LOW FREQUENCY GRAVITATIONAL WAVES IN COSMOLOGY

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Abstract. An intense non-thermal background of cosmic gravitational radiation in the Megaparsec wave band could be detected by its influence on many astrophysical processes. In particular, it may give an explanation of the so-called redshift anomalies.

1. Introduction

An intense experimental search for cosmic gravitational radiation has been started by many groups, based on the pioneering work by Weber (1969, 1970a, b, 1972) and by Braginski (1972). Although the results of these experiments are still under discussion, and positive effects could be explained perhaps more successfully by solar-terrestrial or associated geophysical effects (Tyson *et al.*, 1973), the possible existence and effect of cosmic gravitational radiation in astrophysical processes remain an interesting problem. If strong time-dependent gravitational fields with GM/Rc^2 of order 1 occur in relativistic objects such as black holes, pulsars and possibly also quasars, we expect appreciable amounts of gravitational radiation from these sources. Both the gravitational wavelengths and the wave amplitudes are subject to upper bounds: since only in highly relativistic objects is the output expected to be large, the wavelengths do not exceed the geometrical dimensions of the sources by a very large factor. The intensity is limited by the available amount of rest mass – at least as long as gravitational theories with matter creation are excluded.

A possible source of non-thermal gravitational radiation of both much larger wavelengths and presumably much higher intensity is the fireball state of the metagalaxy (Zel'dovich, 1966; Zipoy, 1966; Ruffini and Wheeler, 1969; Dautcourt, 1969a, b; Rees, 1971, 1972a, b; Gowdy, 1971). The radiation may either result from turbulent motion of primordial matter or the transverse degrees of freedom of the cosmological gravitational field are a priori excited. In the latter case the active gravitational mass of background radiation fields may be high enough to fill the cosmological density gap and create – assuming homogeneity and isotropy of the radiation in the mean – a Tolman radiation universe, a simple example of an unstable 'gravitational geon' in Wheeler's language. There is in general no upper bound to the range of wavelengths in an open universe, the amplitudes only being limited for long wavelengths to ensure sufficient homogeneity on the large scale. Rees (1971) was the first to face the interesting astrophysical consequences of this possibility.

The intensity of the gravitational background radiation can be measured either by its energy density ϱc^2 (which according to Isaacson (1968) can be defined in a covariant way for sufficiently high-frequency radiation) or by the dimensionless amplitudes h of the perturbation in the metric tensor, representing the wave fields. Both

M. S. Longair (ed.), Confrontation of Cosmological Theories with Observational Data, 299–315. *All Rights Reserved. Copyright © 1974 by the IAU.*

https://doi.org/10.1017/S007418090023564X Published online by Cambridge University Press

quantities are connected by an order-of-magnitude relation

$$\varrho c^2 \simeq \frac{c^4 h^2}{G\lambda^2} = \frac{c^4 h_0^2}{G\lambda_0^2} \quad (1+z)^4, \tag{1}$$

if the wave amplitudes are measured in a locally Minkowskian coordinate system. λ is the dominant wave length. With cosmological epoch h and λ change as $h \simeq (1+z)\, h_0$, $\lambda = (1+z)^{-1}\, \lambda_0$, where the subscript zero refers to the present epoch $z=0$. Thus, as must be expected for cosmic background radiation, the energy density increases as $\sim (1+z)^4$ into the past.

The dynamical effect of the wave field on matter depends on the field strength h, while its contribution to the smoothed-out cosmological background metric is measured by the equivalent active gravitational mass density ϱ. From Equation (1) it is seen that for a given energy density – which can not of course greatly exceed the critical energy density $\varrho_c = 3H_0^2/\kappa$ – the amplitude h and therefore the influence on the motion of matter and radiation increases with wavelength and becomes appreciable if λ reaches some fractional value of the world horizon distance $\lambda_H \simeq 3000$ $(100/H_0)$ Mpc. For ultra low-frequency gravitational waves with λ between 1 and 3000 Mpc, h ranges from 3×10^{-4} to 1. These values may be compared with the much lower metric amplitudes $h \simeq 10^{-18}$ detectable by Weber's experiment (Press and Thorne, 1972). They may also be compared with the dimensionless Newtonian gravitational potential of galaxies, $\phi/c^2 \simeq 10^{-6}$. Thus, if an appreciable amount of intergalactic gravitational radiation is stored in the Megaparsec wave band, rather remarkable observational effects must be expected. It is suggested here that the effects may be those known as redshift anomalies and discussed extensively in recent years (Burbidge and Sargent, 1969; Burbidge, 1968; Arp, 1970, 1971; Burbidge and Sargent, 1971; Burbidge and O'Dell, 1972; Tifft, 1972, 1973). Independent of this possible relation, a study of observable effects produced by low-frequency waves leads to new upper limits for the low-frequency end of the spectrum of gravitational background radiation.

2. Random Wave Fields

The possibility of intergalactic gravitational radiation with wavelengths comparable to the Hubble distance was apparently first considered by Kristian and Sachs (1966). A description of wave fields in terms of a spatially homogeneous and isotropic random process analogous to methods employed in optical coherence theory (Mandel and Wolf, 1965) seems appropriate. Solutions of the first order wave equation for perturbations $\underset{1}{g}_{00} = \underset{1}{g}_{0i} = 0$, $\underset{1}{g}_{ik} \neq 0$ with $\underset{1}{g}_{ii} = 0$, $\underset{1}{g}_{ik,k} = 0$ to a flat Robertson-Walker model $\underset{0}{g}_{00} = -1$, $\underset{0}{g}_{0i} = 0$, $\underset{0}{g}_{ik} = (t/t_0)^n\, \delta_{ik}$ will be employed (Sachs and Wolfe, 1967). The general solution of the wave equation in this cosmological model may be represented by a spatial Fourier integral

$$\underset{1}{g}_{ik} = \int \gamma_{ik}(\mathbf{k}, t)\, e^{i\phi}\, \mathrm{d}\mathbf{k} + \text{complex conjugate}, \tag{2}$$

