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INDECOMPOSABLE POSITIVE QUADRATIC FORMS

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ABSTRACT. Let F be a formally real field. A quadratic form q is called positive if $\mathrm{sgn}_P \geqq 0$ for all orderings P of F. A positive q is called decomposable if there exist positive forms q_1, q_2 such that $q = q_1 \perp q_2$. Otherwise it is called indecomposable. In a first part we ask for which F there exist indecomposable three dimensional forms over F. We show that such forms exist iff F does not satisfy the property (A) defined in (J. K. Arason, A. Pfister: Zur Theorie der quadratischen Formen über formal reellen Körpern, Math Z. 153, 289–296 (1977)). We use an indecomposable three dimensional form defined by Arason and Pfister to construct indecomposable forms of arbitrary dimension. Then we examine the question for which fields F every positive form over F represents a nonzero sum of squares.

Let F be a formally real field and $X = X_F$ the space of orderings of F. A quadratic form φ over F is called positive if $\mathrm{sgn}_P\varphi \geqq 0$ for all $P \in X$. A positive form φ is called decomposable if there exist positive forms ψ_1, ψ_2 such that $\varphi = \psi_1 \perp \psi_2$. Otherwise it is called indecomposable.

In the first part of this paper we ask for which F there exist indecomposable three dimensional forms over F. We show that such forms exist iff F does not satisfy the property (A) defined in the paper [1]. Then we use an indecomposable three dimensional form defined by Arason and Pfister to construct indecomposable forms of arbitrary dimension. In a third part we examine the question for which fields F every positive form over F represents a nonzero sum of squares.

1. **The property** (P_3). Let F be a formally real field. Let $F^{\cdot} = F - \{0\}$ and for $a_1, \dots, a_n \in F^{\cdot}$ let

$$H(a_1, \dots, a_n) = \{P \in X \mid a_i \in P \text{ for } i = 1, \dots, n\}.$$

Let $T = T_F$ denote the sums of squares of F and let $\dot{T} = T - \{0\}$.

First examples of indecomposable forms are obviously $\langle 1 \rangle$ and $\langle 1, -1 \rangle$. But one sees soon that finding an indecomposable three dimensional form is a far more difficult and interesting problem. