequence $\beta = (b_1, \ldots, b_n) \in \mathcal{B} \cup \{(0)\}$, define $J(\beta)$ to be the smallest integer j such that $1 \leq j \leq n - 1$ and $b_j, b_{j+1} \not\equiv 1 \pmod 3$. If there is no such j, we define $J(\beta) = n$. One can check that $m - I(\alpha)$ and $J(\beta)$ are odd. Moreover, $a_{I(\alpha)+1} > a_{I(\alpha)+2} < \cdots < a_m$ and $b_1 > b_2 < \cdots < b_{J(\beta)}$.

We define the map φ as follows. For $\alpha = (a_1, \ldots, a_m) \in \mathcal{A}$ and $\beta = (b_1, \ldots, b_n) \in \mathcal{B} \cup \{(0)\}$,

1. define $\varphi(\alpha, \beta) = ((a_1, \ldots, a_m, b_1, \ldots, b_{J(\beta)}), (b_{J(\beta)+1}, \ldots, b_n))$ if one of the following conditions is satisfied:

$$\begin{cases} m = 0, \\ a_m \equiv 0 \pmod 3 \quad \text{and} \quad b_1 \equiv 0 \pmod 3, \\ a_m \equiv 2 \pmod 3 \quad \text{and} \quad b_1 \equiv 0 \pmod 3 \text{ with } a_m > b_1, \\ a_m \equiv 0 \pmod 3 \quad \text{and} \quad b_1 \equiv 2 \pmod 3 \text{ with } a_m < b_1, \end{cases}$$

2. define $\varphi(\alpha, \beta) = ((a_1, \ldots, a_{I(\alpha)}), (a_{I(\alpha)+1}, \ldots, a_m, b_1, \ldots, b_n))$ if one of the following conditions is satisfied:

$$\begin{cases} n = 0, \\ a_m \equiv 2 \pmod 3 \quad \text{and} \quad b_1 \equiv 2 \pmod 3, \\ a_m \equiv 2 \pmod 3 \quad \text{and} \quad b_1 \equiv 0 \pmod 3 \text{ with } a_m < b_1, \\ a_m \equiv 0 \pmod 3 \quad \text{and} \quad b_1 \equiv 2 \pmod 3 \text{ with } a_m > b_1. \end{cases}$$

Then it is not hard to see that the map φ is a sign-reversing involution with unique fixed point $(\emptyset, \emptyset)$, which proves the claim. For example, let $\alpha = (a_1, a_2, a_3, a_4, a_5) = (2, 0, 3, 1, 5) \in \mathcal{A}$ and $\beta = (b_1, b_2, b_3, b_4) = (2, 3, 1, 2) \in \mathcal{B}$. Then $I(\alpha) = 2$ since $a_2, a_3 \not\equiv 1 \pmod 3$, and $J(\beta) = 1$ since $b_1, b_2 \not\equiv 1 \pmod 3$. Since $a_5 \equiv 2 \pmod 3$ and $b_1 \equiv 2 \pmod 3$, it satisfies the second condition of the case (2), so $\varphi(\alpha, \beta) = ((2, 0), (3, 1, 5, 2, 3, 1, 2))$. Moreover, one can easily check that $\varphi((2, 0), (3, 1, 5, 2, 3, 1, 2)) = ((2, 0, 3, 1, 5), (2, 3, 1, 2)) = (\alpha, \beta)$.

By the claim (12), we have

$$
-\sum_{\alpha \in \mathcal{A}} \mathrm{wt}(\alpha)(-x)^{|\alpha|} = \frac{1}{-V_0 x - \sum_{\beta \in \mathcal{B}} \mathrm{wt}(\beta) x^{|\beta|}}
$$

$$
= \cfrac{1}{-V_0 x - 1 - \cfrac{1}{-V_1 x - 1 - \cfrac{1}{\ddots - \cfrac{1}{-V_{3k} x - 1}}}},
$$

which completes the proof.   □

Similar to Theorem 4.2, using Proposition 4.4, we can find a combinatorial interpretation for $\mu_{-n}^{\leq 3k}(\boldsymbol{b}, \boldsymbol{b}^2)$. We omit the proof.

Theorem 4.5. *Let $\boldsymbol{b} = (b_i)_{i \geq 0}$ and $\boldsymbol{\lambda} = (\lambda_i)_{i \geq 1}$ be the sequences given by $b_i = -V_i^{-1}$ and $\lambda_i = V_i^{-1} V_{i-1}^{-1}$ for all i. Then we have*

$$
\mu_{-n}^{\leq 3k}(\boldsymbol{b}, \boldsymbol{\lambda}) = V_0 \sum_{\pi \in \widetilde{\mathrm{PV}}_{n-1}^{3,3k}} \mathrm{wt}(\pi).
$$

5. Negative moments using inverse matrices

In this section, we generalize Theorems 4.2 and 4.5 using inverse matrices.

For integers k and i with $0 \leq i \leq k$, let $\epsilon_i^{\leq k}$ be the standard basis vector in $\mathbb{R}^{k+1}$ such that the ith entry is equal to 1 and the other entries are all 0. If the size of $\epsilon_i^{\leq k}$ is clear from the context, we will simply write it as ϵ_i. We also define the tridiagonal matrix $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ by

$$
A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \begin{pmatrix} b_0 & 1 & & & \\ \lambda_1 & b_1 & 1 & & \\ & & \ddots & & \\ & & \lambda_{k-1} & b_{k-1} & 1 \\ & & & \lambda_k & b_k \end{pmatrix}. \tag{13}
$$

By the definition of $\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$, it is easy to see that

$$
\mu_{n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) \right)^n \epsilon_s. \tag{14}
$$

The next proposition shows that $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ can be computed similarly using the inverse of $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$. This is essentially the same as [8, Lemma 2.7] due to Hopkins and Zaimi, which was first presented in [20].

Proposition 5.1 [8, Lemma 2.7]. *For nonnegative integers r, s, k, n with $r, s \leq k$ and $n \geq 1$, if $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ is invertible, then*

$$
\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) \right)^{-n} \epsilon_s. \tag{15}
$$

Proof. Let $x^m + c_{m-1} x^{m-1} + \cdots + c_0$ be the minimal polynomial of $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ so that

$$
(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}))^m + c_{m-1}(A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}))^{m-1} + \cdots + c_0 I = O,
$$

where I (resp. O) is the identity matrix (resp. zero matrix). For each $N \in \mathbb{Z}$, multiplying $(A^{\leq k}(\boldsymbol{b}, \lambda))^{N-m}$ and then multiplying ϵ_r^T and ϵ_s on the left and right, respectively, in the above equation, we obtain

$$\epsilon_r^T (A^{\leq k}(\boldsymbol{b}, \lambda))^N \epsilon_s + c_{m-1} \epsilon_r^T (A^{\leq k}(\boldsymbol{b}, \lambda))^{N-1} \epsilon_s + \cdots + c_0 \epsilon_r^T (A^{\leq k}(\boldsymbol{b}, \lambda))^{N-m} \epsilon_s = 0.$$

Therefore, $(\epsilon_r^T (A^{\leq k}(\boldsymbol{b}, \lambda))^N \epsilon_s)_{N \in \mathbb{Z}}$ is the sequence that is extended from $(\mu_{N,r,s}^{\leq k}(\boldsymbol{b}, \lambda))_{N \geq 0} = (\epsilon_r^T (A^{\leq k}(\boldsymbol{b}, \lambda))^N \epsilon_s)_{N \geq 0}$ by the above linear recurrence relation, which implies (15). $\square$

Usmani [17] found a formula for the inverse of a general tridiagonal matrix. Specializing Usmani's result to the tridiagonal matrix $A^{\leq k}(\boldsymbol{b}, \lambda)$, we obtain the following lemma.

Lemma 5.2. *Suppose that $A^{\leq k}(\boldsymbol{b}, \lambda)$ is invertible and let $\left(A^{\leq k}(\boldsymbol{b}, \lambda)\right)^{-1} = (\alpha_{i,j})_{0 \leq i,j \leq k}$. Then*

$$\alpha_{i,j} = \begin{cases} (-1)^{i+j} \theta_i \phi_{j+2}/\theta_{k+1} & \text{if } i \leq j, \\ (-1)^{i+j} \lambda_j \cdots \lambda_{i-1} \theta_j \phi_{i+2}/\theta_{k+1} & \text{if } i > j, \end{cases} \tag{16}$$

where θ_i and ϕ_i are defined by

$$\begin{aligned} \theta_i &= b_{i-1} \theta_{i-1} - \lambda_{i-1} \theta_{i-2}, & i &= 1, 2, \ldots, k+1, \\ \phi_i &= b_{i-1} \phi_{i+1} - \lambda_i \phi_{i+2}, & i &= k+1, k, \ldots, 1, \end{aligned}$$

with initial conditions $\phi_{k+2} = \theta_0 = 1$ and $\phi_{k+3} = \theta_{-1} = 0$.

The next lemma shows that if $\lambda = \boldsymbol{b}^2$, then there is a simple explicit formula for $\alpha_{i,j}$ in Lemma 5.2.

Lemma 5.3. *Let $\boldsymbol{b} = (b_i)_{i \geq 0}$ and $\lambda = (\lambda_i)_{i \geq 1}$ be the sequences given by $b_i = -V_i^{-1}$ and $\lambda_i = V_i^{-1} V_{i-1}^{-1}$ for all i. Suppose that $A^{\leq k}(\boldsymbol{b}, \lambda)$ is invertible and let $\left(A^{\leq k}(\boldsymbol{b}, \lambda)\right)^{-1} = (\alpha_{i,j})_{0 \leq i,j \leq k}$. Then*

$$\alpha_{i,j} = (-1)^{\lfloor \frac{i}{3} \rfloor + \lfloor \frac{j}{3} \rfloor} \frac{V_0 \cdots V_j}{V_0 \cdots V_{i-1}} \chi_{i,j},$$

where for $k \equiv -1 \pmod 3$,

$$\chi_{i,j} = \begin{cases} 0 & \text{if } i \equiv -1 \pmod 3 \text{ and } i \leq j, \\ 0 & \text{if } j \equiv 0 \pmod 3 \text{ and } i \leq j, \\ 0 & \text{if } i \equiv 0 \pmod 3 \text{ and } i > j, \\ 0 & \text{if } j \equiv -1 \pmod 3 \text{ and } i > j, \\ 1 & \text{otherwise,} \end{cases}$$

and for $k \equiv 0 \pmod 3$,

$$\chi_{i,j} = \begin{cases} 0 & \text{if } i \equiv -1 \pmod 3 \text{ and } i \leq j, \\ 0 & \text{if } j \equiv 1 \pmod 3 \text{ and } i \leq j, \\ 0 & \text{if } i \equiv 1 \pmod 3 \text{ and } i > j, \\ 0 & \text{if } j \equiv -1 \pmod 3 \text{ and } i > j, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By induction on i, one can easily verify that the θ_i's and ϕ_i's in Lemma 5.2 are given by

$$\begin{aligned} \theta_{3i} &= V_0^{-1} \cdots V_{3i-1}^{-1}, & \phi_{k+1-3i} &= -V_k^{-1} \cdots V_{k-3i}^{-1}, \\ \theta_{3i+1} &= -V_0^{-1} \cdots V_{3i}^{-1}, & \phi_{k+2-3i} &= V_k^{-1} \cdots V_{k+1-3i}^{-1}, \\ \theta_{3i+2} &= 0, & \phi_{k+3-3i} &= 0. \end{aligned}$$