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where the phase $\phi = \mathbf{k}\ \mathbf{x} + k(\lambda_H nz/(1-n)(1+z))$ describes null planes in the background metric. The Fourier components γ_{ik} are time dependent and satisfy, apart from the algebraic conditions $\gamma_{ik}k^k = 0$, $\gamma_{ii} = 0$, an ordinary linear differential equation

$$\ddot{\gamma}_{ik} - \dot{\gamma}_{ik}\left(\frac{2ik}{R} + \frac{\dot{R}}{R}\right) - \gamma_{ik}\left(\frac{6\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} - \frac{2ik\dot{R}}{R^2}\right) = 0 \tag{3}$$

with $R = (t/t_0)^n$.

An interpretation of the right-hand-side of Equation (2) as stochastic Fourier integrals accounts for the expected random behaviour of intergalactic waves. The average structure of the wave field is described by the mutual covariance functions (mcf's) for the perturbation of the metric tensor. The most general mcf which could be formed from the random components $\underset{1}{g}_{ik}(\mathbf{x}, t)$ is given by

$$\langle \underset{1}{g}_{ik}(\mathbf{x}, t)\, \underset{1}{g}_{lm}(\mathbf{x}', t')\rangle = 2\int \mathrm{d}\mathbf{k}\ \gamma_{iklm}(\mathbf{k}, t, t') \times \\ \times \cos\left[\mathbf{k}(\mathbf{x}-\mathbf{x}') + \frac{k\lambda_H n}{1-n}\left(\frac{z}{1+z} - \frac{z'}{1+z'}\right)\right] \tag{4}$$

where according to standard theorems in spectral theory

$$\begin{aligned} \langle \gamma_{ik}(\mathbf{k}, t)\, \gamma^*_{lm}(\mathbf{k}', t')\rangle &= \gamma_{iklm}(\mathbf{k}, t, t')\, \delta(\mathbf{k}-\mathbf{k}') \\ \langle \gamma_{ik}(\mathbf{k}, t)\, \gamma_{lm}(\mathbf{k}', t')\rangle &= 0 \end{aligned} \tag{5}$$

has been used. Requirements of isotropy and homogeneity for the tensor quantities $\underset{1}{g}_{ik}$ restrict the form of the spectral density

$$\begin{aligned} &\gamma_{iklm}(\mathbf{k}, t, t') = \alpha(k, t, t')\, \delta_{iklm} \\ &\delta_{iklm} = \bar{\delta}_{il}\bar{\delta}_{km} + \bar{\delta}_{im}\bar{\delta}_{kl} - \bar{\delta}_{ik}\bar{\delta}_{lm} \\ &\bar{\delta}_{ik} = \delta_{ik} - k_i k_k/k^2 . \end{aligned} \tag{6}$$

$\alpha = \alpha(k, t, t')$ is a single real function of k, $k = \sqrt{\mathbf{kk}}$, as well as of t and t'. Apart from problems of stochastic particle acceleration by wave fields – which will be discussed elsewhere – the mcf's are needed at time $t = t'$ only; thus only $\alpha(k, t) = \alpha(k, t, t)$ is considered below. One may also introduce mcf's involving first derivatives of $\underset{1}{g}_{ik}$; these quantities can be reduced to the corresponding spectral densities

$$\langle \dot{\gamma}_{ik}\dot{\gamma}^*_{lm}\rangle = \beta\delta_{iklm}\delta(\mathbf{k}-\mathbf{k}') \tag{7}$$

$$\langle \dot{\gamma}^*_{ik}\gamma_{lm}\rangle = \Gamma\delta_{iklm}\delta(\mathbf{k}-\mathbf{k}') \tag{8}$$

with real $\beta(k, t)$ and complex $\Gamma(k, t)$. Since every realization of $\underset{1}{g}_{ik}$ satisfies the wave equation, second time derivatives of $\underset{1}{g}_{ik}$ can be replaced by at least first-order time derivatives. Thus spectral densities corresponding t to higher derivatives may be reduced to the basic spectral densities α, β and Γ. Differentiating the defining Equa-

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tions (5), (7) and (8) with respect to t and using again the generalized wave equation, one obtains a coupled system of equations for α, β, Γ, which can be reduced to a non-linear differential equation

$$\ddot{\alpha}-\frac{\dot{\alpha}^2}{2\alpha}-\frac{\dot{R}}{R}\dot{\alpha}+\frac{2\alpha k^2}{R^2}-\frac{4\dot{R}^2\alpha}{R^2}-\frac{12\ddot{R}}{R}\alpha=\frac{2k^2\alpha_0^2R^2}{\alpha} \tag{9}$$

for α alone, with $\alpha_0=\text{const}$. Solving this equation in the high-frequency approximation $k\lambda_H \gg 1$, or more generally, for $\dot{R}/R \ll ck$,

$$\alpha=\frac{\alpha_0(k)}{(1+z)^2}$$

follows. Similarly one obtains

$$\beta=k(\alpha_1+2\alpha_0 k) \tag{11}$$

$$\gamma\equiv\mathrm{Re}(\Gamma)=(\alpha_1+2\alpha_0 k)/(1+z), \tag{12}$$

where α_1 is a further arbitrary function of k. In the same high-frequency approximation the energy-momentum tensor of the gravitational wave fields can be calculated according to the description given by Isaacson (1968), with the averaging process in this procedure corresponding to an ensemble average. The energy-momentum tensor is that of ideal fluid matter with an ultra-relativistic equation of state, $p=\varrho/3$, where

$$\kappa\varrho=8\pi(1+z)^4\int\alpha_0(k)\,k^4\,\mathrm{d}k. \tag{13}$$

Equation (1) essentially follows from Equation (13) in the approximation of monochromatic radiation, where the spectral density α_0 is replaced by an expression proportional to the Dirac delta function $\delta(k-k_0)$ around a wavelength $\lambda_0=2\pi/k_0$.

