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Periodic Solutions of an Indefinite Singular Equation Arising from the Kepler Problem on the Sphere

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Abstract. We study a second-order ordinary differential equation coming from the Kepler problem on $\mathbb{S}^2$. The forcing term under consideration is a piecewise constant with singular nonlinearity that changes sign. We establish necessary and sufficient conditions to the existence and multiplicity of T-periodic solutions.

1 Introduction and Main Result

The purpose of this paper is to investigate the existence of T-periodic solutions to equations of the type

$$\ddot{u} = \frac{h(t)}{\cos^2 u}, \tag{1.1}$$

where, generally speaking, $h:\mathbb{R} \to \mathbb{R}$ is a T-periodic (nontrivial) locally integrable function. Note that if there exists a T-periodic solution to (1.1), then h necessarily changes its sign. Obviously, the right-hand side of (1.1) has singularities at the points $u = \pi/2 + k\pi$ ($k \in \mathbb{Z}$). We will focus on T-periodic solutions u lying in the strip $(-\pi/2, \pi/2)$, i.e., $u(t) \in (-\pi/2, \pi/2)$ for all $t \in [0, T]$.

It is worth mentioning here that we are dealing with a class of indefinite singular equations with two singularities. The problems with more than one singularity have received little attention in literature; however, they are very interesting from a theoretical point of view.

One of important goals of this paper is to establish criteria guaranteeing the solvability of an indefinite periodic problem associated with a singular differential equation with two singularities.

In an applied situation, the idea of exploring the Kepler problem on a non-Euclidean space started around 1840. Lobachevsky [6] and Bolyai [1] studied independently the Kepler problem where the attraction force is inversely proportional to the surface of $\mathbb{S}^2$. In 1860, Serret resolved the Kepler problem defined on $\mathbb{S}^2$ by using a suitable extension to the motion of a gravitational force (see [9]). Recently, some mathematicians of the Russian school, including V. Kozlov and A. Harin [5] and

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A. Shchepetilov [8], have made important contributions to this area using the cotangent potential defined on $\mathbb{S}^2$. From this point of view the motion of a particle subjected to the influence of an electric field created by a charge of a time-depending magnitude fixed in the north pole can be modelled by the equations

$$\begin{aligned}\ddot{u} &= \frac{-h(t)uw}{(1-w^2)^{\frac{3}{2}}} - (\dot{u}^2+\dot{v}^2+\dot{w}^2)u,\\ \ddot{v} &= \frac{-h(t)vw}{(1-w^2)^{\frac{3}{2}}} - (\dot{u}^2+\dot{v}^2+\dot{w}^2)v,\\ \ddot{w} &= \frac{h(t)}{(1-w^2)^{\frac{1}{2}}} - (\dot{u}^2+\dot{v}^2+\dot{w}^2)w,\end{aligned}$$

where h is an integrable T-periodic function corresponding to the magnetic interaction between the charges. When our free-particle has angular moment equal to 0 (if we restrict our movement to a circle), by using a change of variable to polar coordinates, we obtain that the above system of differential equations can be written in the form

$$\ddot{u} = h(t)\frac{\cos u}{|\cos u|^3}. \tag{1.2}$$

This problem models the dynamical behaviour of a particle moving on the $\mathbb{S}^1$ under the influence of the Newton's law (Kepler problem on $\mathbb{S}^1$).

From the above-mentioned point of view, our purpose is to investigate the existence of T-periodic motions to (1.2). If we restrict the movement of our particle to $\mathbb{S}^1_\pm := \{(x,y) \in \mathbb{S}^1 : \operatorname{sgn} x = \pm 1\}$, then equation (1.2) can be viewed as two independent equations, namely, equation (1.1) if $u \in (-\pi/2, \pi/2)$ and

$$\ddot{u} = -\frac{h(t)}{\cos^2 u} \tag{1.3}$$

if $u \in (\pi/2, 3\pi/2)$. Note that the existence of a T-periodic solution to (1.1) in the case when $u \in (-\pi/2, \pi/2)$ implies the existence of (another) T-periodic solution of (1.3), and vice versa.

As it was previously mentioned, in this paper we are going to study the solvability of the T-periodic problem depending on the parameter λ associated with the equation

$$\ddot{u} = \frac{\lambda^2 h(t)}{\cos^2 u}, \tag{1.4}$$

under the following assumptions: $T > 0$ is a fixed period and h is a T-periodic piecewise-constant function composed by two weights, *i.e.,*

$$h(t) = \begin{cases} -h_1, & \text{for } t \in [kT, kT+a),\\ h_2, & \text{for } t \in [kT+a, (k+1)T),\end{cases} \qquad k \in \mathbb{Z},$$

where $a \in (0,T)$, and h_1, h_2 are positive constants. These conditions simplify the physical interpretation to model given by (1.4), making it clearer and less complicated. Furthermore, the study of the existence of T-periodic solutions to an indefinite singular equation is a difficult problem to solve, especially under the presence of more than one singularity at the spatial variable. It therefore makes sense to consider the external force h defined as above, which is the case studied in [2] (one of the first works dealing with indefinite singular problems). By a solution to equation (1.4) we understand a function $u: \mathbb{R} \to (-\pi/2, \pi/2)$ that is locally absolutely continuous together with its first derivative and satisfies (1.4) almost everywhere in $\mathbb{R}$.

The main result gives the following relation.

Theorem 1.1 *There exists a critical parameter $\lambda^* > 0$ such that*

- *there is no T-periodic solution to* (1.4) *provided $\lambda > \lambda^*$;*
- *there exists at least one T-periodic solution to* (1.4) *provided $\lambda = \lambda^*$;*
- *there exist at least two T-periodic solutions to* (1.4) *provided $\lambda < \lambda^*$.*

It is worth mentioning here that the both analytical and numerical estimates of the value λ^* are established in Section 5.

The following physical interpretation of the result obtained can be deduced: the existence of an electrically charged particle moving periodically on $\mathbb{S}^1$ under a fixed electrical field with alternating charge periods of attractive and repulsive interaction occurs only in the case when the electrical field has low voltage or its polarity reverses quickly.