If $k \equiv -1 \pmod 3$, then these can be written as

$$
\theta_i = \begin{cases} 0 & \text{if } i \equiv -1 \pmod 3, \\ (-1)^{i-3\lfloor \frac{i}{3} \rfloor} V_0^{-1} \cdots V_{i-1}^{-1} & \text{otherwise,} \end{cases}
$$

$$
\phi_i = \begin{cases} 0 & \text{if } i \equiv -1 \pmod 3, \\ (-1)^{i-3\lfloor \frac{i}{3} \rfloor +1} V_{i-1}^{-1} \cdots V_k^{-1} & \text{otherwise.} \end{cases}
$$

Substituting the formulas for ϕ_i and θ_i to (16) completes the proof for $k \equiv -1 \pmod 3$. One can obtain the result for $k \equiv 0 \pmod 3$ in the same way. □

Note that if $k \equiv 1 \pmod 3$, then $\alpha_{i,j}$ in (16) is not defined (i.e., $A^{\leq k}(\boldsymbol{b}, \lambda)$ is not invertible). Using Proposition 5.1 and Lemma 5.2, we can give a combinatorial interpretation for $\mu_{-n,r,s}^{\leq k}(\boldsymbol{b}, \boldsymbol{b}^2)$. To do this, we first need to define (ℓ, r, s)-peak-valley sequences, which are a generalization of ℓ-peak-valley sequences in Definition 3.1.

Definition 5.4. An (ℓ, r, s)-*peak-valley sequence* is a sequence $(a_1, \ldots, a_n)$ of nonnegative integers such that for $i = 0, \ldots, n+1$,

○ if $a_i \equiv 0 \pmod \ell$, then a_i is a valley; that is, $a_{i-1} > a_i < a_{i+1}$,
○ if $a_i \equiv -1 \pmod \ell$, then a_i is a peak; that is, $a_{i-1} < a_i > a_{i+1}$,

where we set $a_0 = r$ and $a_{n+1} = s$. Here, we ignore the inequalities involving a_t for $t = -1$ or $t = n+2$. Denote by $\mathrm{PV}_{n,r,s}^{\ell,k}$ the set of (ℓ, r, s)-peak-valley sequences $(a_1, \ldots, a_n)$ with bound k, i.e., $0 \leq a_i \leq k$ for all $i = 1, \ldots, n$.

Note that Definition 5.4 reduces to Definition 3.1 when $r = s = 0$. Cigler and Krattenthaler [3, Theorem 28] found a combinatorial description of $\mu_{-n,r,s}^{\leq 2k-1}(\boldsymbol{0}, \lambda)$. The next theorem gives a combinatorial interpretation for $\mu_{-n,r,s}^{\leq 3k-1}(\boldsymbol{b}, \lambda)$ when $b_i = -V_i^{-1}$ and $\lambda_i = V_i^{-1} V_{i-1}^{-1}$ for all i. Note that this theorem reduces to Theorem 4.2 if $r = s = 0$.

Theorem 5.5. *Suppose that* $\boldsymbol{b} = (b_i)_{i \geq 0}$ *and* $\lambda = (\lambda_i)_{i \geq 1}$ *are the sequences given by* $b_i = -V_i^{-1}$ *and* $\lambda_i = V_i^{-1} V_{i-1}^{-1}$ *for all* i. *Then*

$$
\mu_{-n,r,s}^{\leq 3k-1}(\boldsymbol{b}, \lambda) = (-1)^{\lfloor r/3 \rfloor + \lfloor s/3 \rfloor} \frac{V_0 \cdots V_s}{V_0 \cdots V_{r-1}} \sum_{\pi \in \mathrm{PV}_{n-1,r,s}^{3,3k-1}} \mathrm{wt}(\pi).
$$

Here, we set $V_0 \cdots V_{r-1} = 1$ *if* $r = 0$.

Proof. Let $a_0 = r$, $a_n = s$ and $(\alpha_{i,j})_{0 \leq i,j \leq 3k-1} = \left(A^{\leq 3k-1}(\boldsymbol{b}, \lambda)\right)^{-1}$. By Proposition 5.1,

$$
\mu_{-n,r,s}^{\leq 3k-1}(\boldsymbol{b}, \lambda) = \epsilon_r^T \left(A^{\leq 3k-1}(\boldsymbol{b}, \lambda)\right)^{-n} \epsilon_s = \sum_{(a_1, \ldots, a_{n-1}) \in X} \prod_{i=0}^{n-1} \alpha_{a_i, a_{i+1}}, \tag{17}
$$

where X is the set of sequences $(a_1, \ldots, a_{n-1})$ of integers with $0 \leq a_i \leq 3k-1$ for all i.

For $(a_1, \ldots, a_{n-1}) \in X$, we claim that $\prod_{i=0}^{n-1} \alpha_{a_i, a_{i+1}} = 0$ unless $(a_1, \ldots, a_{n-1}) \in \mathrm{PV}_{n-1,r,s}^{3,3k-1}$. To see this, suppose $(a_1, \ldots, a_{n-1}) \notin \mathrm{PV}_{n-1,r,s}^{3,3k-1}$. Then there is an integer $0 \leq j \leq n$ satisfying one of the following two conditions:

○ $a_j \equiv 0 \pmod 3$ and a_j is not a valley,
○ $a_j \equiv -1 \pmod 3$ and a_j is a not peak.

First, suppose $a_j \equiv 0 \pmod 3$ and a_j is not a valley. Then $a_{j-1} \leq a_j$ or $a_j \geq a_{j+1}$. By Lemma 5.3, $a_{j-1} \leq a_j$ implies $\alpha_{a_{j-1}, a_j} = 0$ and each of $a_j = a_{j+1}$ and $a_j > a_{j+1}$ implies $\alpha_{a_j, a_{j+1}} = 0$. Hence,

we always have $\prod_{i=0}^{n-1} \alpha_{a_i,a_{i+1}} = 0$. Similarly, one can prove $\prod_{i=0}^{n-1} \alpha_{a_i,a_{i+1}} = 0$ in the second case that $a_j \equiv -1 \pmod 3$ and a_j is not a peak for some integer j.

By (17) and the claim, we have

$$\mu_{-n,r,s}^{\leq 3k-1}(\boldsymbol{b}, \lambda) = \sum_{(a_1,\ldots,a_{n-1}) \in \mathrm{PV}_{n-1,r,s}^{3,3k-1}} \prod_{i=0}^{n-1} \alpha_{a_i,a_{i+1}}. \tag{18}$$

By Lemma 5.3, for $\pi = (a_1, \ldots, a_{n-1}) \in \mathrm{PV}_{n-1,r,s}^{3,3k-1}$, we have

$$\prod_{i=0}^{n-1} \alpha_{a_i,a_{i+1}} = (-1)^{\lfloor \frac{a_0}{3} \rfloor + 2\lfloor \frac{a_1}{3} \rfloor + \cdots + 2\lfloor \frac{a_{n-1}}{3} \rfloor + \lfloor \frac{a_n}{3} \rfloor} \prod_{i=0}^{n-1} \frac{V_0 \cdots V_{a_{i+1}}}{V_0 \cdots V_{a_i}} V_{a_i}$$

$$= (-1)^{\lfloor \frac{r}{3} \rfloor + \lfloor \frac{s}{3} \rfloor} \frac{V_0 \cdots V_s}{V_0 \cdots V_{r-1}} V_{a_1} \cdots V_{a_{n-1}},$$

which together with (18) gives the theorem. □

Putting $V_i = -1$ in Theorem 5.5 gives the following corollary.

Corollary 5.6. *We have*

$$\mu_{-n,r,s}^{\leq 3k-1}(\mathbf{1}, \mathbf{1}) = (-1)^{\lfloor r/3 \rfloor + \lfloor s/3 \rfloor + r + s + n} \left| \mathrm{PV}_{n-1,r,s}^{3,3k-1} \right|.$$

Similarly, we can find a combinatorial interpretation for $\mu_{-n,r,s}^{\leq 3k}(\boldsymbol{b}, \lambda)$. To do this, we introduce modified peak-valley sequences.

Definition 5.7. A *modified (ℓ, r, s)-peak-valley sequence* is a sequence $(a_1, \ldots, a_n)$ of nonnegative integers such that for $i = 0, \ldots, n+1$,

○ if $a_i \equiv 1 \pmod \ell$, then a_i is a valley; that is, $a_{i-1} > a_i < a_{i+1}$,
○ if $a_i \equiv -1 \pmod \ell$, then a_i is a peak; that is, $a_{i-1} < a_i > a_{i+1}$,

where we set $a_0 = r$ and $a_{n+1} = s$. Here, we ignore the inequalities involving a_t for $t = -1$ or $t = n+2$. Denote by $\widetilde{\mathrm{PV}}_{n,r,s}^{\ell,k}$ the set of modified (ℓ, r, s)-peak-valley sequences $(a_1, \ldots, a_n)$ with bound k (i.e., $0 \leq a_i \leq k$ for all $i = 0, \ldots, n+1$).

Theorem 5.8. *Suppose that $\boldsymbol{b} = (b_i)_{i \geq 0}$ and $\lambda = (\lambda_i)_{i \geq 1}$ are the sequences given by $b_i = -V_i^{-1}$ and $\lambda_i = V_i^{-1} V_{i-1}^{-1}$ for all i. Then*

$$\mu_{-n,r,s}^{\leq 3k}(\boldsymbol{b}, \lambda) = (-1)^{\lfloor (r+1)/3 \rfloor + \lfloor (s+1)/3 \rfloor + n} \frac{V_0 \cdots V_s}{V_0 \cdots V_{r-1}} \sum_{\pi \in \widetilde{\mathrm{PV}}_{n-1,r,s}^{3,3k}} \mathrm{wt}(\pi).$$

Proof. This can be proved by the same arguments as in the proof of Theorem 5.5. We omit the details. □

Putting $V_i = -1$ in Theorem 5.8, we obtain the following corollary.

Corollary 5.9. *We have*

$$\mu_{-n,r,s}^{\leq 3k}(\mathbf{1}, \mathbf{1}) = (-1)^{\lfloor (r+1)/3 \rfloor + \lfloor (s+1)/3 \rfloor + r + s} \left| \widetilde{\mathrm{PV}}_{n-1,r,s}^{3,3k} \right|.$$

6. A general reciprocity theorem

In this section, we prove a general reciprocity theorem, Theorem 6.1. Using this theorem, we will prove the Cigler–Krattenthaler conjectures, Theorems 1.4 and 1.5, in the next section.