The simple description of random wave fields in a spatially flat universe, given by Equations (2)–(13) may be used to predict several observable effects, as discussed below. Some restrictions on this description should be kept in mind.

(i) While the approach appears acceptable for the wave amplitudes, appreciable errors could be introduced in the phases, which are calculated to zero order only, using the cosmological background metric. As must be expected from the geometrical interpretation of the gravitational wave fields, a phase-amplitude relation exists, which causes the first-order phase correction to depend on the metric fluctuations $\underset{1}{g}_{ik}$. The effect distorts phase coherence properties and may have an influence on observable effects, even for small phase corrections.

(ii) The solutions (10)–(12) for the mcf's are confined to sufficiently high frequencies or to wavelengths small compared to the horizon distance. Since the ratio wavelength to horizon distance changes as $(\lambda/\lambda_H)/(\lambda/\lambda_H)_0=(1+z)^{1/n-1}$, this approach breaks down for $z>z^*$, where $(\lambda/\lambda_H)_0(1+z^*)^{1/n-1}\simeq 1$. In this case more general solutions can be found, in general, only by numerical calculation.

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Note also that a basic postulate which should be fulfilled for real wave fields is the property of ergodicity. It should be possible to replace ensemble averages by space averages. Otherwise, a determination of the spectral density from observations in a limited space-time domain would not be possible.

3. Variations in the Microwave Background

The intensity of electromagnetic radiation from distant sources will show spatial and temporal fluctuations due to an interaction with gravitational background radiation, see Zipoy (1966) and Kaufmann (1970). In particular, electromagnetic background radiation is influenced and should vary in intensity across the sky. If at any time t^* in the past the intensity I_ν of background radiation can be described by a Planck spectrum, the same description applies for $t>t^*$, with a space-time and direction depending temperature field T. Its anisotropic part $\tau=-T_0+T$ has at the present instant of time a Fourier decomposition

$$\tau=\int \hat{\tau}(\mathbf{k})\, e^{i\mathbf{kx}-ik\xi}+\text{complex conjugate}. \tag{14}$$

The integration of the equation of radiative transfer for τ, with Thomson scattering as well as interaction with gravitational waves taken into account, leads to an explicit expression for $\hat{\tau}$, given by ($\hat{\tau}$ is supposed to be zero at some initial time t_1):

$$\hat{\tau}(\mathbf{k})=-\frac{kT_0\gamma_{ik}n^i n^k}{2(\mathbf{kn}+k)}\left(1-[1+z_1]\exp\left[-\frac{\lambda_H}{\lambda_c}\,\bar{q}+\right.\right.$$
$$\left.\left.+\frac{in\lambda_H(\mathbf{kn}+k)}{(1-n)}\left(1-[1+z_1]^{1-1/n}\right)\right]\right). \tag{15}$$

Here

$$\bar{q}=\int_0^{z_1} q(z')\,(1+z')^{2-1/n}\,\mathrm{d}z', \qquad \lambda_c=1/n_0\sigma_T, \tag{16}$$

where $q(z')$ is the degree of ionization of intergalactic matter, n_0 the present density of matter and σ_T the Thompson cross section.

It is seen from Equation (15) that $\hat{\tau}$ contains two parts. One component varies slowly across the sky, the other component, proportional to the exponential function, oscillates rapidly with the direction of observation. The slowly varying component results from the local gravitational wave field in the neighbourhood of the observer; the fluctuating part arises from the interaction of the blackbody radiation with the gravitational wave field at some early instant of time prior to the pre-galactic plasma recombination (Dautcourt, 1974).

The temperature variation on a large angular scale may be decomposed into spherical harmonics. The complex Fourier components γ_{ik} of the local wave field can

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be represented algebraically by

$$\gamma_{ik} = P(l_i l_k - m_i m_k) + Q(l_i m_k + l_k m_i), \tag{17}$$

where l_i, m_k are two unit directions orthogonal to each other and to $\mathbf{k}/k$, and P, Q are two complex functions of $\mathbf{k}$, describing the amplitude and phase of the local wave field. The resulting large-scale temperature variation depends only on the real part of the arbitrary functions P and Q. It should be noted that P and Q may take any value: their ensemble mean only is restricted by the requirement of yielding no contribution to the energy density and pressure of the wave field that exceeds the cosmological limit. Furthermore, since $P_1 = \mathrm{Re}(P)$ and $Q_1 = \mathrm{Re}(Q)$ are independent of each other, their ratio is also arbitrary. Nothing is known of the probability distribution of P and Q.

If P_1 and Q_1 depend only on the wave number k but not on the wave direction $\mathbf{k}/k$, a numerical evaluation of the coefficients of the dipole and quadrupole components of τ has shown (Dautcourt, 1974), that the dipole contribution is only about 1% of the anisotropic part in T, provided P_1 and Q_1 are of comparable order. The main component of τ would be of quadrupole type, given by

$$\tau/T_0 \simeq 3.50 \sin^2\theta \int k^2 P_1(k)\, \mathrm{d}k \tag{18}$$

(θ is an azimuthal angle in a wave orientated coordinate system). Several measurements of possible variations of the 3 K radiation on large angular scales have been made (Partridge and Wilkinson, 1967; Conklin and Bracewell, 1967a, b; Conklin, 1972), only Conklin (1972) reports a possibly positive results, $(\tau/T_0)_{\mathrm{dipole}} = (8.5-3.4) \times \times 10^{-4}$ and $(\tau/T_0)_{\mathrm{quadrupole}} = (5.3-3.0) \times 10^{-4}$, respectively. The dipole component, if real, may correspond to the Earth's motion with respect to the cosmological frame of reference (Sciama, 1967). The quadrupole component gives an upper limit

$$\int k^2 P_1\, \mathrm{d}k \lesssim 2.4 \times 10^{-4} \tag{19}$$

for the local wave field.