Theorem 1.1 motivates the question on the exact number of T-periodic solutions in each of the cases $\lambda = \lambda^*$ and $\lambda < \lambda^*$. This interesting problem remains open.

The rest of the paper is devoted to the proof of Theorem 1.1, the estimation of λ^*, and examples illustrating the result. In Section 2, we construct a continuous function $\Psi: \mathbb{R}^2 \to \mathbb{R}^2$ so that there is a one-to-one correspondence between T-periodic solutions to (1.4) and zeroes of Ψ. The properties of the function Ψ are discussed in Section 3. Conditions guaranteeing the existence, resp. non-existence of zeroes of the function Ψ are established in Section 4 using the topological degree method. In Section 5, we give the estimates of λ^*, *i.e.,* of the critical value such that equation (1.4) has no T-periodic solution provided $\lambda > \lambda^*$. In this context we discuss a problem with an important physical interpretation. Finally, the conclusions and some open problems are presented in Section 6.

2 Construction of T-periodic Solutions

The model under consideration has a special symmetric aspects due to the external force h. In fact, equation (1.4) can be studied as two alternating autonomous equations

(2.1) $$\ddot{u} = -\frac{\lambda^2 h_1}{\cos^2 u}, \qquad t \in [0, a),$$

(2.2) $$\ddot{u} = \frac{\lambda^2 h_2}{\cos^2 u}, \qquad t \in [a, T),$$

subjected to the boundary value conditions

(2.3) $$u(0) = u(T), \qquad u'(0) = u'(T).$$

In this context the following lemma plays a key role in the proof of Theorem 1.1, establishing that if u is a T-periodic solution to (1.4), then its value at a is necessarily equal to its values at the extreme points of the interval $[0, T]$.

Lemma 2.1 *If u is a T-periodic solution to (1.4), then*

$$\tan u(0) = \tan u(a) = \frac{h_1 \tan x_0 - h_2 \tan y_0}{h_1 + h_2},$$

where

$$x_0 = \max\{u(t) : t \in [0, T]\}, \qquad y_0 = -\min\{u(t) : t \in [0, T]\}. \tag{2.4}$$

Proof According to sign properties of the function h, there exist $t_M \in (0, a)$ and $t_m \in (a, T)$ such that $x_0 = u(T_M)$ and $y_0 = -u(t_m)$. Multiplying both sides of (2.1) by $\dot{u}$ and integrating on $[t_M, a]$, we obtain

$$\dot{u}^2(a) = -2\lambda^2 h_1\big(\tan u(a) - \tan x_0\big). \tag{2.5}$$

Further, multiplying both sides of (2.2) by $\dot{u}$ and integrating on $[a, t_m]$, we get

$$\dot{u}^2(a) = 2\lambda^2 h_2\big(\tan y_0 + \tan u(a)\big). \tag{2.6}$$

In view of (2.5) and (2.6), we conclude that

$$\tan u(a) = \frac{h_1 \tan x_0 - h_2 \tan y_0}{h_1 + h_2}.$$

Arguing analogously as before on the intervals $[0, t_M]$ and $[t_m, T]$, respectively, we can prove, using (2.3), that

$$h_1\big(\tan x_0 - \tan u(0)\big) = h_2\big(\tan u(0) + \tan y_0\big). \tag{2.7}$$

From (2.7) it follows that $\tan u(0) = \tan u(a)$, concluding the proof. ∎

As a consequence of the previous lemma we get the following assertion that any T-periodic solution to (1.4) attains its maximum and minimum values at the points $a/2$ and $(a+T)/2$, respectively.

Lemma 2.2 *If u is a T-periodic solution to (1.4), then*

$$\max\{u(t) : t \in [0, T]\} = u\Big(\frac{a}{2}\Big), \qquad \min\{u(t) : t \in [0, T]\} = u\Big(\frac{a+T}{2}\Big).$$

Proof Define x_0 and y_0 by (2.4) and let $t_M \in (0, a)$, $t_m \in (a, T)$ be such that $x_0 = u(T_M)$, $y_0 = -u(t_m)$. We will prove that $t_M = a/2$. Indeed, since $\ddot{u}(t) < 0$ for $t \in [0, t_M)$, $\dot{u}(t) > 0$ for $t \in [0, t_M)$ (note that $\dot{u}(t_M) = 0$). Multiplying both sides of (2.1) by $\dot{u}$ and integrating over $[t, t_M]$, we arrive at

$$\dot{u}(t) = \lambda\sqrt{2h_1}\sqrt{\tan x_0 - \tan u(t)} \qquad \text{for } t \in [0, t_M],$$

and, consequently,

$$\int_{u(0)}^{x_0} \frac{ds}{\sqrt{\tan x_0 - \tan s}} = \lambda t_M \sqrt{2h_1}.$$

Analogously, the double integration over the interval $[t_M, t]$, with respect to the fact that $\dot{u}(t) < 0$ for $t \in (t_M, a]$, yields

$$\int_{u(a)}^{x_0} \frac{ds}{\sqrt{\tan x_0 - \tan s}} = \lambda(a - t_M)\sqrt{2h_1}.$$

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According to Lemma 2.1, by subtracting the above-mentioned identities we obtain that $t_M = a/2$.