Following the notation in [3], let $R^{(n)}$ be the operator defined on polynomials in b_i's and λ_i's that replaces each b_i by b_{n-i} and each λ_i by λ_{n+1-i}. For example, $R^{(5)}(b_1 + \lambda_2 + b_3^2 \lambda_1) = b_4 + \lambda_4 + b_2^2 \lambda_5$.

Recall the matrix $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$ given in (13). We now state the general reciprocity theorem.

Theorem 6.1. *For positive integers k and m, we have*

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)_{i,j=0}^{k-1}$$
$$= \left(\prod_{i=1}^{k+m-1} \lambda_i^{k-i}\right) \det\left(A^{\leq k+m-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)^{n+2m-2} R^{(k+m-1)}\left(\det\left(\mu_{-n-i-j}^{\leq k+m-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)_{i,j=0}^{m-1}\right).$$

Before proving Theorem 6.1, we define several terminologies and prove some auxiliary results.

Definition 6.2. For a matrix $A = (A_{i,j})_{i,j=0}^{k}$ and two subsets $I, J \subseteq \{0, \dots, k\}$ of the same cardinality, we define

$$[A]_{I,J} = \det(A_{i,j})_{i \in I, j \in J}.$$

The following well-known lemma is an important tool in our proofs.

Lemma 6.3. *Suppose that $A = (A_{i,j})_{i,j=0}^{k}$ is an invertible matrix. For subsets $I, J \subseteq \{0, \dots, k\}$ of the same cardinality, we have*

$$\left[A^{-1}\right]_{I,J} = (-1)^{\|I\| + \|J\|} \frac{[A]_{J',I'}}{\det(A)},$$

where $\|I\| = \sum_{i \in I} i$ and $I' = \{0, \dots, k\} \setminus I$.

Definition 6.4. Let $A = (A_{i,j})_{i,j=0}^{k}$. We define the weighted directed graph $G(A)$ whose vertex set is $V(A) = \{(i, j) : i \in \mathbb{Z}, j \in \{0, \dots, k\}\}$ and edge set is $E(A) = \{(i, j) \to (i+1, j') : i \in \mathbb{Z}, j, j' \in \{0, \dots, k\}\}$. We assign the weight $A_{j,j'}$ to each edge $(i, j) \to (i+1, j')$ and ignore the edges with zero weights.

For $u, v \in V(A)$, let $P(G(A); u \to v)$ be the set of paths in $G(A)$ from u to v. The weight $\mathrm{wt}_A(\pi)$ of a path π is defined to be the product of weights on its edges. For $u_0, \dots, u_n, v_0, \dots, v_n \in V(A)$, we define $\mathrm{NI}(G(A); (u_0, \dots, u_n) \to (v_0, \dots, v_n))$ to be the set of tuples $\boldsymbol{\pi} = (\pi_0, \dots, \pi_n)$ of nonintersecting paths (i.e., no two paths meet at a vertex in $V(A)$), such that each π_i is a path in $G(A)$ from u_i to v_j for some j. For such a path tuple $\boldsymbol{\pi}$, there exists a permutation σ of $\{0, \dots, n\}$ such that each π_i is a path from u_i to $v_{\sigma(i)}$. The weight $\mathrm{wt}_A(\boldsymbol{\pi})$ of the path tuple is defined to be $\mathrm{sgn}(\sigma) \prod_{i=0}^{n} \mathrm{wt}_A(\pi_i)$.

Note that if A is the tridiagoanl matrix $A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$, then, for any $i \in \mathbb{Z}$, a path π in $P(G(A); (i, 0) \to (i+n, 0))$ can be identified with a Motzkin path in $\mathrm{Mot}_n^{\leq k}$, and we have

$$\mu_n^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \sum_{\pi \in \mathrm{Mot}_n^{\leq k}} \mathrm{wt}(\pi) = \sum_{\pi \in P(G(A); (i,0) \to (i+n,0))} \mathrm{wt}_A(\pi). \tag{19}$$

Moreover, by Proposition 5.1,

$$\mu_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda}) = \epsilon_0^T \left(A^{-1}\right)^n \epsilon_0 = \sum_{\pi \in P(G(A^{-1}); (i,0) \to (i+n,0))} \mathrm{wt}_{A^{-1}}(\pi). \tag{20}$$

Lemma 6.5. *For the tridiagoanl matrix $A = A^{\leq k}(\boldsymbol{b}, \boldsymbol{\lambda})$, the following statements hold.*

1. *Given $I, J \subseteq \{0, \dots, k\}$ of the same cardinality and $\{0, \dots, i\} \subseteq I$ for some i, if J misses two or more elements in $\{0, \dots, i+1\}$, then $[A]_{I,J} = 0$.*

2. *Given $I \subseteq \{0, \ldots, i-1\}$, $J \subseteq \{0, \ldots, i\}$ of the same cardinality, we have*

$$[A]_{I \cup \{i,\ldots,k-1\}, J \cup \{i+1,\ldots,k\}} = [A]_{I,J},$$

$$[A]_{I \cup \{i+1,\ldots,k\}, J \cup \{i,\ldots,k-1\}} = [A]_{I,J} \prod_{j=i+1}^{k} \lambda_j.$$

Proof. (1) Let B be the submatrix of A consisting of the rows indexed by $0, \ldots, i$. Since A is a tridiagonal matrix, the jth column of B is zero if $j > i + 1$. So if J misses two or more elements in $0, \ldots, i + 1$, the submatrix of B consisting of the columns indexed by J has rank at most i. We conclude $[A]_{I,J} = 0$.

(2) The submatrix of A with rows indexed by $i, \ldots, k-1$ and columns indexed by $i+1, \ldots, k$ is a lower triangular matrix with diagonal entries all 1. So the first identity follows. Likewise, the submatrix of A with rows indexed by $i+1, \ldots, k$ and columns indexed by $i, \ldots, k-1$ is an upper triangular matrix with diagonal entries $\lambda_{i+1}, \ldots, \lambda_k$. This gives the second identity. $\square$

Lemma 6.6. *Letting $A = A^{\leq k+m-1}(\boldsymbol{b}, \lambda)$, we have*

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \lambda)\right)_{i,j=0}^{k-1} = \left(\prod_{i=1}^{k-1} \lambda_i^{k-i}\right) \sum_{(I_0,\ldots,I_{n+2m-2}) \in X} \prod_{j=0}^{n+2m-3} [A]_{I_j, I_{j+1}},$$

where X is the set of all tuples $(I_0, \ldots, I_{n+2m-2})$ of k-element subsets of $\{0, \ldots, k+m-1\}$ such that $I_j, I_{n+2m-2-j} \subseteq \{0, \ldots, k-1+j\}$ for all $0 \leq j \leq m-1$.

Proof. By (19), we have

$$\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \lambda) = \sum_{\pi \in P(G(A); (-i,0) \to (n+2m-2+j,0))} \mathrm{wt}_A(\pi).$$

Thus, the Lindström–Gessel–Viennot lemma [6, 13] gives

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \lambda)\right)_{i,j=0}^{k-1} = \sum_{\pi \in \mathrm{NI}(G(A); R \to S)} \mathrm{wt}_A(\pi),$$

where $R = ((0,0), (-1,0), \ldots, (-k+1,0))$ and $S = ((n+2m-2,0), (n+2m-1,0), \ldots, (n+2m+k-3,0))$.

Suppose $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_{k-1}) \in \mathrm{NI}(G(A); R \to S)$. Since $\pi_0, \ldots, \pi_{k-1}$ are nonintersecting Motzkin paths and each π_i is from $(-i, 0)$ to $(n + 2m - 2 + \sigma(i), 0)$, for some permutation σ, the first i steps (resp. last $\sigma(i)$ steps) of π_i are up steps (resp. down steps) whose weights are 1's (resp. $\lambda_1, \ldots, \lambda_{\sigma(i)}$). Considering the subpath obtained from π_i by deleting the first i steps and last $\sigma(i)$ steps, we obtain

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \lambda)\right)_{i,j=0}^{k-1} = \left(\prod_{i=1}^{k-1} \lambda_i^{k-i}\right) \sum_{\pi \in \mathrm{NI}(G(A); R_1 \to S_1)} \mathrm{wt}_A(\pi), \tag{21}$$

where $R_1 = ((0,0), (0,1), \ldots, (0,k-1))$ and $S_1 = ((n+2m-2,0), (n+2m-2,1), \ldots, (n+2m-2,k-1))$.

For each $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_{k-1}) \in \mathrm{NI}(G(A); R_1 \to S_1)$, we define $I(\boldsymbol{\pi}) = (I_0, \ldots, I_{n+2m-2})$, where each I_j is the k-element subset of $\{0, \ldots, k+m-1\}$ consisting of the y-coordinates of the points of $\pi_0, \ldots, \pi_{k-1}$ on the line $x = j$. For brevity we write $\boldsymbol{I} = (I_0, \ldots, I_{n+2m-2})$. Observe that, since $\boldsymbol{\pi} \in \mathrm{NI}(G(A); R_1 \to S_1)$, we have $I_0 = I_{n+2m-2} = \{0, \ldots, k-1\}$. Moreover, since $\pi_0, \ldots, \pi_{k-1}$ are Motzkin paths, we also have $I_j, I_{n+2m-2-j} \subseteq \{0, \ldots, k-1+j\}$ for all $0 \leq j \leq m-1$; that is, $\boldsymbol{I} \in X$.