Turning to small-scale variations, the observable quantities are the rms temperature fluctuation

$$\sigma = \langle \tau^2 \rangle^{1/2} \tag{20}$$

and the angular correlation function

$$\Gamma(\theta) = \langle \tau(x, y)\, \tau(x+u, y+v) \rangle / \sigma^2, \tag{21}$$

where $\theta^2 = u^2 + v^2$ and x, y are rectangular coordinates in a small region on the sky.

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From Equation (15), the fractional temperature fluctuation is given by

$$\sigma/T_0=(1+z_1)\,r_1\left(\frac{8\pi}{3}\int \alpha_0 k^2\,\mathrm{d}k\;\Psi(k,z_1)\right)^{1/2},$$
$$r_1=\exp\left(-\frac{\lambda_H}{\lambda_c}\,\bar{q}\right), \tag{22}$$

where $\Psi(k, z_1)$ is a function accounting for a deviation of the spectral density α from a $\sim(1+z)^{-2}$ dependence for large z, as mentioned above. z_1 has to be chosen sufficiently large, $z_1 \gtrsim 2\times10^3$, to ensure that those variations in the background radiation which might be present at $z=z_1$ are nearly completely damped out-by Compton scattering in pre-galactic matter – at the present instant of time. The remaining fluctuations in the 3 K radiation arise at redshifts smaller than $z=z_1$ and are given by Equation (22). Their amplitude depends critically on the exponential damping factor r_1 in Equation (22). r_1 is a function of the ratio of the present matter density to the Hubble constant H_0 and also depends on the scale factor index n. In a low-density universe, the existing upper limits (Parijskij and Pyatunina, 1970; Boynton and Partridge, 1973), in particular the extremely low value of $\tau/T_0 \lesssim 3\times10^{-5}$ found recently by Parijskij (1973) on angular scales between 3′ and 1°, confine gravitational background radiation to small amplitudes with energy densities below the critical cosmological density. On the other hand, primordial small-scale variations are damped out if the present matter density is sufficiently large, $\varrho_m > \varrho^*$. The matter density ϱ^* for which small-scale variations induced by gravitational radiation with critical cosmological energy density could just have been detected by Parijskij's measurement, depends on the gravitational wavelengths and on the value of the Hubble constant. For the range $\lambda = 3 \ldots 100$ Mpc and $H_0 = 50 \ldots 100$ km s^{-1} Mpc^{-1}, ϱ^* has† a value between 10^{-30} and 10^{-29} g cm^{-3} (Dautcourt, 1973).

The angular correlation length $\Delta\theta$ for small-scale temperature fluctuations arising at $z \lesssim z_1$ is approximately given by

$$\Delta\theta \simeq \frac{\lambda}{\lambda_H}\,\frac{1+z_1}{z_1}, \tag{23}$$

which gives $\Delta\theta^\circ \simeq 2\times10^{-2}\,\lambda$ for $z_1 = 2\times10^3$. This covers the range of scales for which Parijskij gives his limit.

To summarize, the microwave anisotropy measurements do not necessarily exclude the existence of gravitational background radiation, even if it reaches the critical energy density and has large mean wavelengths. They would do so, however, if the microwave background is not of primordial origin but arises from the superposition of radiation from many discrete sources, since here the reduction factor r_1 in Equation (22) is no longer small compared to 1.

† Provided that the phase perturbation mentioned above – which corrects the zero-order phase calculation – is not large enough to cast doubt on the application of the method of stationary phase, that has been used to derive Equation (15).

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4. Redshift Fluctuations

The particular interest in low-frequency gravitational waves comes from the existence of a fluctuating component in the redshifts of galaxies and other distant objects. It is well known that gravitational waves cause a beam of photons to experience fluctuations in frequency (Zipoy, 1966; Kaufmann, 1970). Let V_μ^Q, V_μ^P be the four-velocities of a light source and an observer, respectively. The redshift measured by the observer at the world point P is given by

$$1+z_{\text{total}}=P^\mu V_\mu^Q/P^\nu V_\nu^P, \tag{24}$$

where P^μ is the ray direction.

Provided that source and observer have no peculiar velocity, a simple calculation gives for z_{total}

$$1+z_{\text{total}}=1+z+\delta z. \tag{25}$$

Here z is the non-random mean of the redshift, connected with distance by the Hubble relation $D=cz/H_0$ for small z. The second component δz is given by

$$\begin{aligned}\delta z&=-it_0\int_0^{z_1}\frac{\mathrm{d}z}{1+z}\int \mathbf{dk}\,\gamma e^{i\phi}(\mathbf{kn}+k)+\text{complex conjugate}\simeq\\ &\simeq\int \mathbf{dk}\,\gamma_1(1-[1+z_1]\cos\phi)+(1+z_1)\int \mathbf{dk}\,\gamma_2\sin\phi,\end{aligned} \tag{26}$$

$$\begin{aligned}\gamma&=\gamma_1+i\gamma_2=\gamma_{ik}n^in^k,\\ \phi&=\frac{\lambda_H(\mathbf{kn}+k)\,z}{1+z},\end{aligned} \tag{27}$$