The proof of the relation $t_m = (a+T)/2$ is analogous, and it will be omitted. ∎

To conclude this section we characterize the existence of a T-periodic solution to (1.4) in terms of the existence of a solution to algebraic equations involving the functions F_1 and F_2 defined on $\Delta = \{(x,y) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) : x + y > 0\}$ by

$$F_1(x,y) = \int_0^{\sqrt{\frac{h_2(\tan x + \tan y)}{h_1+h_2}}} \frac{2dz}{1 + (\tan x - z^2)^2},$$

$$F_2(x,y) = \int_0^{\sqrt{\frac{h_1(\tan x + \tan y)}{h_1+h_2}}} \frac{2dz}{1 + (\tan y - z^2)^2}.$$

It can be easily verified that

$$F_1(x,y) > 0, \qquad F_2(x,y) > 0 \qquad \text{for } (x,y) \in \Delta. \tag{2.8}$$

Proposition 2.3 *Let $(x_0, y_0) \in \Delta$. Then the following assertions are equivalent:*

(i)

$$F_1(x_0,y_0) = \lambda a\sqrt{\frac{h_1}{2}}, \qquad F_2(x_0,y_0) = \lambda(T-a)\sqrt{\frac{h_2}{2}}; \tag{2.9}$$

(ii) *there exists a T-periodic solution to* (1.4) *such that* (2.4) *is fulfilled.*

Proof Note that, using the substitutions $\tan s = \tan x - z^2$ and $\tan s = z^2 - \tan y$, respectively, the functions F_1 and F_2 can be equivalently rewritten in the forms

$$F_1(x,y) = \int_{\gamma(x,y)}^{x} \frac{ds}{\sqrt{\tan x - \tan s}}, \tag{2.10}$$

$$F_2(x,y) = \int_{-y}^{\gamma(x,y)} \frac{ds}{\sqrt{\tan y + \tan s}}, \tag{2.11}$$

where $\gamma(x,y) = \arctan[(h_1 \tan x - h_2 \tan y)/(h_1 + h_2)]$.

[(i) ⇒ (ii)]. Define functions ρ_1 and ρ_2 by

$$\int_{\rho_1(t)}^{x_0} \frac{ds}{\sqrt{\tan x_0 - \tan s}} = \lambda\Big(\frac{a}{2} - t\Big)\sqrt{2h_1} \qquad \text{for } t \in [0, a/2],$$

$$\int_{-y_0}^{\rho_2(t)} \frac{ds}{\sqrt{\tan y_0 + \tan s}} = \lambda\Big(t - \frac{a+T}{2}\Big)\sqrt{2h_2} \qquad \text{for } t \in [(a+T)/2, T],$$

and put

$$u(t) = \begin{cases} \rho_1(t) & \text{for } t \in [0, a/2], \\ \rho_1(a-t) & \text{for } t \in (a/2, a], \\ \rho_2(a+T-t) & \text{for } t \in (a, (a+T)/2), \\ \rho_2(t) & \text{for } t \in [(a+T)/2, T]. \end{cases}$$

It can be easily verified that the function $u: [0,T] \to (-\pi/2, \pi/2)$ is absolutely continuous together with its first derivative, and it satisfies (2.1)–(2.4). Consequently, its

T-periodic prolongation to the whole real axis is a T-periodic solution to (1.4) satisfying (2.4).

[(ii) ⇒ (i)]. Let u be a T-periodic solution to (1.4) satisfying (2.4). Then, according to Lemmas 2.1 and 2.2, we have

$$u(0) = \gamma(x_0, y_0), \qquad u\Big(\frac{a}{2}\Big) = x_0, \qquad \dot{u}\Big(\frac{a}{2}\Big) = 0.$$

Multiplying both sides of (2.1) by $\dot{u}$ and integrating over $[t, a/2]$ we obtain

$$\frac{\dot{u}(t)}{\sqrt{\tan x_0 - \tan u}} = \lambda\sqrt{2h_1} \qquad \text{for } t \in [0, a/2).$$

The integration over $[0, a/2]$ then implies

$$\int_{\gamma(x_0,y_0)}^{x_0} \frac{ds}{\sqrt{\tan x_0 - \tan s}} = \lambda a \sqrt{\frac{h_1}{2}}.$$

Consequently, according to (2.10) we obtain that the first equality in (2.9) holds.

If we handle with the equation (2.2) on the interval $[(a+T)/2, T]$ in analogous way, we arrive at

$$\int_{-y_0}^{\gamma(x_0,y_0)} \frac{ds}{\sqrt{\tan y_0 + \tan s}} = \lambda(T-a)\sqrt{\frac{h_2}{2}}.$$

Therefore, according to (2.11), we get the second equality in (2.9). ∎

3 Some Properties of the Functions F_1 and F_2

To prove that the algebraic system (2.9) possesses (or not) a solution on Δ, we will use some important properties of the above-defined functions F_1 and F_2. However, we first introduce a function $F\colon [-\pi/2, \pi/2] \to [0, +\infty)$ defined by

$$F(x) = \pi\sqrt{\frac{(1+\sin x)\cos x}{2}},$$

which plays an important role throughout the paper. Now, we will state some basic properties of F_1 and F_2 that can be verified by direct calculation; therefore, the proofs are omitted.

Lemma 3.1 *The following identities hold:*

(i) $$\lim_{y\to\pi/2} F_1(x,y) = F(x) \qquad \textit{for every } x \in (-\pi/2, \pi/2);$$

(ii) $$\lim_{x\to\pi/2} F_2(x,y) = F(y) \qquad \textit{for every } y \in (-\pi/2, \pi/2);$$

(iii) $$\lim_{x+y\to 0} F_1(x,y) = \lim_{x+y\to 0} F_2(x,y) = 0;$$

(iv) $$\frac{\partial F_1}{\partial y}(x,y) > 0, \qquad \frac{\partial F_2}{\partial x}(x,y) > 0 \qquad \textit{for } (x,y) \in \Delta.$$

From Lemma 3.1 we immediately obtain the following lemma.

Lemma 3.2 *The following relations hold:*

(i) $\lim_{x\to\pi/2} F_1(x,y) = 0$ *uniformly on* $(-\pi/2,\pi/2)$;

(ii) $\lim_{y\to\pi/2} F_2(x,y) = 0$ *uniformly on* $(-\pi/2,\pi/2)$.