Therefore, we can rewrite (21) as

$$\det\left(\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b},\lambda)\right)_{i,j=0}^{k-1} = \left(\prod_{i=1}^{k-1} \lambda_i^{k-i}\right) \sum_{\boldsymbol{I} \in X} \sum_{\substack{\pi \in \mathrm{NI}(G(A);R_1 \to S_1) \\ I(\pi)=\boldsymbol{I}}} \mathrm{wt}_A(\boldsymbol{\pi}). \tag{22}$$

For a fixed tuple $\boldsymbol{I} = (I_0, \ldots, I_{n+2m-2}) \in X$, applying the Lindström–Gessel–Viennot lemma repeatedly to the paths (of length 1) starting from the points (j,r) for $r \in I_j$ to the points $(j+1,s)$ for $s \in I_{j+1}$, we obtain

$$\sum_{\substack{\pi \in \mathrm{NI}(G(A);R_1 \to S_1) \\ I(\pi)=\boldsymbol{I}}} \mathrm{wt}_A(\boldsymbol{\pi}) = \prod_{j=0}^{n+2m-3} [A]_{I_j,I_{j+1}}. \tag{23}$$

Combining (22) and (23) completes the proof. $\qquad\square$

Lemma 6.7. *Letting* $A = A^{\leq k+m-1}(\boldsymbol{b},\lambda)$, *we have*

$$\det\left(\mu_{-n-i-j}^{\leq k+m-1}(\boldsymbol{b},\lambda)\right)_{i,j=0}^{m-1}$$

$$= \sum_{(J_0,\ldots,J_{n+2m-2}) \in Y} \prod_{j=0}^{m-2} [A^{-1}]_{J_j,J_{j+1}\setminus\{0\}} \prod_{j=m-1}^{n+m-2} [A^{-1}]_{J_j,J_{j+1}} \prod_{j=n+m-1}^{n+2m-3} [A^{-1}]_{J_j\setminus\{0\},J_{j+1}},$$

where Y is the set of all tuples $(J_0, \ldots, J_{n+2m-2})$ of subsets of $\{0, \ldots, k+m-1\}$ satisfying the following conditions:

1. *Both J_j and $J_{n+2m-2-j}$ have cardinality $j+1$ and contain 0 for all $0 \leq j \leq m-1$.*
2. *$|J_j| = m$ for all $m-1 \leq j \leq n+m-1$.*
3. *$J_j \cap \{1,\ldots,m-1-j\} = J_{n+2m-2-j} \cap \{1,\ldots,m-1-j\} = \emptyset$ for all $0 \leq j \leq m-2$.*

Proof. By (20) and the Lindström–Gessel–Viennot lemma, we have

$$\det\left(\mu_{-n-i-j}^{\leq k+m-1}(\boldsymbol{b},\lambda)\right)_{i,j=0}^{m-1} = \sum_{\pi \in \mathrm{NI}(G(A^{-1});R \to S)} \mathrm{wt}_{A^{-1}}(\boldsymbol{\pi}), \tag{24}$$

where $R = ((m-1,0), (m-2,0), \ldots, (0,0))$ and $S = ((n+m-1,0), (n+m,0), \ldots, (n+2m-2,0))$.

For each $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_{m-1}) \in \mathrm{NI}(G(A^{-1}); R \to S)$, we define $J(\boldsymbol{\pi}) = (J_0, \ldots, J_{n+2m-2})$, where J_j is the subset of $\{0, \ldots, k+m-1\}$ consisting of the y-coordinates of the points of $\pi_0, \ldots, \pi_{m-1}$ on the line $x = j$. For brevity, we write $\boldsymbol{J} = (J_0, \ldots, J_{n+2m-2})$. It is easy to check that the tuple $\boldsymbol{J}$ satisfies the first two conditions for the elements in Y. Let Z be the set of such tuples $\boldsymbol{J}$. Then we can rewrite (24) as

$$\det\left(\mu_{-n-i-j}^{\leq k+m-1}(\boldsymbol{b},\lambda)\right)_{i,j=0}^{m-1} = \sum_{\boldsymbol{J} \in Z} \sum_{\substack{\pi \in \mathrm{NI}(G(A^{-1});R \to S) \\ J(\pi)=\boldsymbol{J}}} \mathrm{wt}_{A^{-1}}(\boldsymbol{\pi}). \tag{25}$$

For a fixed tuple $\boldsymbol{J} = (J_0, \ldots, J_{n+2m-2}) \in Z$, applying the Lindström–Gessel–Viennot lemma repeatedly to the paths (of length 1) starting from the points (j,r) for $r \in J_j$ to the points $(j+1,s)$ for $s \in J_{j+1}$, where we ignore the point $(j+1,0)$ (resp. $(j,0)$) if $0 \leq j \leq m-2$ (resp. $n+m-1 \leq j \leq n+2m-3$), we obtain

$$\sum_{\substack{\pi \in \mathrm{NI}(G(A^{-1});R \to S) \\ J(\pi)=\boldsymbol{J}}} \mathrm{wt}_{A^{-1}}(\boldsymbol{\pi}) = w(\boldsymbol{J}), \tag{26}$$

where

$$w(\boldsymbol{J}) = \prod_{j=0}^{m-2} [A^{-1}]_{J_j, J_{j+1}\setminus\{0\}} \prod_{j=m-1}^{n+m-2} [A^{-1}]_{J_j, J_{j+1}} \prod_{j=n+m-1}^{n+2m-3} [A^{-1}]_{J_j\setminus\{0\}, J_{j+1}}.$$

Combining (25) and (26) gives

$$\det\left(\mu_{-n-i-j}^{\le k+m-1}(\boldsymbol{b}, \lambda)\right)_{i,j=0}^{m-1} = \sum_{\boldsymbol{J}\in Z} w(\boldsymbol{J}).$$

Since Z is the set of tuples $(J_0, \ldots, J_{n+2m-2})$ satisfying the conditions (1) and (2) (but not necessarily (3)) for the elements in Y, it remains to show the following claim.

Claim. *Let $\boldsymbol{J} = (J_0, \ldots, J_{n+2m-2}) \in Z$. If $J_j \cap \{1, \ldots, m-1-j\} \ne \emptyset$ or $J_{n+2m-2-j} \cap \{1, \ldots, m-1-j\} \ne \emptyset$ for some $0 \le j \le m-2$, then $w(\boldsymbol{J}) = 0$.*

To prove the claim, suppose that $J_j \cap \{1, \ldots, m-1-j\} \ne \emptyset$ for some $0 \le j \le m-2$. Take the largest j so that $J_{j+1} \cap \{1, \ldots, m-2-j\} = \emptyset$, which is clearly true if $j = m-2$. By Lemma 6.3, we have

$$[A^{-1}]_{J_j, J_{j+1}\setminus\{0\}} = (-1)^{\|J_j\| + \|J_{j+1}\setminus\{0\}\|} \frac{[A]_{(J_{j+1}\setminus\{0\})', (J_j)'}}{\det(A)}.$$

By the assumption on j, we have $\{0, \ldots, m-2-j\} \subseteq (J_{j+1} \setminus \{0\})'$ and $0, t \notin (J_j)'$ for some $t \in \{1, \ldots, m-1-j\}$. Thus, by Lemma 6.5 (1), we have $[A]_{(J_{j+1}\setminus\{0\})', (J_j)'} = 0$, which implies $w(\boldsymbol{J}) = 0$. Similarly, one can prove that if $J_{n+2m-2-j} \cap \{1, \ldots, m-j-1\} \ne \emptyset$ for some $0 \le j \le m-2$, then $w(\boldsymbol{J}) = 0$. This settles the claim and the proof is completed. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Abusing the notation, let $R^{(k+m-1)}$ also denote the operator acting on the subsets K of $\{0, \ldots, k+m-1\}$ by

$$R^{(k+m-1)}(K) = \{k+m-1-i : i \in K\}.$$

Recall the sets X and Y given in Lemmas 6.6 and 6.7, respectively. We define the map $f : X \to Y$ by $f(I_0, \ldots, I_{n+2m-2}) = (J_0, \ldots, J_{n+2m-2})$, where

$$J_j = \begin{cases} \{0\} \cup R^{(k+m-1)}(\{0, \ldots, k-1+j\} \setminus I_j) & \text{if } 0 \le j \le m-1, \\ R^{(k+m-1)}(\{0, \ldots, k+m-1\} \setminus I_j) & \text{if } m-1 \le j \le n+m-1, \\ \{0\} \cup R^{(k+m-1)}(\{0, \ldots, k+n+2m-3-j\} \setminus I_j) & \text{if } n+m-1 \le j \le n+2m-2. \end{cases} \tag{27}$$

Note that J_j is well-defined when $j = m-1$ or $j = n-m-1$. It is not hard to see that the map f is a bijection.

We claim that, for $0 \le j \le m-2$,

$$[A^{-1}]_{J_j, J_{j+1}\setminus\{0\}} = (-1)^{\|J_j\| + \|J_{j+1}\setminus\{0\}\|} \frac{R^{(k+m-1)}([A]_{I_{j+1}, I_j})}{\det(A)}. \tag{28}$$

To prove the claim, we first use Lemma 6.3 to obtain

$$[A^{-1}]_{J_j, J_{j+1}\setminus\{0\}} = (-1)^{\|J_j\| + \|J_{j+1}\setminus\{0\}\|} \frac{[A]_{(J_{j+1}\setminus\{0\})', (J_j)'}}{\det(A)}. \tag{29}$$

Note that, for $0 \leq j \leq m - 2$, (27) implies

$$(J_j)' = \{0, \ldots, k + m - 1\} \setminus J_j = R^{(k+m-1)}(I_j \cup \{k + j, \ldots, k + m - 2\}),$$
$$(J_{j+1} \setminus \{0\})' = \{0, \ldots, k + m - 1\} \setminus (J_{j+1} \setminus \{0\})$$
$$= R^{(k+m-1)}(I_{j+1} \cup \{k + j + 1, \ldots, k + m - 1\}).$$

Thus, the right-hand side of (29) is equal to

$$(-1)^{\|J_j\| + \|J_{j+1} \setminus \{0\}\|} \frac{[A]_{R^{(k+m-1)}(I_j \cup \{k+j,\ldots,k+m-2\}), R^{(k+m-1)}(I_{j+1} \cup \{k+j+1,\ldots,k+m-1\})}}{\det(A)}$$
$$= (-1)^{\|J_j\| + \|J_{j+1} \setminus \{0\}\|} \frac{R^{(k+m-1)}([A]_{I_j \cup \{k+j,\ldots,k+m-2\}), I_{j+1} \cup \{k+j+1,\ldots,k+m-1\}})}{\det(A)},$$

which, by Lemma 6.5 (2), is equal to the right-hand side of (28), and the claim is proved.