where $\mathbf{n}$ is the source direction and the approximation holds for small redshifts or, for arbitrary z, after using the method of stationary phase (Copson, 1965) to carry out the z integration. Equations (26) and (27) as well as the formula below hold for a Tolman universe ($R\sim t^{1/2}$). From the random character of the metric quantities it follows that the fluctuating redshift component is also a random quantity, changing irregularly with the source position. If the wavelengths are confined to a small range around the mid frequency k (this corresponds to quasimonochromatic gravitational radiation) δz shows periodicities with periods in source distance and in angular distance of the order

$$\Delta z\simeq\frac{\lambda}{\lambda_H}(1+z), \tag{28}$$

$$\Delta\theta\simeq\frac{\lambda}{\lambda_H}\frac{(1+z)}{z}, \tag{29}$$

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respectively. Another interpretation of Δz, $\Delta\theta$ as given by Equations (28), (29) is in terms of a correlation length with respect to depth and to angular separation. The basic quantity in this connexion is the redshift fluctuation autocovariance function

$$\langle \delta z_1\, \delta z_2 \rangle = S(z_1, z_2, \theta). \tag{30}$$

S is equal to the ensemble averaged mean of the product $\delta z_1\, \delta z_2$ of two fluctuating redshift components $\delta z_1, \delta z_2$, associated with two sources within different depths, corresponding to the mean redshifts z_1, z_2, and separated by an angular distance $\theta = \operatorname{arc}\cos(n_1^i n_2^i)$ on the sky. In full generality S turns out to be complicated. For particular cases, analytic expressions are available. For zero angular lag$(\theta=0)$;

$$\begin{aligned} S(z_1, z_2, 0) = \frac{16\pi}{15} \int_0^\infty & \alpha_0 k^2\, \mathrm{d}k \,[1 - (1+z_1)\, f(x_1) - \\ & - (1+z_2)\, f(x_2) + (1+z_1)(1+z_2)\, f(x_1 - x_2)], \end{aligned} \tag{31}$$
$$x_1 = k\lambda_H z_1/(1+z_1), \qquad x_2 = k\lambda_H z_2/(1+z_2),$$

with $f(x)$ as an oscillating function tending to 1 for $x \to 0$:

$$f(x) = \frac{15}{x^3} \cos x \left[3\, \frac{\sin x}{x^2} - 3\, \frac{\cos x}{x} - \sin x \right]. \tag{32}$$

The amplitude $\langle \delta z^2 \rangle$ is given by

$$\begin{aligned} \langle \delta z^2 \rangle &= \frac{16\pi}{15} \int_0^\infty \alpha_0 k^2\, \mathrm{d}k \,[1 + (1+z_1)^2 - 2(1+z_1)\, f(x_1)] \simeq \\ &\simeq \frac{32\pi}{15} \left(1 + z + \frac{z^2}{2} \right) \int_0^\infty \alpha_0 k^2\, \mathrm{d}k, \end{aligned} \tag{33}$$

where the approximation is valid for large values of $k\lambda_H z/(1+z)$, with the oscillating terms being damped out. Finally, in the case of quasi-monochromatic background radiation,

$$\langle \delta z^2 \rangle^{1/2} \simeq \frac{\bar{\lambda}}{\lambda_H} \sqrt{\frac{4}{5} \left(1 + z + \frac{z^2}{2} \right) \frac{\varrho}{\varrho_c}}, \quad \bar{\lambda} = \frac{\lambda}{2\pi} \tag{34}$$

From Equations (33) and (34) an important conclusion can be derived: distances to extragalactic objects – if determined by redshift measurement – are generally uncertain by an amount of the order c/H_0. $\langle \delta z^2 \rangle^{1/2} \simeq \bar{\lambda}$ in the mean, that is of the order of the dominant wavelength, if the wave energy reaches the critical density.

With a wave induced fluctuating redshift component, a number of anomalous redshift effects can be explained, although not all of them, in particular not those in-

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volving luminosity changes. A connexion between a fluctuating redshift component and a corresponding intensity variation exists, both produced by the same gravitational waves. The relative intensity change is, however, only of relative order λ/λ_H. This is too small to explain, for instance, the band pattern in the magnitude-redshift plot of galaxies in the Coma cluster found by Tifft (1972, 1973a, b). As noted above, the theory employed here is incomplete and should be supplemented by a more accurate treatment of phases. It is still an open question if a more accurate wave theory could describe all redshift anomalies. In the following, some effects are noted, which could be explained by the theory already in its present form.

5. Mass Discrepancy in Galaxy Clusters

The fluctuating redshift component (26) increases the velocity dispersion of galaxies in clusters and groups and may be the cause of the mass discrepancy. Introducing a mass weighted average distance r of galaxies in the cluster by $r = M^2/\sum_{A,B} m_A m_B/r_{AB}$ ($M = \sum_A m_A$, m_A the individual masses, $A = 1 \ldots N$), and defining $V^2 = \sum m_A V_A'^2/M$ with V_A'/c as the observed redshifts V_A/c minus a mass weighted average of V_A/c (thus $\sum V_A' m_A = 0$), the 'virial mass' M_{VT} is usually defined by

$$M_{VT}/M = V^2 r/MG \tag{35}$$

(projections factors are neglected here). If gravitational waves are present, the value of M_{VT}/M, defined by the operation described above, is given by

$$M_{VT}/M = 1 + \frac{rc^2\sqrt{3}}{GM^2}\left(\sum_A m_A\,\delta z_A^2 - \frac{1}{M}\left[\sum_A m_A\,\delta z_1\right]^2\right), \tag{36}$$

if the intrinsic dynamical motion of galaxies satisfies the virial theorem. For a given cluster of galaxies, δz_A varies as a function of both the distance and the apparent sky position of the galaxy (a slight position displacement, also caused by the low-frequency waves as discussed below, may be neglected in this context). If the coherence length of the waves is of the order of or small compared with the mean distance between galaxies, the redshift fluctuations are only partly correlated for neighbouring galaxies. In this case the term involving $\sum m_A\,\delta z_A$ becomes small, if the number N of cluster members is sufficiently large, and the virial discrepancy attains its maximal value, with an ensemble average given by

$$M_{VT}/M \simeq 1 + \frac{Rc^2\sqrt{3}}{GM}\,\langle \delta z^2 \rangle. \tag{37}$$

Note, δz in Equation (37) refers to the fluctuating component in the redshift change (a redshift component equal for all galaxies does not contribute to the discrepancy).