Proof According to Lemma 3.1 and (2.8) we have

$$0 < F_1(x,y) \le \lim_{z\to\frac{\pi}{2}} F_1(x,z) = F(x) \qquad \text{for } (x,y)\in\Delta,$$
$$0 < F_2(x,y) \le \lim_{z\to\frac{\pi}{2}} F_1(z,y) = F(y) \qquad \text{for } (x,y)\in\Delta.$$

Thus, taking the limits as $x\to\pi/2$ and $y\to\pi/2$, respectively, we get the assertion. ∎

Remark 3.3 By virtue of Lemmas 3.1 and 3.2 we can also extend continuously the functions F_1 and F_2 to $\partial\Delta$ and define them to be zero on

$$\{(x,y)\in[-\pi/2,\pi/2]\times[-\pi/2,\pi/2] : x+y<0\}.$$

4 Conditions for the Existence and Multiplicity of T-periodic Solutions

In this section we complete the proof of Theorem 1.1. We use Proposition 2.3 to establish efficient conditions guaranteeing the existence (resp. nonexistence) of a T-periodic solution to (1.4). With this aim, we introduce the set

$$A = \{(x,y)\in\Delta : (T-a)F_1(x,y)\sqrt{h_2} = aF_2(x,y)\sqrt{h_1}\}.$$

Obviously, every solution (x_0,y_0) to the algebraic system (2.9) belongs to A. Note also that on the set A, for arbitrary $\lambda>0$, the following relation holds:

$$\operatorname{sgn}\left[F_1(x,y) - \lambda a\sqrt{\frac{h_1}{2}}\right] = \operatorname{sgn}\left[F_2(x,y) - \lambda(T-a)\sqrt{\frac{h_2}{2}}\right]. \tag{4.1}$$

Lemma 4.1 *The following statements hold:*

(i) *If $F_1(x,y) < \lambda a\sqrt{h_1/2}$ for all $(x,y)\in A$, then there is no T-periodic solution to* (1.4).

(ii) *If $F_1(x_0,y_0) = \lambda a\sqrt{h_1/2}$ for some $(x_0,y_0)\in A$, then there exists at least one T-periodic solution to* (1.4).

(iii) *If $F_1(x_0,y_0) > \lambda a\sqrt{h_1/2}$ for some $(x_0,y_0)\in A$, then there exist at least two T-periodic solutions to* (1.4).

Proof Assertions (i) and (ii) immediately follow from relation (4.1) and Proposition 2.3. As for assertion (iii), with respect to Remark 3.3, we introduce the functions

G_1 and G_2 defined on $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ as follows:

$$G_1(x, y) = F_1(x, y) - \lambda a \sqrt{\frac{h_1}{2}},$$

$$G_2(x, y) = F_2(x, y) - \lambda(T - a) \sqrt{\frac{h_2}{2}}.$$

According to Proposition 2.3, it is necessary to show that the mapping $G = (G_1, G_2)$ has at least two different zeroes in Δ. In view of (4.1), we have $G_1(x_0, y_0) > 0$ and $G_2(x_0, y_0) > 0$. According to Lemma 3.1(iv) and Lemma 3.2, we have

$$G_1(x_0, y) > 0, \qquad G_1\Big(\frac{\pi}{2}, y\Big) < 0 \qquad \text{for } y \in [y_0, \pi/2],$$

$$G_2(x, y_0) > 0, \qquad G_2\Big(x, \frac{\pi}{2}\Big) < 0 \qquad \text{for } x \in [x_0, \pi/2].$$

Put $\Omega_0 = (x_0, \pi/2) \times (y_0, \pi/2)$ and let $(\alpha, \beta) \in \Omega_0$ be arbitrary but fixed. Define the homotopy as follows:

$$H_1 \colon [0, 1] \times \Omega_0 \to \mathbb{R}^2, \qquad H_1(\mu, x, y) := (1 - \mu) G(x, y) + \mu(\alpha - x, \beta - y).$$

Obviously the homotopy is admissible, because $H_1(\mu, x, y) \neq 0$ for all $(\mu, x, y) \in [0, 1] \times \partial\Omega_0$. Thus,

$$d_B[G, \Omega_0, 0] = d_B\big[H_1(0, \cdot, \cdot), \Omega_0, 0\big] = d_B\big[H_1(1, \cdot, \cdot), \Omega_0, 0\big] = 1$$

(d_B denotes the Brouwer degree). On the other hand, according to Lemma 3.2 and Remark 3.3, we have

$$G_2\Big(x, -\frac{\pi}{2}\Big) < 0, \qquad G_2\Big(x, \frac{\pi}{2}\Big) < 0 \qquad \text{for } x \in [-\pi/2, \pi/2],$$

$$G_1\Big(-\frac{\pi}{2}, y\Big) < 0, \qquad G_1\Big(\frac{\pi}{2}, y\Big) < 0 \qquad \text{for } y \in [-\pi/2, \pi/2].$$

Therefore, setting $\Omega = (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, the homotopy

$$H_2 \colon [0, 1] \times \Omega \to \mathbb{R}^2, \qquad H_2(\mu, x, y) = (1 - \mu) G(x, y) + \mu(-1, -1)$$

is admissible and

$$d_B[G, \Omega, 0] = d_B\big[H_2(0, \cdot, \cdot), \Omega, 0\big] = d_B\big[H_2(1, \cdot, \cdot), \Omega, 0\big] = 0.$$

Consequently, $d_B[G, \Omega \setminus \overline{\Omega}_0, 0] = -1$. Hence, it follows that there exist at least two different zeroes of G; one of them belongs to Ω_0 and the other one is in $\Omega \setminus \overline{\Omega}_0$. ∎

Lemma 4.2 *The set A is nonempty.*

Proof For every $c \in (0, \pi)$ we define the function

$$g_c \colon \Big(c - \frac{\pi}{2}, \frac{\pi}{2}\Big) \longrightarrow \mathbb{R}, \quad g_c(x) = \sqrt{h_2}(T - a) F_1(x, c - x) - \sqrt{h_1} a F_2(x, c - x).$$

According to Remark 3.3, we observe that

$$\lim_{x \to \pi/2} g_c(c - x) > 0 \quad \text{and} \quad \lim_{x \to \pi/2} g_c(x) < 0$$

for every $c \in (0, \pi)$. Thus the continuity of g_c yields the existence of a point $x_0 \in (c - \pi/2, \pi/2)$ such that $g_c(x_0) = 0$. However, that means that $(x_0, c - x_0) \in A$. ∎

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Since $\overline{A}$ is a bounded and closed (compact) set with inner points, with respect to Remark 3.3, we can define

$$\lambda^* = \max_{(x,y)\in\overline{A}} F_1(x,y)a^{-1}\sqrt{\frac{2}{h_1}}. \tag{4.2}$$

Obviously, the maximum value is attached at some point of A. Therefore, let $(x_0, y_0) \in A$ be such that

$$\lambda^* = F_1(x_0, y_0)a^{-1}\sqrt{\frac{2}{h_1}}.$$

Then the assertion of Theorem 1.1 immediately follows from Lemma 4.1 and (4.2).