A similar argument shows that, for $m - 1 \leq j \leq n + m - 2$,

$$[A^{-1}]_{J_j, J_{j+1}} = (-1)^{\|J_j\| + \|J_{j+1}\|} \frac{R^{(k+m-1)}([A]_{I_{j+1}, I_j})}{\det(A)}, \tag{30}$$

and, for $n + m - 1 \leq j \leq n + 2m - 3$,

$$[A^{-1}]_{J_j \setminus \{0\}, J_{j+1}} = (-1)^{\|J_j \setminus \{0\}\| + \|J_{j+1}\|} \frac{\left(\prod_{i=1}^{j-n-m+1} \lambda_{k+m-i}\right) R^{(k+m-1)}([A]_{I_{j+1}, I_j})}{\det(A)}. \tag{31}$$

By (28), (30), and (31) and using the fact that $\|J_0\| = \|J_{n+2m-2}\| = 0$ and $\|J \setminus \{0\}\| = \|J\|$ for any set J, we obtain

$$\prod_{j=0}^{m-2} [A^{-1}]_{J_j, J_{j+1} \setminus \{0\}} \prod_{j=m-1}^{n+m-2} [A^{-1}]_{J_j, J_{j+1}} \prod_{j=n+m-1}^{n+2m-3} [A^{-1}]_{J_j \setminus \{0\}, J_{j+1}}$$
$$= \frac{\prod_{i=k}^{k+m-1} \lambda_i^{k-i}}{\det(A)^{n+2m-2}} R^{(k+m-1)}\left(\prod_{j=1}^{n+2m-2} [A]_{I_{j+1}, I_j}\right). \tag{32}$$

Since $(I_0, \ldots, I_{n+2m-2}) \in X$ if and only if $(I_{n+2m-2}, \ldots, I_0) \in X$, combining Lemma 6.6, Lemma 6.7 and (32) completes the proof. □

7. Proof of Cigler–Krattenthaler conjectures

In this section, we show that Theorem 6.1 implies the Cigler–Krattenthaler conjectures, Theorems 1.4 and 1.5. To obtain Theorem 1.5, we can simply put $\boldsymbol{b} = \boldsymbol{1}$ and $\lambda = \boldsymbol{1}$ in Theorem 6.1. However, it is more difficult to derive Theorem 1.4 from Theorem 6.1 because the left-hand side of the equation in Theorem 1.4 is not of the form as written in Theorem 6.1. To remedy this, we find suitable sequences $\boldsymbol{b}$ and λ such that

$$\mu_{n+i+j+2m-2}^{\leq k+m-1}(\boldsymbol{b}, \lambda) = \sum_{s=0}^{2k+2m-1} \mu_{n+i+j+2m-1,0,s}^{\leq 2k+2m-1}(\boldsymbol{0}, \boldsymbol{1}). \tag{33}$$

Indeed, we will show in Lemma 7.3 that (33) holds if $\boldsymbol{b} = \boldsymbol{b}^{(k+m-1)}$ and $\lambda = -\boldsymbol{1}$, where the sequences $\boldsymbol{b}^{(\ell)}$ and $-\boldsymbol{1}$ are defined by

$$\boldsymbol{b}^{(\ell)} = \left(b_i^{(\ell)}\right)_{i \geq 0}, \qquad b_i^{(\ell)} = \begin{cases} (-1)^i 2 & \text{if } 0 \leq i < \ell, \\ (-1)^i & \text{if } i \geq \ell, \end{cases} \tag{34}$$

$$-\mathbf{1} = (-1, -1, \dots). \tag{35}$$

We will mostly consider $\mu_n^{\leq \ell}(\boldsymbol{b}^{(\ell)}, \lambda)$, which does not depend on the values $b_i^{(\ell)}$ for $i > \ell$.

Remark 7.1. One may wonder how to guess that $\boldsymbol{b} = \boldsymbol{b}^{(k+m-1)}$ and $\lambda = -\mathbf{1}$ is a solution to (33). Such a solution can be found by computer once we fix the value of $k + m$. For example, if $k + m = 3$, then the sequences $\boldsymbol{b}$ and λ must satisfy

$$\mu_n^{\leq 2}(\boldsymbol{b}, \lambda) = \sum_{s=0}^{5} \mu_{n+1,0,s}^{\leq 5}(\mathbf{0}, \mathbf{1}). \tag{36}$$

Substituting $n = 1, \dots, 5$ in (36) gives five equations with variables $b_0, b_1, b_2, \lambda_1, \lambda_2$. One can check by computer that there is a unique solution to these equations, which is $b_0 = 2, b_1 = -2, b_2 = 1, \lambda_1 = -1, \lambda_2 = -1$. After computing more solutions for different choices of $k + m$, one can guess that $\boldsymbol{b} = \boldsymbol{b}^{(k+m-1)}$ and $\lambda = -\mathbf{1}$ is a solution to (33).

Now we begin with a simple lemma.

Lemma 7.2. *We have*

$$\det\left(A^{\leq k-1}(\mathbf{1}, \mathbf{1})\right) = \begin{cases} 0 & \text{if } k \equiv 2 \pmod 3, \\ (-1)^{\lfloor k/3 \rfloor} & \text{otherwise}, \end{cases} \tag{37}$$

$$\det\left(A^{\leq k-1}(\boldsymbol{b}^{(k-1)}, -\mathbf{1})\right) = (-1)^{\lfloor k/2 \rfloor}. \tag{38}$$

Proof. To prove (37), we expand the determinant with respect to the first row to get

$$\det\left(A^{\leq k-1}(\mathbf{1}, \mathbf{1})\right) = \det\left(A^{\leq k-2}(\mathbf{1}, \mathbf{1})\right) - \det\left(A^{\leq k-3}(\mathbf{1}, \mathbf{1})\right).$$

Then (37) follows easily by induction on k.

For the second identity, let $U = (U_{i,j})_{i,j=0}^{k-1}$ and $L = (L_{i,j})_{i,j=0}^{k-1}$ be the matrices defined by

$$U_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ (-1)^{i-1} & \text{if } i = j - 1, \\ 0 & \text{otherwise}, \end{cases} \qquad L_{i,j} = \begin{cases} (-1)^i & \text{if } i = j, \\ -1 & \text{if } i = j + 1, \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that $A^{\leq k-1}(\boldsymbol{b}^{(k-1)}, -\mathbf{1}) = UL$. Since U is an upper-triangular matrix with diagonal entries all 1 and L is a lower-triangular matrix with diagonal entries $1, -1, 1, -1, \dots$, we obtain (38). □

Since the proof of Theorem 1.5 is simpler than that of Theorem 1.4, we present it first.

Proof of Theorem 1.5. We put $\boldsymbol{b} = \mathbf{1}$ and $\lambda = \mathbf{1}$ in Theorem 6.1. Then Theorem 1.5 immediately follows from (37) and

$$R^{(k+m-1)}\left(\mu_{-n-i-j}^{\leq k+m-1}(\boldsymbol{b}, \lambda)\right)\Big|_{\boldsymbol{b}=\mathbf{1}, \lambda=\mathbf{1}} = \mu_{-n-i-j}^{\leq k+m-1}(\mathbf{1}, \mathbf{1}). \qquad \qquad \square$$

In order to prove Theorem 1.4, we need the following two technical lemmas whose proofs will be given later.

Lemma 7.3. *For $n \geq 0$ and $k \geq 1$, we have*

$$\mu_n^{\leq k}(\boldsymbol{b}^{(k)}, -\boldsymbol{1}) = \sum_{s=0}^{2k+1} \mu_{n+1,0,s}^{\leq 2k+1}(\boldsymbol{0}, \boldsymbol{1}).$$

Lemma 7.4. *For $n \geq 0$ and $k \geq 1$, we have*

$$(-1)^{kn}\left(R^{(k)}\left(\mu_{-n}^{\leq k}(\boldsymbol{b}, \lambda)\right)\right)\Big|_{\boldsymbol{b}=\boldsymbol{b}^{(k)}, \lambda=-1} = \left|\mathrm{Alt}_n^{\leq k+1}\right|.$$

Proof of Theorem 1.4. We put $\boldsymbol{b} = \boldsymbol{b}^{(k+m-1)}$ and $\lambda = -\boldsymbol{1}$ in Theorem 6.1 and apply Lemma 7.3, (38) and Lemma 7.4, which gives

$$\det\left(\sum_{s=0}^{2k+2m-1} \mu_{n+i+j+2m-1,0,s}^{\leq 2k+2m-1}(\boldsymbol{0}, \boldsymbol{1})\right)_{i,j=0}^{k-1}$$

$$= \left(\prod_{i=1}^{k+m-1}(-1)^{k-i}\right)(-1)^{\lfloor \frac{k+m}{2}\rfloor(n+2m-2)} \det\left((-1)^{(k+m-1)(n+i+j)}|\mathrm{Alt}_{n+i+j}^{\leq k+m}|\right)_{i,j=0}^{m-1}. \quad (39)$$

Observe that $\lfloor \frac{k+m}{2}\rfloor \equiv \binom{k+m}{2} \pmod{2}$ and

$$\det\left((-1)^{(k+m-1)(n+i+j)}|\mathrm{Alt}_{n+i+j}^{\leq k+m}|\right)_{i,j=0}^{m-1} = (-1)^{(k+m-1)nm}\det\left(|\mathrm{Alt}_{n+i+j}^{\leq k+m}|\right)_{i,j=0}^{m-1},$$

and $(-1)^{(k+m-1)nm} = (-1)^{kmn}$. Therefore, we can rewrite (39) as

$$\det\left(\sum_{s=0}^{2k+2m-1} \mu_{n+i+j+2m-1,0,s}^{\leq 2k+2m-1}(\boldsymbol{0}, \boldsymbol{1})\right)_{i,j=0}^{k-1}$$

$$= s \cdot \det\left(|\mathrm{Alt}_{n+i+j}^{\leq k+m}|\right)_{i,j=0}^{m-1},$$

where $s = (-1)^{k(k+m-1)-\binom{k+m}{2}}(-1)^{n\binom{k+m}{2}}(-1)^{knm}$, which is equal to

$$(-1)^{km+(n+1)\binom{k+m}{2}+knm} = (-1)^{\left(km+\binom{k+m}{2}\right)(n+1)} = (-1)^{\left(\binom{k}{2}+\binom{m}{2}\right)(n+1)}.$$

This completes the proof. □

Now it remains to prove Lemmas 7.3 and 7.4.

Proof of Lemma 7.3. Using (14), we can restate the lemma as

$$\epsilon_0^T (A^{\leq k}(\boldsymbol{b}^{(k)}, -\boldsymbol{1}))^n \epsilon_0 = \epsilon_0^T (A^{\leq 2k+1}(\boldsymbol{0}, \boldsymbol{1}))^{n+1} v, \quad (40)$$

where $v = \sum_{s=0}^{2k+1} \epsilon_s$.

Fix the integer $k \geq 1$ and, for $n \geq 0$, let

$$(A^{\leq k}(\boldsymbol{b}^{(k)}, -\boldsymbol{1}))^n \epsilon_0 = (a_{n,0}, \ldots, a_{n,k})^T,$$
$$(A^{\leq 2k+1}(\boldsymbol{0}, \boldsymbol{1}))^n v = (d_{n,0}, \ldots, d_{n,2k+1})^T.$$

We also define $a_{n,j} = 0$ for $j \notin \{0, \ldots, k\}$ and $d_{n,i} = 0$ for $i \notin \{0, \ldots, 2k+1\}$. Then, by definition, we have

$$a_{n+1,i} = -a_{n,i-1} + (-1)^i 2a_{n,i} + a_{n,i+1}, \qquad 0 \le i \le k-1, \tag{41}$$

$$a_{n+1,k} = -a_{n,k-1} + (-1)^k a_{n,k}, \tag{42}$$

$$d_{n+1,i} = d_{n,i-1} + d_{n,i+1}, \qquad 0 \le i \le 2k+1, \tag{43}$$

$$d_{n+1,i} = d_{n+1,2k+1-i}, \qquad 0 \le i \le 2k+1, \tag{44}$$

where (44) follows from the symmetry of the matrix $A^{\le 2k+1}(\mathbf{0}, \mathbf{1})$.