In the other case, if the wave coherence lengths are considerably larger than the distances between galaxies, there is a high probability that all galaxies will attain the same value of the anomalous redshift. There is then some cancellation of terms in

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Equation (36), M_{VT}/M tends to 1, and the discrepancy vanishes. Thus very low-frequency waves with wavelengths greatly exceeding the cluster diameters will not contribute to the velocity dispersion. The possible existence of 100 Mpc waves – which might explain a number of other effects (see below) – is not restricted by the virial data.

From Equation (36) it is seen that instead of M_{VT}/M, the quantity $M_{VT}-M$ is a useful measure of the virial discrepancy, since this quantity should be a function of the cluster extension only, independent of the cluster mass M. To obtain an approximate value for $M_{VT}-M$ with the assumption $m_A = m$ we note that

(i) averaging the square of δz_A over the cluster members gives a sample estimate of $\langle \delta z^2 \rangle$, and

(ii) averaging $\delta z_A\, \delta z_B$ with $A \neq B$ over the cluster members gives an approximate sample estimate of the covariance function $S(z, z+H\Delta r/c, \theta)$, where Δr is the mean space distance between the galaxies and θ the mean angular distance of galaxies. Thus

$$M_{VT}-M \simeq \frac{rc^2\sqrt{3}}{G}\left(\langle \delta z^2 \rangle - \langle \delta z_1 \delta z_2 \rangle\right) \simeq$$
$$\simeq \frac{rc^2\sqrt{3}}{G}\left(S[z,z,0] - S\left[z, z+\frac{H\Delta r}{c}, \theta\right]\right). \tag{38}$$

In Figure 1 this function is plotted as a function of r for monochromatic gravitational background radiation with a wave length $\lambda = 10$ Mpc at the critical cosmological energy density, together with data for some clusters of Abell richness class 2, compiled by Silk and Tarter (1973). The radii r are taken from this publication, the unweighted radii Δr and $\theta = \Delta r/\Delta$ (Δ the distance to the cluster) are from the work by Rood *et al.* (1971). Although these data should be considered only as an illustration of the general idea, it appears that gravitational wave induced redshift components could account for the observed increase of $M_{VT}-M$ with the cluster diameters. Also, since $M_{VT}-M$

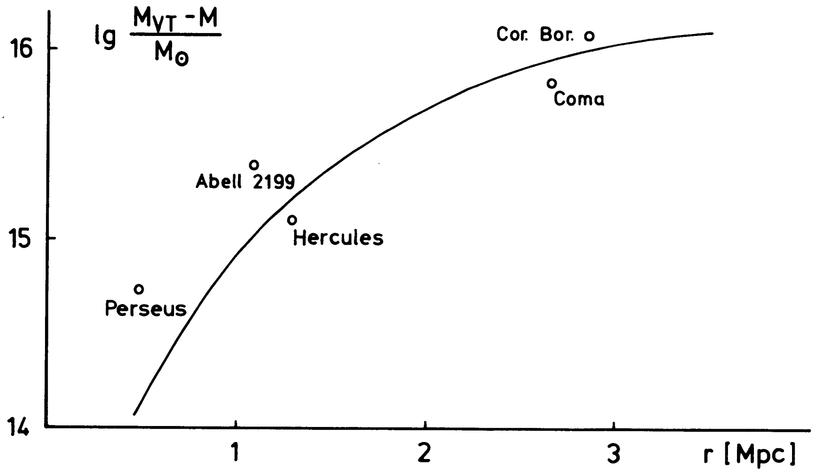

Fig. 1. Virial discrepancy according to Equation (38), for monochromatic gravitational background radiation of critical cosmological density, with $\lambda = 10$ Mpc.

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is essentially mass independent, the high velocity dispersion found for less massive groups (Rood, 1971) may be explained as caused by the same spectral band (this band is probably not restricted to a single line but may have a rather broad appearance).

6. Systematic Redshifts in Chains of Galaxies

An effect related to the mass discrepancy in groups of galaxies is a variation of the measured redshift across a group or cluster. A systematic change of the wave induced redshift component over a cluster should appear, if the apparent cluster diameter is comparable with the angular correlation length $\Delta\theta$ given by Equation (28).

Recently Gregory and Connolly (1973) reported redshift measurement of two groups which belong to the cluster Zw Cl 1609.0+82°.12. One group, listed separately as Abell cluster A 2247, contains a chain of galaxies, which extends over a distance of approximately $\simeq 9'$. The redshifts seem to change systematically along the chain, leading to a difference of the order $\simeq 750$ km s^{-1} at the ends. The mean redshift of the cluster A 2247 is of the order $z \simeq 4 \times 10^{-4}$, giving $D = 120$ Mpc for $H_0 = 100$ km s^{-1} Mpc^{-1}. Application of the virial theorem gives an M/L ratio 200 times the solar value. According to Gregory and Connolly there exists no convincing interpretation of the systematic redshift variation in terms of peculiar motion. The hypothesis of a gravitational wave induced redshift component may account for the observations. The correlation of redshifts suggests an angular correlation length $\Delta\theta'$ of the order of or exceeding the angular extension $l' \simeq 9'$ of the chain. Thus from Equation (29), $\bar{\lambda} > 0.1$ Mpc should hold for the dominant wave length λ. If the local values of the wave amplitudes at the cluster are just equal to the root mean square value, then $\bar{\lambda} \simeq 8$ Mpc, if a critical energy density is assumed for the waves.