5 Estimations of λ^*

In the previous section we have defined the number λ^* (see (4.2)). However, the set A was defined implicitly, which may cause problems in computation of the explicit value of λ^*. The aim of this section is to prove some lemmas in order to establish the estimates for this value.

Our first task will consist of finding an upper bound for λ^*. We introduce the notation

$$f_1 \overset{\text{def}}{=} \max_{\varphi} F_1(x,y), \qquad f_2 \overset{\text{def}}{=} \max_{\varphi} F_2(x,y), \tag{5.1}$$

where φ is a curve given parametrically by $\varphi = (\varphi_1, \varphi_2)\colon [0,1] \to \mathbb{R}^2$ such that

$$\varphi(0) \in \{(x,-x) : x \in [-\pi/2, \pi/2]\}, \qquad \varphi(1) = (\pi/2, \pi/2),$$
$$\varphi(t) \in \Delta \qquad \text{for } t \in (0,1),$$

and it separates Δ into two nonempty connected parts.

Lemma 5.1 *The inequality*

$$\lambda^* \le \sqrt{2}\max\Big\{\frac{f_1}{a\sqrt{h_1}}, \frac{f_2}{(T-a)\sqrt{h_2}}\Big\} \tag{5.2}$$

holds, where f_1 and f_2 are defined by (5.1).

Proof We define the sets

$$B_1 := \Big\{(x,y) \in \Delta : F_1(x,y) = \lambda^* a\sqrt{\frac{h_1}{2}}\Big\},$$
$$B_2 := \Big\{(x,y) \in \Delta : F_2(x,y) = \lambda^*(T-a)\sqrt{\frac{h_2}{2}}\Big\}.$$

Note that from the definition of λ^*, we have

$$B_1 \cap B_2 \ne \varnothing. \tag{5.3}$$

Assume, on the contrary, that (5.2) does not hold. Then

$$(B_1 \cup B_2) \cap \varphi = \varnothing. \tag{5.4}$$

Indeed, if there exists $(x_0, y_0) \in B_1 \cap \varphi$, then

$$a^{-1}\sqrt{\frac{2}{h_1}}F_1(x_0, y_0) = \lambda^* > a^{-1}\sqrt{\frac{2}{h_1}}f_1.$$

However, this contradicts the definition of f_1 as $(x_0, y_0) \in \varphi$. Analogously, we conclude that $B_2 \cap \varphi = \varnothing$.

As, according to our assumptions, the curve φ divides Δ into two disjoint parts, we denote by Φ_+ (resp. by Φ_-) the upper part (resp. the below part) of Δ with respect to the curve. The remainder of the proof is devoted to checking that

$$B_1 \subseteq \Phi_+, \qquad B_2 \subseteq \Phi_-. \tag{5.5}$$

Assume on the contrary, that there exists $(x_0, y_0) \in B_1 \cap \overline{\Phi}_-$. According to Lemma 3.1(i), (iv), we have

$$F(x_0) > \lambda^* a\sqrt{\frac{h_1}{2}}, \qquad F\Big(\frac{\pi}{2}\Big) = 0.$$

Thus, there exist $x_1 \in (x_0, \pi/2)$ such that $F(x_1) = \lambda^* a\sqrt{h_1/2}$ and a connected set $\Sigma_1 \subseteq B_1$ joining (x_0, y_0) with $(x_1, \pi/2)$. This contradicts (5.4) (see left side of Figure 1).

If there exists $(x_0, y_0) \in B_2 \cap \overline{\Phi}_+$, according to Lemma 3.1(ii), (iv), we have

$$F(y_0) > \lambda^*(T-a)\sqrt{\frac{h_2}{2}}, \qquad F\Big(\frac{\pi}{2}\Big) = 0.$$

Thus, there exist $y_1 \in (y_0, \pi/2)$ such that $F(y_1) = \lambda^*(T-a)\sqrt{h_2/2}$ and a connected set $\Sigma_2 \subseteq B_2$ joining (x_0, y_0) with $(\pi/2, y_1)$. This contradicts (5.4) (see right side of Figure 1).

Figure 1: The triangle represents the region Δ.

Finally, according to (5.5) we conclude that $B_1 \cap B_2 = \varnothing$, which contradicts (5.3). ∎

The next result shows a lower bound for λ^*.

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Lemma 5.2 *The following inequality holds:*

$$\lambda^* \geq \sqrt{2}\min\left\{\frac{F_1(x,y)}{a\sqrt{h_1}}, \frac{F_2(x,y)}{(T-a)\sqrt{h_2}}\right\} \qquad \textit{for } (x,y) \in \Delta. \tag{5.6}$$

Proof It is sufficient to show that for every $(x,y) \in \Delta$, there exists $(x_0,y_0) \in A$ (depending on (x,y)) such that

$$\min\left\{\frac{F_1(x,y)}{a\sqrt{h_1}}, \frac{F_2(x,y)}{(T-a)\sqrt{h_2}}\right\} \leq \frac{F_1(x_0,y_0)}{a\sqrt{h_1}};$$

then the assertion follows from (4.2).

Let, therefore, $(x,y) \in \Delta$ be arbitrary but fixed. If

$$\frac{F_1(x,y)}{a\sqrt{h_1}} = \frac{F_2(x,y)}{(T-a)\sqrt{h_2}},$$

then $(x,y) \in A$, and we can put $(x_0,y_0) \stackrel{\text{def}}{=} (x,y)$.