We claim that, for all $n \ge 0$ and $0 \le i \le k$, (with $k \ge 1$ fixed)

$$d_{n+1,i} - d_{n+1,i-1} = a_{n,i} + (-1)^{i-1} a_{n,i-1}. \tag{45}$$

Note that if $i = 0$, we have $d_{n+1,0} = a_{n,0}$, which is equivalent to (40). Thus, it suffices to prove the claim.

To prove the claim (45), we proceed by induction on n. The base case $n = 0$ is easily checked by

$$(d_{1,0}, \ldots, d_{1,2k+1})^T = (A^{\le 2k+1}(\mathbf{0}, \mathbf{1}))^1 v = (1, 2, \ldots, 2, 1)^T,$$
$$(a_{0,0}, \ldots, a_{0,k})^T = \epsilon_0 = (1, 0, \ldots, 0)^T.$$

Now assume (45) is true for n and consider the case $n + 1$. Suppose $0 \le i \le k - 1$. By (43) and the induction hypothesis,

$$d_{n+2,i} - d_{n+2,i-1} = (d_{n+1,i-1} + d_{n+1,i+1}) - (d_{n+1,i-2} + d_{n+1,i})$$
$$= a_{n,i-1} + (-1)^{i-2} a_{n,i-2} + a_{n,i+1} + (-1)^i a_{n,i}.$$

However, by (41),

$$a_{n+1,i} + (-1)^{i-1} a_{n+1,i-1}$$
$$= (-a_{n,i-1} + (-1)^i 2a_{n,i} + a_{n,i+1}) + (-1)^{i-1}(-a_{n,i-2} + (-1)^{i-1} 2a_{n,i-1} + a_{n,i})$$
$$= a_{n,i-1} + (-1)^{i-2} a_{n,i-2} + a_{n,i+1} + (-1)^i a_{n,i}.$$

Thus, $d_{n+2,i} - d_{n+2,i-1} = a_{n+1,i} + (-1)^{i-1} a_{n+1,i-1}$. Suppose $i = k$. By (43), (44) and the induction hypothesis,

$$d_{n+2,k} - d_{n+2,k-1} = (d_{n+1,k-1} + d_{n+1,k+1}) - (d_{n+1,k-2} + d_{n+1,k})$$
$$= d_{n+1,k-1} - d_{n+1,k-2}$$
$$= a_{n,k-1} + (-1)^{k-2} a_{n,k-2}.$$

However, by (42),

$$a_{n+1,k} + (-1)^{k-1} a_{n+1,k-1}$$
$$= (-a_{n,k-1} + (-1)^k a_{n,k}) + (-1)^{k-1}(-a_{n,k-2} + (-1)^{k-1} 2a_{n,k-1} + a_{n,k})$$
$$= a_{n,k-1} + (-1)^{k-2} a_{n,k-2}.$$

Thus, we also have $d_{n+2,i} - d_{n+2,i-1} = a_{n+1,i} + (-1)^{i-1} a_{n+1,i-1}$. This settles (45) by induction and the proof is completed. $\qquad \square$

In order to prove Lemma 7.4, we need the following two lemmas.

Lemma 7.5. *We have*

$$| \text{Alt}_n^{\leq k+1} | = \epsilon_0^T (A')^n v,$$

where $v = \sum_{s=0}^{2k+1} \epsilon_s$ *and* $A' = (A'_{i,j})_{i,j=0}^{2k+1}$ *is the matrix defined by*

$$A'_{i,j} = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2, j \equiv 1 \pmod 2, \text{ and } i < j, \\ 1 & \text{if } i \equiv 1 \pmod 2, j \equiv 0 \pmod 2, \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From the definition of A', the value $\epsilon_0^T (A')^n \epsilon_s$ equals the number of sequences $(a_1, \ldots, a_n)$ satisfying the following three conditions:

1. $0 \leq a_i \leq 2k + 1$ and $a_i \equiv i \pmod 2$,
2. $a_1 > a_2 < a_3 > a_4 < \cdots$,
3. $a_n = s$.

So $\epsilon_0^T (A')^n v$ counts the number of sequences $(a_1, \ldots, a_n)$ satisfying the conditions (1) and (2). Using the bijection in Proposition 3.3, such sequences are in bijection with the elements of $\text{Alt}_n^{\leq k+1}$.   □

Lemma 7.6. *We have*

$$R^{(k)}\left(A^{\leq k}(\boldsymbol{b}, \lambda)\right)\Big|_{\boldsymbol{b}=\boldsymbol{b}^{(k)}, \lambda=-1} = B^{-1},$$

where $B = (B_{i,j})_{i,j=0}^k$ *is the matrix defined by*

$$B_{i,j} = (-1)^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k+1-j}{2} \rfloor}(k + 1 - \max(i, j)).$$

Proof. Denote $\bar{A} = (\bar{A}_{i,j})_{i,j=0}^k = (R^{(k)}(A^{\leq k}(\boldsymbol{b}, \lambda)))|_{\boldsymbol{b}=\boldsymbol{b}^{(k)}, \lambda=-1}$. In other words,

$$\bar{A}_{i,j} = \begin{cases} (-1)^k & \text{if } i = j = 0, \\ (-1)^{k-i} 2 & \text{if } i = j \geq 1, \\ -1 & \text{if } i = j + 1, \\ 1 & \text{if } i = j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We must show $(B\bar{A})_{i,j} = \delta_{i,j}$ for $0 \leq i, j \leq k$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. To this end, we consider the following three cases.

First, suppose $j = 0$. Then $(B\bar{A})_{i,0} = (-1)^k B_{i,0} - B_{i,1}$. Since $k = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k+1}{2} \rfloor$,

$$(B\bar{A})_{i,0} = (-1)^k(-1)^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k+1}{2} \rfloor}(k + 1 - i) - (-1)^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k}{2} \rfloor}(k + 1 - \max(i, 1))$$
$$= (-1)^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k}{2} \rfloor}(-i + \max(i, 1)) = \delta_{i,0}.$$

Second, suppose $1 \leq j \leq k - 1$. Then

$$(B\bar{A})_{i,j} = B_{i,j-1} + (-1)^{k-j} 2 B_{i,j} - B_{i,j+1}.$$

Letting $s = (-1)^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k-j}{2} \rfloor}$, we have

$$B_{i,j-1} = -s(k + 1 - \max(i, j - 1)),$$

$$(-1)^{k-j}2B_{i,j} = (-1)^{\lfloor \frac{k-j}{2} \rfloor + \lfloor \frac{k+1-j}{2} \rfloor}2B_{i,j} = 2s(k + 1 - \max(i, j)),$$

$$-B_{i,j+1} = -s(k + 1 - \max(i, j + 1)).$$

Thus,

$$\left(B\bar{A}\right)_{i,j} = s(\max(i, j - 1) - 2\max(i, j) + \max(i, j + 1)) = \delta_{i,j}.$$

Finally, suppose $j = k$. In this case, we have

$$\begin{aligned}\left(B\bar{A}\right)_{i,k} &= B_{i,k-1} + 2B_{i,k} \\ &= (-1)^{\lfloor \frac{k-i}{2} \rfloor + 1}(k + 1 - \max(i, k - 1)) + 2(-1)^{\lfloor \frac{k-i}{2} \rfloor}(k + 1 - k) \\ &= (-1)^{\lfloor \frac{k-i}{2} \rfloor}(\max(i, k - 1) - k + 1) = \delta_{i,k}.\end{aligned}$$

Therefore, $\left(B\bar{A}\right)_{i,j} = \delta_{i,j}$ for all $0 \le i, j \le k$ and the lemma follows. □

Now we are ready to prove Lemma 7.4.

Proof of Lemma 7.4. Recall the matrices A' and B in Lemmas 7.5 and 7.6, respectively. By these lemmas and Proposition 5.1, it is enough to show

$$\epsilon_0^T (A')^n v = (-1)^{kn} \epsilon_0^T B^n \epsilon_0, \tag{46}$$

where $v = \sum_{s=0}^{2k+1} \epsilon_s$.

Denote $(A')^n v = (a_{n,0}, \ldots, a_{n,2k+1})^T$ and $B^n \epsilon_0 = (b_{n,0}, \ldots, b_{n,k})^T$. We have $a_{n,i} = a_{n,2k+1-i}$ for $0 \le i \le k$ due to the symmetry of the matrix A'.

We claim that for $0 \le i \le k$,

$$a_{n,i} - a_{n,i-1} = \begin{cases} (-1)^{n-1+\lfloor \frac{i-1}{2} \rfloor} b_{n,i} & \text{if } k \equiv 1 \pmod 2, \\ (-1)^{\lfloor \frac{i}{2} \rfloor} b_{n,i} & \text{if } k \equiv 0 \pmod 2, \end{cases} \tag{47}$$

where $a_{n,i} = b_{n,i} = 0$ if $i < 0$. Observe that if $i = 0$ in (47), we have $a_{n,0} = (-1)^n b_{n,0}$ if $k \equiv 1 \pmod 2$ and $a_{n,0} = b_{n,0}$ if $k \equiv 0 \pmod 2$, which is equivalent to (46). Hence, it suffices to prove (47).

To prove the claim (47), we proceed by induction on n. The base case $n = 0$ is trivial. Now assume (47) is true for n and consider the case $n + 1$. We will only prove the case when k is even because the other case can be proved similarly.

For $0 \le i \le k$, using the symmetry $a_{n,i} = a_{n,2k+1-i}$, we get

$$a_{n+1,i} = \sum_{j=0}^{2k+1} A'_{i,j} a_{n,j} = \sum_{j=0}^{k} \left(A'_{i,j} + A'_{i,2k+1-j} \right) a_{n,j},$$

which implies

$$a_{n+1,i} = \begin{cases} \sum_{j=0}^{k} a_{n,j} - \sum_{j=0}^{i/2} a_{n,2j-1} & \text{if } i \equiv 0 \pmod 2, \\ \sum_{j=0}^{(i-1)/2} a_{n,2j} & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

Therefore, we have

$$a_{n+1,i} - a_{n+1,i-1} = (-1)^i \sum_{j=i}^{k} a_{n,j}. \tag{48}$$

Using the induction hypothesis, we sum the identities (47), where the index i takes the values $i, i-1, \dots, 1$, to obtain

$$a_{n,i} = \sum_{j=0}^{i} (-1)^{\lfloor \frac{j}{2} \rfloor} b_{n,j}. \tag{49}$$

Now we compare both sides of (47) for the case $n + 1$. By (48) and (49), we have

$$a_{n+1,i} - a_{n+1,i-1} = \sum_{j=0}^{i} (-1)^{i+\lfloor \frac{j}{2} \rfloor} (k+1-i) b_{n,j} + \sum_{j=i+1}^{k} (-1)^{i+\lfloor \frac{j}{2} \rfloor} (k+1-j) b_{n,j}. \tag{50}$$

However,

$$(-1)^{\lfloor \frac{i}{2} \rfloor} b_{n+1,i} = (-1)^{\lfloor \frac{i}{2} \rfloor} \sum_{j=0}^{k} B_{i,j} b_{n,j} = \sum_{j=0}^{k} (-1)^{\lfloor \frac{i}{2} \rfloor + \lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k+1-j}{2} \rfloor} (k+1-\max(i,j)) b_{n,j},$$

which is equal to the right-hand side of (50) because the assumption that k is even implies

$$(-1)^{\lfloor \frac{i}{2} \rfloor + \lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k+1-j}{2} \rfloor} = (-1)^{\lfloor \frac{i}{2} \rfloor + \lfloor \frac{-i}{2} \rfloor + \lfloor \frac{1-j}{2} \rfloor} = (-1)^{i+\lfloor \frac{j}{2} \rfloor}.$$

Therefore, (47) is also true for $n + 1$. By induction, the claim is settled, which completes the proof. □

We finish this section by presenting an interesting consequence of Lemma 7.4.