Similar remarks may be made for the more distant ($z = 0.1$) Zwicky cluster of compact galaxies, Zw Cl 0152+33 (Sargent, 1972), which also shows a redshift anomaly.

There are also some well-known cases of a single very discrepant redshift in groups of obviously physically related galaxies (Arp, 1971; Burbidge and Sargent, 1971). The gravitational wave explanation also possibly covers these cases. It should be stressed that nothing is known of the probability distribution for a wave amplitude or a single redshift component δz. Since one deals with random quantities, large deviations from the mean are not excluded.

7. Local Supercluster

A suggestive explanation for the anisotropy and non-linearity of the redshift distribution of nearby galaxies in terms of a differential rotation and expansion of the local supercluster has been given by de Vaucouleurs (1953, 1968, 1972) and others (Cooper-Rubin, 1951; Ogorodnikov, 1952). If gravitational radiation with wavelengths of order 100 Mpc exists with not too small amplitudes – at least locally – this picture should be modified. The resulting redshift distribution for galaxies in the extreme near field ($D < \lambda$) of the waves has some similarity with the observed pattern. The general

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expression for the anomalous redshift distribution is given by

$$\begin{aligned}\delta z = &\int P_1(\mathbf{k})\,\mathrm{d}\mathbf{k}\,(l^2 - m^2)\,(1-\cos\phi) + \\ &+2\int Q_1(\mathbf{k})\,\mathrm{d}\mathbf{k}\;lm(1-\cos\phi) + \\ &+\int P_2(\mathbf{k})\,\mathrm{d}\mathbf{k}\,(l^2-m^2)\sin\phi + \\ &+2\int Q_2(\mathbf{k})\,\mathrm{d}\mathbf{k}\;lm\sin\phi ,\end{aligned}$$

where ϕ is the phase

$$\phi = k\lambda_H z\left(1+\frac{\mathbf{kn}}{k}\right) = \frac{2\pi D}{\lambda}\left(1+\frac{\mathbf{kn}}{k}\right) \tag{40}$$

and

$$l = l_i n^i, \qquad m = m_i n^i .$$

It is seen from Equation (39), that the change of δz along the supergalactic equator is mainly of quadrupole type. This is the main difference from the de Vaucouleurs interpretation suggesting a dipole-like variation of the non-Hubble redshift component with supergalactic longitude. A clear decision between both possibilities seems to be difficult at present, in view of the scarcity of southern hemisphere data and because of obscuration by the Galaxy.

Equation (39) also predicts a non-linearity of the redshift-distance relation. If

$$H_{\mathrm{eff}} = H_0(1+\delta z/z) \tag{41}$$

is defined as the effective Hubble constant, with H_0 as the asymptotic value behind the local supercluster, this quantity is a function of the distance $D = cz/H_0$ as well as of the direction of observation. Its variation with D and $\mathbf{n}$ depends strongly on the wave vector dependence of the amplitudes P_1, P_2, Q_1 and Q_2 as well as on the relative weight of these quantities.

As a simple example, which has been discussed in detail elsewhere (to be published), we consider P_1 through Q_2 as depending on the wave number k only. The terms involving Q_1 and Q_2 give no contribution to δz, the remaining terms lead to

$$\delta z = \sqrt{\frac{\varrho}{\varrho_c}}\,\frac{\lambda}{2\sqrt{3}\,\lambda_H}\,(3\cos^2\theta - 1)\,(\pi_1 h_1(a) + \pi_2 h_2(a)),$$

with π_1, π_2 as constants, $a = 2\pi D/\lambda$ and

$$h_1(a) = 1 + \frac{3\cos a}{a^2}\left(\cos a - \frac{\sin a}{a}\right),$$

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$$h_2(a) = -\frac{\sin a}{a^2}\left(\cos a - \frac{\sin a}{a}\right). \tag{43}$$

The observations (de Vaucouleurs, 1972) may be represented by choosing $\pi_2 \simeq 0$ and π_1 roughly of order 1. The effective Hubble constant increases up to a maximum at $a \simeq 1.8$ and shows a slow decrease for a larger a. There are other peaks in the theoretical expression for H_{eff} for $a \simeq 1.8 + 2n\pi$, $n = 1, 2, 3\ldots$, but with strongly reduced amplitude ($H_{\max} \sim 1/D^2$). For $a \gg 1$, H_{eff} tends to its asymptotic value H_0. The observational data suggest that the first maximum of H_{eff} corresponds to a distance of $\gtrsim 25$ Mpc. Thus the local wave field should have a wavelength of the order $\gtrsim 100$ Mpc.

A large local gravitational wave field also predicts a large-scale anisotropy in the microwave background radiation, mainly of quadrupole type, as discussed above. The variations reported by Conklin (1972) are only marginally compatible with the numerical data required to explain the local redshift anomaly within the simplified model. It is an open question at present if a more refined model of the local wave field gives a better representation of all observational data. It may be noted that redshift data from the Local Group also impose some upper limits to the wave amplitudes in Equation (39).