Now assume that

$$\frac{F_1(x,y)}{a\sqrt{h_1}} < \frac{F_2(x,y)}{(T-a)\sqrt{h_2}}.$$

According to Lemma 3.1(iv) and Lemma 3.2(ii), with respect to the continuity of the functions F_1 and F_2, there exists $y_0 \in (y,\pi/2)$ such that

$$\frac{F_1(x,y)}{a\sqrt{h_1}} < \frac{F_1(x,y_0)}{a\sqrt{h_1}} = \frac{F_2(x,y_0)}{(T-a)\sqrt{h_2}},$$

and we put $(x_0,y_0) \stackrel{\text{def}}{=} (x,y_0)$.

Analogously, if

$$\frac{F_1(x,y)}{a\sqrt{h_1}} > \frac{F_2(x,y)}{(T-a)\sqrt{h_2}},$$

then, in view of Lemma 3.1(iv) and Lemma 3.2(i) and with respect to the continuity of the functions F_1 and F_2, there exists $x_0 \in (x,\pi/2)$ such that

$$\frac{F_2(x,y)}{(T-a)\sqrt{h_2}} < \frac{F_2(x_0,y)}{(T-a)\sqrt{h_2}} = \frac{F_1(x_0,y)}{a\sqrt{h_1}}.$$

Consequently, we put $(x_0,y_0) \stackrel{\text{def}}{=} (x_0,y)$. ■

The definition of λ^* in (4.2) and the estimation (5.6) yield the identity where the set A is not involved.

Proposition 5.3

$$\lambda^* = \sqrt{2}\max\left\{\min\left\{\frac{F_1(x,y)}{a\sqrt{h_1}}, \frac{F_2(x,y)}{(T-a)\sqrt{h_2}}\right\} : (x,y) \in \overline{\Delta}\right\}.$$

In what follows we establish some efficient estimations for λ^* using the above-proven results. Define the function $\widetilde{F}\colon [0,\pi/2] \to [0,+\infty)$:

$$\widetilde{F}(x) = \int_0^{\sqrt{\tan x}} \frac{2dz}{1+(\tan x - z^2)^2}$$

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and put $\varphi(t) = (\frac{\pi t}{2}, \arctan(\frac{h_1}{h_2}\tan\frac{\pi t}{2}))$. Then it can be easily verified that

$$\max_{\varphi} F_1(x,y) = \max_{\varphi} F_2(x,y) = \max_{x\in[0,\pi/2]} \widetilde{F}(x),$$

and from Lemma 5.1 we obtain the following consequence.

Corollary 5.4 *The following estimate holds:*

$$\lambda^* \leq \sqrt{2} \max_{x\in[0,\frac{\pi}{2}]} \widetilde{F}(x) \max\left\{\frac{1}{a\sqrt{h_1}}, \frac{1}{(T-a)\sqrt{h_2}}\right\}.$$

Remark 5.5 Numerically, we have

$$\max_{x\in[0,\frac{\pi}{2}]} \widetilde{F}(x) \approx 1.3821093976,$$

and the maximum value is attained at some $x \in (5\pi/21, \pi/4)$.

In the case where $a = T/2$ and $h_1 = h_2$, the equality $F_1(x,y) = F_2(x,y)$ holds if and only if $x = y$. Therefore, in that case we have $A = \{(x,x) : x \in (0,\pi/2)\}$ and $F_1(x,x) = F_2(x,x) = \widetilde{F}(x)$. Thus, directly from (4.2) we obtain

Corollary 5.6 *If $a = T/2$ and $h_1 = h_2$, then*

$$\lambda^* = \frac{2^{\frac{3}{2}}}{T\sqrt{h_1}} \max_{x\in[0,\frac{\pi}{2}]} \widetilde{F}(x) \approx \frac{3.90919571}{T\sqrt{h_1}}.$$

The following example illustrates the obtained results and conjectures the exact number of solutions.

Example 5.7 Consider the case $h_1 = h_2 = 1$, $T = 2\pi$, and $a = \pi$. According to Corollary 5.6, we have $\lambda^* \approx 0.622167821$. Based on numerical evidence we find:

- if $\lambda > \lambda^*$, then (1.4) has no 2π-periodic solution (see Figure 2);
- if $\lambda = \lambda^*$, then (1.4) has a unique 2π-periodic solution (see Figure 3);
- if $\lambda < \lambda^*$, then (1.4) has exactly two 2π-periodic solutions (see Figure 4).

The exact number of 2π-periodic solutions is a conjecture based on numerical experiments.

We proceed to the following problem. Given parameters h_1, h_2, λ, and T, we will study the existence of T-periodic solutions based on the parameter a. The physical interpretation is the following: how long should the electrical field generated by the particles preserve positive or negative charge in order to a free-particle moves T-periodically at the time variable on $\mathbb{S}^1$? Note that a corresponds to the time in which the electrical potential field is negative, and $T - a$ when it is positive. In other words, is there a value $a \in (0,T)$ such that, for given h_1, h_2, λ, and T, the equation (1.4) has a T-periodic solution?

Obviously, according to the numerical evidence, it seems that such a value a does not exist for any inputs (*e.g.*, the case when $h_1 = 1$, $h_2 = 10$, $\lambda = 1$, and $T = 2\pi$).

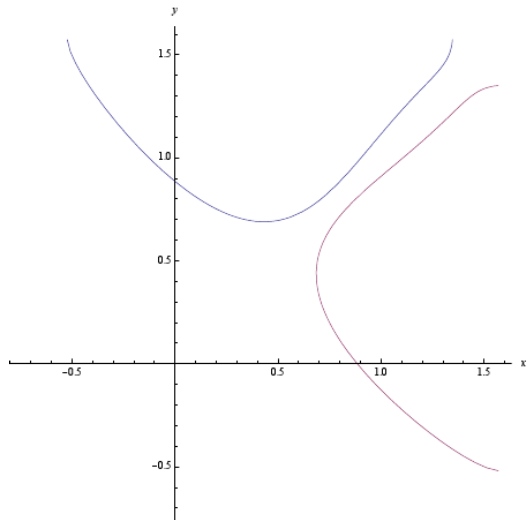

Figure 2: The level curves $F_1(x,y) = \lambda a\sqrt{h_1/2}$ and $F_2(x,y) = \lambda(T-a)\sqrt{h_2/2}$ with $h_1 = h_2 = 1$, $T = 2\pi$, $a = \pi$, and $\lambda > \lambda^*$.