Corollary 7.7. *We have*

$$\sum_{n \geq 1} |\operatorname{Alt}_n^{\leq k+1}| x^n = \cfrac{-y}{y - b_0 - \cfrac{-1}{y - b_1 - \cfrac{-1}{y - b_2 - \cfrac{\ddots}{ - \cfrac{-1}{y - b_k}}}}},$$

where $y = (-1)^k x$ and

$$b_i = \begin{cases} (-1)^k & \text{if } i = 0, \\ (-1)^{k-i} 2 & \text{if } 1 \leq i \leq k. \end{cases}$$

Proof. This is immediate from Proposition 2.7 and Lemma 7.4. □

It would be interesting to find a direct combinatorial proof of Corollary 7.7.

8. Application of the general reciprocity theorem

In this section, we show that the general reciprocity theorem (Theorem 6.1) implies the following result of Cigler and Krattenthaler [3, Theorem 34]. Using this theorem, we give a generalization of a

result on reverse plane partitions, which was conjectured by Morales, Pak and Panova [14] and proved independently by Hwang et al. [9] and Guo et al. [7].

Theorem 8.1 [3, Theorem 34]. *We have*

$$\det\left(\mu_{2n+2i+2j+4m-2}^{\leq 2k+2m-1}(\mathbf{0}, \boldsymbol{\lambda})\right)_{i,j=0}^{k-1}$$
$$= \left(\prod_{i=1}^{k+m-1} \lambda_{2i}^{k-i} \prod_{i=1}^{k+m} \lambda_{2i-1}^{k-i+n+2m-1}\right) R^{(2k+2m-1)}\left(\det\left(\mu_{-2n-2i-2j}^{\leq 2k+2m-1}(\mathbf{0}, \boldsymbol{\lambda})\right)_{i,j=0}^{m-1}\right).$$

We note that the statement in [3, Theorem 34] uses the change of variables $\lambda_{2i-1} = A_i^{-1} V_i^{-1}$ and $\lambda_{2i} = A_i^{-1} V_{i+1}^{-1}$.

As before, let $\boldsymbol{b} = (b_i)_{i\geq 0}$ and $\boldsymbol{\lambda} = (\lambda_i)_{i\geq 1}$ be sequences of indeterminates. We define $\lambda_0 = 0$ and the following sequences:

$$\begin{aligned}
\boldsymbol{b}' &= (b_i')_{i\geq 0}, & b_i' &= \lambda_{2i} + \lambda_{2i+1}, \\
\boldsymbol{\lambda}' &= (\lambda_i')_{i\geq 1}, & \lambda_i' &= \lambda_{2i-1}\lambda_{2i}, \\
\boldsymbol{b}'' &= (b_i'')_{i\geq 0}, & b_i'' &= \lambda_{2i+1} + \lambda_{2i+2}, \\
\boldsymbol{\lambda}'' &= (\lambda_i'')_{i\geq 1}, & \lambda_i'' &= \lambda_{2i}\lambda_{2i+1}.
\end{aligned}$$

Lemma 8.2. *We have*

$$\mu_{2n}^{\leq 2k-1}(\mathbf{0}, \boldsymbol{\lambda}) = \mu_n^{\leq k-1}(\boldsymbol{b}', \boldsymbol{\lambda}') = \lambda_1 \mu_{n-1}^{\leq k-1}(\boldsymbol{b}'', \boldsymbol{\lambda}'')|_{\lambda_{2k}=0}.$$

Proof. This can be proved by the same method in the proof of [4, Proposition 4.2]. □

Lemma 8.3. *We have*

$$\det\left(A^{\leq k-1}(\boldsymbol{b}', \boldsymbol{\lambda}')\right) = \prod_{i=1}^k \lambda_{2i-1}.$$

Proof. Expanding the determinant along the last row gives a simple recurrence for the left-hand side. Then the lemma follows easily by induction. □

Lemma 8.4. *We have*

$$\left(R^{(k-1)}\left(\mu_{-n}^{\leq k-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)\right)\Big|_{\boldsymbol{b}=\boldsymbol{b}', \boldsymbol{\lambda}=\boldsymbol{\lambda}'} = R^{(2k-1)}\left(\lambda_1^{-1} \mu_{-2n+2}^{\leq 2k-1}(\mathbf{0}, \boldsymbol{\lambda})\right).$$

Proof. It is easy to see that

$$\left(R^{(k-1)}\left(\mu_n^{\leq k-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)\right)\Big|_{\boldsymbol{b}=\boldsymbol{b}', \boldsymbol{\lambda}=\boldsymbol{\lambda}'} = R^{(2k-1)}\left(\mu_n^{\leq k-1}(\boldsymbol{b}'', \boldsymbol{\lambda}'')\right)\Big|_{\lambda_{2k}=0},$$

where in the right-hand side $\mu_n^{\leq k-1}(\boldsymbol{b}'', \boldsymbol{\lambda}'')$ is a polynomial in λ_i's and the operator $R^{(2k-1)}$ replaces λ_i to λ_{2k-i}. Thus, by Lemma 8.2, we obtain

$$\left(R^{(k-1)}\left(\mu_n^{\leq k-1}(\boldsymbol{b}, \boldsymbol{\lambda})\right)\right)\Big|_{\boldsymbol{b}=\boldsymbol{b}', \boldsymbol{\lambda}=\boldsymbol{\lambda}'} = R^{(2k-1)}\left(\lambda_1^{-1} \mu_{2n+2}^{\leq 2k-1}(\mathbf{0}, \boldsymbol{\lambda})\right).$$

Extending both sides to the negative indices completes the proof. □

Proof of Theorem 8.1. We put $\boldsymbol{b} = \boldsymbol{b}'$ and $\lambda = \lambda'$ in Theorem 6.1. By Lemmas 8.2, 8.3 and 8.4, we get

$$
\det\left(\mu^{\leq 2k+2m-1}_{2n+2i+2j+4m-4}(\mathbf{0}, \lambda)\right)^{k-1}_{i,j=0}
$$
$$
= \left(\prod_{i=1}^{k+m-1} \lambda^{k-i}_{2i-1}\lambda^{k-i}_{2i} \prod_{i=1}^{k+m} \lambda^{n+2m-2}_{2i-1}\right) R^{(2k+2m-1)}\left(\det\left(\lambda_1^{-1}\mu^{\leq 2k+2m-1}_{-2n-2i-2j+2}(\mathbf{0}, \lambda)\right)^{m-1}_{i,j=0}\right).
$$

Pulling out the factor λ_1^{-1} in the determinant and replacing n by $n + 1$ gives the desired equation.   □

Now we will give an application of Theorem 8.1 to reverse plane partitions. We will use the definitions of partitions and reverse plane partitions in Hwang et al. [9].

We denote by $\mathrm{RPP}(\lambda/\mu)$ the set of reverse plane partitions of shape λ/μ. We also denote by $\mathrm{RPP}^{\leq k}(\lambda/\mu)$ the set of reverse plane partitions in $\mathrm{RPP}(\lambda/\mu)$ whose entries are contained in $\{0, \dots, k\}$. For a reverse plane partition T, we define $|T|$ to be the sum of all entries in T.

Recall that Alt_{2n+1} is the set of sequences $a_1 \leq a_2 \geq a_3 \leq \cdots \geq a_{2n+1}$ of positive integers and $\mathrm{Alt}^{\leq k}_{2n+1}$ is the set of sequences in Alt_{2n+1} whose entries are contained in $\{1, \dots, k\}$. We define $\overline{\mathrm{Alt}}_{2n+1}$ to be the set of sequences $a_1 \geq a_2 \leq a_3 \geq \cdots \leq a_{2n+1}$ of positive integers and define $\overline{\mathrm{Alt}}^{\leq k}_{2n+1}$ to be the set of sequences in $\overline{\mathrm{Alt}}_{2n+1}$ whose entries are contained in $\{1, \dots, k\}$. For a sequence $s = (a_1, \dots, a_{2n+1})$ in Alt_{2n+1} or $\overline{\mathrm{Alt}}_{2n+1}$, let $|s| = a_1 + \cdots + a_{2n+1}$.

Morales, Pak and Panova [14] conjectured the following identity, which was proved independently by Hwang et al. [9] and Guo et al. [7]:

$$
\sum_{\pi \in \mathrm{RPP}(\delta_{n+2m}/\delta_n)} q^{|\pi|} = q^{-\frac{m(m+1)(6n+8m-5)}{6}} \det\left(\sum_{s \in \overline{\mathrm{Alt}}_{2n+2i+2j+1}} q^{|s|}\right)^{m-1}_{i,j=0}, \tag{51}
$$

where $\delta_n = (n-1, \dots, 0)$. In [9, Theorem 1.2] the matrix entries are generating functions for alternating sequences of nonnegative integers (see [9, (20)]), whereas (51) uses alternating sequences of positive integers. It is easy to check that the two statements are equivalent.

In [3, Theorem 36] Cigler and Krattenthaler gave an equivalent statement of Theorem 8.1 using trapezoidal arrays. They also found a bijection between trapezoidal arrays and bounded plane partitions. Using their bijection, a simple connection between bounded plane partitions and bounded reverse plane partitions, a simple connection between $\mathrm{Alt}^{\leq k+m}_{2n+1}$ and $\overline{\mathrm{Alt}}^{\leq k+m}_{2n+1}$, and the change of variables $V_i \mapsto V_{k+m+1-i}$ and $A_i \mapsto A_{k+m+1-i}$, we can restate [3, Theorem 36] as follows.