8. Redshift Clustering

An apparently non-random distribution of the redshifts of quasistellar objects and related sources has been suggested by many authors (Burbidge, 1968; Cowan, 1968, 1969; Lake and Roeder, 1972; Burbidge and O'Dell, 1972). The gravitational wave hypothesis predicts a periodic redshift clustering for all extragalactic objects. The number density of sources of a given class with redshifts between z and $z + \mathrm{d}z$ is given by

$$n_{\text{total}}(z) = n(z) - \frac{\partial}{\partial z}(n\delta z), \tag{44}$$

where $n(z)$ is the corresponding density without waves. The existence of a broad wave spectrum would tend to smear out any periodicities in n_{total}. If, however, $k^2\alpha_0(k)$ is peaked at some wave number k_0, one expects a redshift clustering on a scale given by Equation (28) with $\lambda = 2\pi/k_0$. A recent analysis by Burbidge and O'Dell (1972) has shown that a power spectrum analysis of the distribution of non-QSO redshifts gives a spectral maximum for a wave length $\Delta z = 0.031$, which is significant at the 97.5% confidence level. According to Equation (28), a gravitational wavelength $\lambda_0 \simeq \simeq 93(100/H_0)$ Mpc may cause the effect. This corresponds to the wavelength required to explain the anomalous redshift-distance relation of nearby galaxies by a local wave field.

9. Scintillation Effects

A further interesting wave effect is the lateral displacement of light rays reaching the observer. This results in a time-dependent random position shift of the source on the sky, $x^A \to x^A + \delta x^A$ ($A = 1, 2$, we use locally cartesian coordinates). The time scale for a

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complete shift reversal is of the order of the wave periods. For ultra low-frequency waves a frozen scintillation would be observed. Displacements at different directions are correlated with each other. Statistically, the displacements δx^A may be considered as the components of a two-dimensional random vector field, whose covariance function is given by

$$\langle \delta x_1^A \delta x_2^B \rangle = F\delta^{AB} + n^A n^B (G-F),$$

if δx^A descibes a locally isotropic and homogeneous random process. F and G are the lateral and longitudinal autocovariance functions. The requirement of local homogeneity and isotropy of δx^A is satisfied if the wave random process that causes the lateral displacements is – as usually assumed – also a homogeneous and isotropic process. In this case F and G depend on the apparent angular distance $\theta = \arccos n_1^i n_2^i$ of two sources as well as on the source distances (or equivalently, on the mean redshifts). The root mean square value of the displacement vector is given by

$$\langle \delta x_1^A \delta x_1^A \rangle^{1/2} = \left[\frac{13}{5\pi} \int \alpha_0 k^2 \, \mathrm{d}k \right]^{1/2} \simeq$$
$$\simeq \left(\frac{39}{40\pi^2} \frac{\varrho}{\varrho_c} \right)^{1/2} \frac{\lambda}{\lambda_H}, \tag{46}$$

the approximation holds for quasimonochromatic radiation at the center wavelength λ. Although the amplitude of the displacement is high for large wavelengths, it must be noted that it is not directly observable, since neighbouring points will in general experience nearly the same shift. What is observable is a differential position shift, which is connected with the lateral derivatives of δx^A.

The angular dependence of the autocovariance functions F and G gives some information on what kind of observable effects could be expected. The mean parts of F and G smoothly decrease with increasing θ, indicating correlation over a large part of the sky. A small fraction ($\sim \lambda/\lambda_H$) of the amplitude oscillates with decreasing peak amplitude, on an angular scale

$$\Delta\theta' \simeq \lambda(1+z)/z \, \frac{H_0}{100}$$

($\Delta\theta'$ in minutes of arc). Thus an increased clustering tendency for distant objects like faint galaxies and quasi-stellar objects can be expected. The correlation length of the clustering as given by Equation (47) is of the order $\Delta\theta^\circ \simeq 2.5^\circ$–$5^\circ$, if $\lambda = 100$ Mpc and z ranges between 0.5 and 2. A clustering tendency for radio sources and quasi-stellar objects on similar scales has been noted by Wagoner (1967) and Arp (1970), who discussed the distribution of distances to the nearest neighbour objects.

An increased apparent clustering must also occur for faint and distant galaxies. An estimate of the index of clumpiness has shown that roughly

$$K \simeq 1 + \frac{l'^2}{4} f(z) \, (\lambda/100)^2 \, \frac{\varrho}{\varrho_c}, \tag{48}$$

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where l' is the extension of the counting cell (in arc seconds) and $f(z) = z^3/(1+z)$ for a Tolman radiation cosmos; the equation is restricted to a counting cell size small compared to the autocorrelation length (47). A dispersion-subdivision curve analysis of counts of galaxies, carried out on plates taken with the Schmidt telescopes at Tautenburg (Dautcourt *et al.*, 1974) and at Palomar (Zwicky, 1957) suggests a rapid increase of the index of clumpiness with counting cell size l.

Zwicky explains the effect by intergalactic obscuration (see, however, Neyman *et al.*, 1954). If there is a contribution from gravitational radiation, it again suggests the presence of an appreciable amount of radiation in the 100 Mpc wave band.

10. Concluding Remarks

In summary, it appears that the hypothesis of extremely low frequency cosmic gravitational radiation could explain a number of puzzling observations. Other closely related questions are still open. The case for gravitational radiation as the source of the redshift anomalies would be strong, if Tifft's band structure in the $m-z$-plot of Coma cluster galacies could be understood – provided, the effect is real. Other redshift anomalies like the systematically higher redshifts of companion galaxies, would follow from an explanation of the Tifft phenomenon.

Attention has been directed to directly observable effects of low-frequency waves. Intense wave fields should have had an influence on matter also at pre-galactic stages. Thus, a stochastic particle acceleration by wave fields – which is small at present time, but increases for large redshifts – might have been an energy source in pre galactic matter, e.g., for maintaining pre-galactic turbulence.

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