However, assuming that

$$\lambda \le \max_{(x,y)\in\overline{\Delta}} \sqrt{2}\left(\frac{F_2(x,y)\sqrt{h_1} + F_1(x,y)\sqrt{h_2}}{T\sqrt{h_1 h_2}}\right), \tag{5.7}$$

we can prove the following assertion.

Corollary 5.8 *Let (5.7) hold. Then there exists $a \in (0,T)$ such that (1.4) has at least one T-periodic solution. If, furthermore, the inequality (5.7) is strict, then (1.4) has at least two T-periodic solutions.*

Proof Let $(x_0, y_0) \in \Delta$ be the point where the maximum value of the right-hand side of (5.7) is attained. We define a function $\gamma\colon (0,T) \to \mathbb{R}$ by

$$\gamma(a) = \sqrt{2}\min\left\{\frac{F_1(x_0,y_0)}{a\sqrt{h_1}}, \frac{F_2(x_0,y_0)}{(T-a)\sqrt{h_2}}\right\}.$$

The function γ is continuous and attains its maximum value at

$$a_M = \frac{F_1(x_0,y_0)T\sqrt{h_2}}{F_2(x_0,y_0)\sqrt{h_1} + F_1(x_0,y_0)\sqrt{h_2}}.$$

Note that

$$\frac{F_2(x_0,y_0)\sqrt{h_1} + F_1(x_0,y_0)\sqrt{h_2}}{T\sqrt{h_1 h_2}} = \frac{F_1(x_0,y_0)}{a_M\sqrt{h_1}}. \tag{5.8}$$

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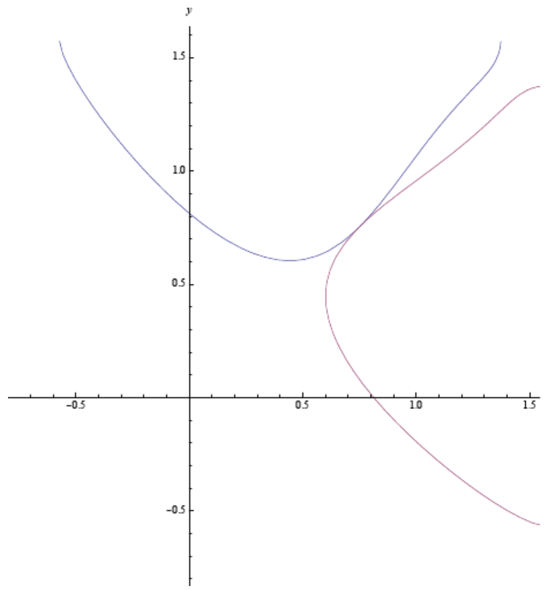

Figure 3: The level curves $F_1(x,y) = \lambda a\sqrt{h_1/2}$ and $F_2(x,y) = \lambda(T-a)\sqrt{h_2/2}$ with $h_1 = h_2 = 1$, $T = 2\pi$, $a = \pi$, and $\lambda = \lambda^*$.

By Lemma 5.2 we have that $\lambda^*(a) \geq \gamma(a)$ for all $a \in (0,T)$ ($\lambda^*(a)$ is the critical value for which (1.4) has at least one T-periodic solution corresponding to the parameter a). On the other hand,

$$\gamma(a_M) = \sqrt{2}\frac{F_1(x_0,y_0)}{a_M\sqrt{h_1}}$$

whence, in view of (5.7), (5.8), and the choice of (x_0,y_0), it follows that $\lambda \leq \gamma(a_M)$. Since $\gamma(a_M) \leq \lambda^*(a_M)$, we obtain $\lambda \leq \lambda^*(a_M)$ and the latter inequality is strict if the inequality in (5.7) is strict. Consequently, the assertion follows from Theorem 1.1 (with $a = a_M$). ∎

Example 5.9 Assume that $h_1 = 10$, $h_2 = 0.1$, $\lambda = 1$, and $T = 2\pi$. It can be numerically verified that

$$1 < \frac{F_2(\pi/6,\pi/6)\sqrt{10} + F_1(\pi/6,\pi/6)\sqrt{10^{-1}}}{2\pi}.$$

By virtue of Corollary 5.8 there exist at least two 2π-periodic solutions to (1.4) for some $a \in (0,2\pi)$ (see Figure 5). This case corresponds to our physical model setting ($\lambda = 1$). Therefore, under these conditions we can obtain two different 2π-periodic smooth movements for an electrical particle on $\mathbb{S}^1$ under the influence of a fixed electrical force field at the north pole.

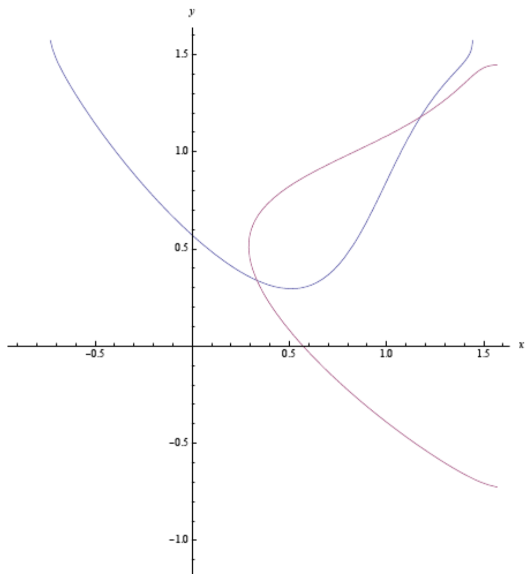

Figure 4: The level curves $F_1(x, y) = \lambda a\sqrt{h_1/2}$ and $F_2(x, y) = \lambda(T - a)\sqrt{h_2/2}$ with $h_1 = h_2 = 1$, $T = 2\pi$, $a = \pi$, and $\lambda < \lambda^*$.