Theorem 8.5 [3, Theorem 36 (restated)]. *We have*

$$
\sum_{T \in \mathrm{RPP}^{\leq k}(\delta_{n+2m}/\delta_n)} \mathrm{wt}(T) = \det\left(\sum_{t \in \overline{\mathrm{Alt}}^{\leq k+m}_{2n+2i+2j+1}} \mathrm{wt}(t)\right)^{m-1}_{i,j=0}, \tag{52}
$$

where for $T \in \mathrm{RPP}^{\leq k}(\delta_{n+2m}/\delta_n)$ and $t = (t_1, \dots, t_{2p+1}) \in \overline{\mathrm{Alt}}^{\leq k+m}_{2p+1}$,

$$
\mathrm{wt}(t) = V_{t_1}V_{t_3}\cdots V_{t_{2n+1}}A_{t_2}A_{t_4}\cdots A_{t_{2n}},
$$
$$
\mathrm{wt}(T) = \prod_{(i,j)\in\delta_{n+2m}/\delta_n} \mathrm{wt}(T(i,j)),
$$
$$
\mathrm{wt}(T(i,j)) = \begin{cases} A_{T(i,j)+\lfloor(i+j-n+1)/2\rfloor} & \text{if } i+j-n \text{ is odd}, \\ V_{T(i,j)+\lfloor(i+j-n+1)/2\rfloor} & \text{if } i+j-n \text{ is even}. \end{cases}
$$

Substituting $V_i = A_i = q^i$ in the equation (52) gives the following corollary.

Corollary 8.6. *We have*

$$\sum_{S \in \mathrm{RPP}^{\leq k}(\delta_{n+2m}/\delta_n)} q^{|S|} = q^{-\frac{m(m+1)(6n+8m-5)}{6}} \det\left(\sum_{s \in \overline{\mathrm{Alt}}^{\leq k+m}_{2n+2i+2j+1}} q^{|s|}\right)_{i,j=0}^{m-1}.$$

If $k \to \infty$ in Corollary 8.6, we get (51).

9. Negative moments of Laurent biorthogonal polynomials

Recall that we have combinatorial reciprocity theorems for the number of Dyck paths of bounded height and for the number of Motzkin paths of bounded height. Therefore, it is natural to ask whether there is a reciprocity theorem for the number of Schröder paths of bounded height. In this section we study the negative version of the number of Schröder paths with bounded height and its connection with the negative moment of Laurent biorthogonal polynomials.

The *Laurent biorthogonal polynomials* $(L_n(x))_{n\geq 0}$ can be defined by a three-term recurrence

$$L_{n+1}(x) = (x - b_n)L_n(x) - a_n x L_{n-1}(x), \quad n \geq 0, \qquad L_{-1}(x) = 0, L_0(x) = 1, \qquad (53)$$

for some sequences $\boldsymbol{b} = (b_n)_{n\geq 0}$ and $\boldsymbol{a} = (a_n)_{n\geq 1}$. To emphasize sequences $\boldsymbol{b}$ and $\boldsymbol{a}$ we will write the polynomials $L_n(x)$ as $L_n(x; \boldsymbol{b}, \boldsymbol{a})$. There is a unique linear functional $\mathcal{L}$ on the space of Laurent polynomials such that $\mathcal{L}(1) = 1$ and

$$\mathcal{L}\left(L_m(x; \boldsymbol{b}, \boldsymbol{a}) \cdot \frac{L_n(x; \boldsymbol{b}, \boldsymbol{a})}{x^n}\right) = 0, \qquad 0 \leq m < n.$$

Since the linear functional $\mathcal{L}$ is defined on the space of Laurent polynomials, we have *positive moments* $\mathcal{L}(x^n)$ and *negative moments* $\mathcal{L}(x^{-n})$. Kamioka [10, 11] showed that both positive and negative moments are generating functions for Schröder paths. To state Kamioka's results, we need the following definitions.

Recall that a lattice path is a finite sequence of points in $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

Definition 9.1. A *Schröder path* is a lattice path in which every step is an *up step* $(1, 1)$, a *double-horizontal step* $(2, 0)$ or a *down step* $(1, -1)$. The set of Schröder paths from $(0, 0)$ to $(n, 0)$ is denoted by Sch_n.

For given sequences $\boldsymbol{b} = (b_0, b_1, \dots)$ and $\boldsymbol{a} = (a_1, a_2, \dots)$, the *weight* $\mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{a})$ of a Schröder path π is defined to be the product of b_i for each double-horizontal step starting at a point with y-coordinate i and a_i for each down step starting at a point with y-coordinate i.

Kamioka [10, 11] showed that the moments $\mathcal{L}(x^n)$, $n \in \mathbb{Z}$, of Laurent biorthogonal polynomials $L_n(x; \boldsymbol{b}, \boldsymbol{a})$ are generating functions for Schröder paths. For $n \geq 0$,

$$\mathcal{L}(x^n) = \sum_{\pi \in \mathrm{Sch}_{2n}} \mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{a}), \qquad (54)$$

$$\mathcal{L}(x^{-n-1}) = b_0^{-1} \sum_{\pi \in \mathrm{Sch}_{2n}} \mathrm{wt}(\pi; \boldsymbol{b}', \boldsymbol{a}'), \qquad (55)$$

where $\boldsymbol{b}'$ and $\boldsymbol{a}'$ are the sequences defined by

$$\boldsymbol{b}' = (b'_i)_{i \geq 0}, \qquad b'_i = b_i^{-1}, \tag{56}$$

$$\boldsymbol{a}' = (a'_i)_{i \geq 1}, \qquad a'_i = a_i b_{i-1}^{-1} b_i^{-1}. \tag{57}$$

Definition 9.2. We define the *bounded moment* $\sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$ of the Laurent biorthogonal polynomials $L_n(x; \boldsymbol{b}, \boldsymbol{a})$ by

$$\sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) = \sum_{\pi \in \mathrm{Sch}_{2n}^{\leq k}} \mathrm{wt}(\pi; \boldsymbol{b}, \boldsymbol{a}),$$

where $\mathrm{Sch}_{2n}^{\leq k}$ is the set of Schröder paths from $(0, 0)$ to $(2n, 0)$ that stay weakly below the line $y = k$.

Note that if the sequence $(\sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a}))_{n \geq 0}$ satisfies a homogeneous linear recurrence relation, then its negative version $(\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}))_{n \geq 1}$ is defined. By definition, we have

$$\mathcal{L}(x^n) = \lim_{k \to \infty} \sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a}), \qquad n \geq 0.$$

The goal of this section is to prove that the negative moment $\mathcal{L}(x^{-n})$ is also the limit of the negative version of $(\sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a}))_{n \geq 0}$.

By specializing the results of Kim and Stanton [12, Corollary 5.4 and Propositions 5.5] on orthogonal polynomials of type R_I, we obtain the following.

Proposition 9.3. *We have*

$$\sum_{n \geq 0} \sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) x^n = \frac{\delta P_k^*(x; \boldsymbol{b}, \boldsymbol{a})}{P_{k+1}^*(x; \boldsymbol{b}, \boldsymbol{a})} = \cfrac{1}{1 - b_0 x - \cfrac{a_1 x}{1 - b_1 x - \cfrac{a_2 x}{1 - b_2 x - \cfrac{\ddots}{\quad - \cfrac{a_k x}{1 - b_k x}}}}},$$

where $\delta P_k^*(x; \boldsymbol{b}, \boldsymbol{a})$ and $P_{k+1}^*(x; \boldsymbol{b}, \boldsymbol{a})$ are defined similarly as in Definition 2.3.

The following proposition can be proved similarly as Propositions 2.10 and 2.11.

Proposition 9.4. *If $P_{k+1}(0; \boldsymbol{b}, \boldsymbol{a}) \neq 0$, then $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$ is well-defined for $n \geq 1$. In particular, if $b_i \neq 0$ for all $i \geq 0$, then $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$ is well-defined for $n \geq 1$.*

By the same argument as the one in the proof of Proposition 2.7, we obtain the generating function for $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$ as follows.

Proposition 9.5. *If $(\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}))_{n \geq 1}$ is defined, we have*

$$\sum_{n \geq 1} \sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) x^n = -\frac{x \delta P_k(x; \boldsymbol{b}, \boldsymbol{a})}{P_{k+1}(x; \boldsymbol{b}, \boldsymbol{a})} = \cfrac{x}{b_0 - x - \cfrac{a_1 x}{b_1 - x - \cfrac{a_2 x}{b_2 - x - \cfrac{\ddots}{\quad - \cfrac{a_k x}{b_k - x}}}}}.$$

Using Proposition 9.5, we can find a combinatorial interpretation for $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$.

Theorem 9.6. *Let n, k be positive integers. We have*

$$\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) = b_0^{-1} \sum_{\pi \in \mathrm{Sch}_{n-1}^{\leq k}} \mathrm{wt}(\pi; \boldsymbol{b}', \boldsymbol{a}'),$$

where $\boldsymbol{b}'$ and $\boldsymbol{a}'$ are defined in (56) and (57).

Proof. Let

$$f_n^{\leq k} = \sum_{\pi \in \mathrm{Sch}_n^{\leq k}} \mathrm{wt}(\pi; \boldsymbol{b}', \boldsymbol{a}').$$

By Proposition 9.3, we have

$$\sum_{n \geq 0} f_n^{\leq k} x^n = \cfrac{1}{1 - b_0^{-1}x - \cfrac{a_1 b_0^{-1} b_1^{-1} x}{1 - b_1^{-1}x - \cfrac{a_2 b_1^{-1} b_2^{-1} x}{1 - b_2^{-1}x - \cfrac{\ddots}{} - \cfrac{a_k b_{k-1}^{-1} b_k^{-1} x}{1 - b_k^{-1}x}}}}$$

$$= \cfrac{b_0}{b_0 - x - \cfrac{a_1 x}{b_1 - x - \cfrac{a_2 x}{b_2 - x - \cfrac{\ddots}{} - \cfrac{a_k x}{b_k - x}}}}.$$

Comparing this with Proposition 9.5, we obtain

$$\sum_{n \geq 1} \sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) x^n = b_0^{-1} x \sum_{n \geq 0} f_n^{\leq k} x^n = \sum_{n \geq 1} b_0^{-1} f_{n-1}^{\leq k} x^n.$$

Therefore, $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}) = b_0^{-1} f_{n-1}^{\leq k}$, which is the desired result. $\qquad \square$

Substituting $\boldsymbol{b} = \boldsymbol{a} = \mathbf{1}$ in Theorem 9.6, we see that the negative version of the number of bounded Schröder paths is also the number of bounded Schröder paths.

Corollary 9.7. *Let $s_n = |\mathrm{Sch}_n^{\leq k}|$ for $n \geq 0$. Then for $n \geq 1$, we have $s_{-n} = s_{n-1}$.*

By Theorem 9.6 and (55), we obtain that the negative moments $\mathcal{L}(x^{-n})$ are the limits of the negative versions $\sigma_{-n}^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$ of the bounded moments $\sigma_n^{\leq k}(\boldsymbol{b}, \boldsymbol{a})$.

Corollary 9.8. *For $n \geq 1$,*

$$\mathcal{L}(x^{-n}) = \lim_{k \to \infty} \sigma_{-n+1}^{\leq k}(\boldsymbol{b}, \boldsymbol{a}).$$

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