6 Conclusions and Final Remarks

We have studied the dynamical behaviour of a mathematical model proposed in the literature for the motion of a particle (with fixed charge) subjected to the influence of an electric field with alternating charge periods of attractive and repulsive interaction (fixed on north pole), which is moving on $\mathbb{S}^1$. In [5], this model is considered to study the motion of a particle moving on $\mathbb{S}^2$ subjected to the gravitational attraction of the Sun that lies at $(0, 0, 1)$, more precisely, a singular Lagrangian system in polar coordinates that leads to the following system of differential equations

$$\begin{aligned}
\ddot{u} &= \frac{-huw}{(1-w^2)^{\frac{3}{2}}} - (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)u, \\
\ddot{v} &= \frac{-hvw}{(1-w^2)^{\frac{3}{2}}} - (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)v, \\
\ddot{w} &= \frac{h}{(1-w^2)^{\frac{1}{2}}} - (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)w.
\end{aligned}$$

Our model arises if we restrict the movement of our free-particle in $\mathbb{S}^2$ to $\mathbb{S}^1$ (that is, the angular momentum is null). We have shown that periodic motions appear only when the electrical field created by the particles has low voltage or when the voltage changes its polarity quickly.

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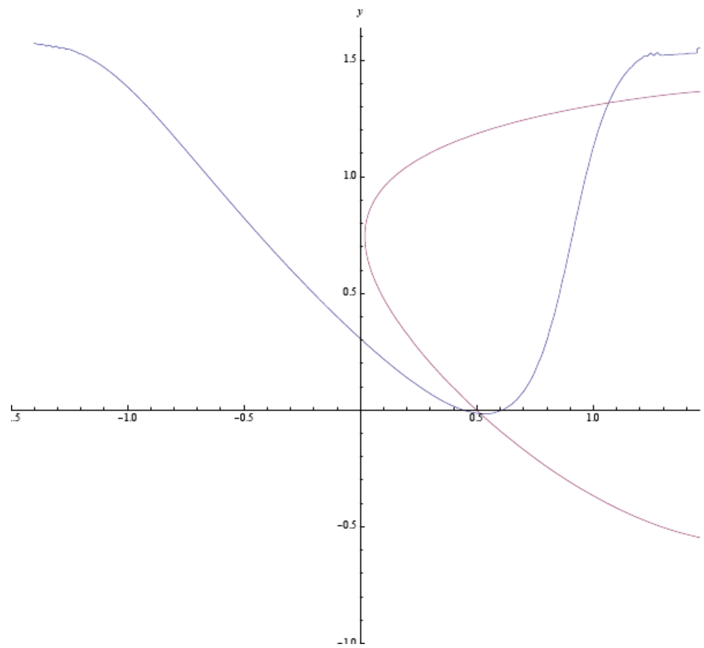

Figure 5: The level curves $F_1(x,y) = \lambda a\sqrt{h_1/2}$ and $F_2(x,y) = \lambda(T-a)\sqrt{h_2/2}$ with $h_1 = 10$, $h_2 = 10^{-1}$, $\lambda = 1$, $a = 0.05$, and $T = 2\pi$.

Actually, there are different models in the literature proposed to describe the dynamical behaviour of a particle moving on the space $\mathbb{S}^1$ under the influence of Newton's gravitational law. For example, in [4] the authors considered two masses m_1 and m_2 with respective position coordinates $(\sin u, \cos u)$ and $(0,1)$. In view of the nature of the space $\mathbb{S}^1$, they assumed that each body was attracted towards two directions. In this way the Kepler problem on $\mathbb{S}^1$ was mathematically modelled as a second-order differential equation

$$\ddot{u} = m_1 m_2\Big(\frac{1}{u^2} - \frac{1}{(2\pi-u)^2}\Big). \tag{6.1}$$

It is important to note that (6.1) and equation (1.4) are not equivalent. The difference between both models is the following: in the first model, the particle with the mass m_1 is attracted in two directions; however, in the second model, it is attracted just in one direction towards the mass m_2.

From the mathematical point of view, both equations, (6.1) and (1.4), are interesting because they present two singularities at the spacial variable. Nevertheless, the equations of such types seem to have received little attention in the literature, except the pioneering work by Fonda, Manásevich and Zanolin [3].

In short, our problem could have been studied from a different point of view considering the family of equations

$$\ddot{u} = \frac{h_1(t)}{u^2} - \frac{h_2(t)}{(2\pi - u)^2},$$

where $h_1, h_2 \in C(\mathbb{R}/T\mathbb{Z})$ are positive functions. This gives reasons for studying the dynamical properties of such a family of equations, which remain insufficiently investigated.

And finally, we would like to point out that the question on the stability and the exact number of solutions is not studied in the paper; this remains as an interesting open problem. In accordance with the numerical experiments, we can establish the following conjectures.

Conjecture 6.1 Let λ^* be the parameter appearing in Theorem 1.1. Then

- there exists a unique T-periodic solution to (1.4) provided $\lambda = \lambda^*$;
- there exist exactly two T-periodic solutions to (1.4) provided $\lambda < \lambda^*$.

It can be easily verified that $A = \{(x, y) \in \Delta : y = x\}$ provided $h_1 = h_2$, $a = T/2$ and the shapes of the curves are such that they have at most two intersections (see Figures 2–4). Moreover, the shapes of the curves do not change significantly in the asymmetric case (see Figure 5). This motivates our Conjecture 6.1. We refer the reader interested in the number of periodic solutions to a system of differential equations to the work of Nakajima and Seifert [7].

Obviously, the exact number of solutions is also related with the question on their stability. In this sense, we can reformulate Conjecture 6.1 in the following manner.

Conjecture 6.2 Let λ^* be the parameter appearing in Theorem 1.1. Then

- there exists a unique T-periodic solution to (1.4), and it is unstable, provided $\lambda = \lambda^*$;
- there exist exactly two T-periodic solutions to (1.4), one of them is unstable and the other one is asymptotically stable, provided $\lambda < \lambda^*$.

It is a classical physical principle that could be proved by using the excision property to the Brouwer degree.

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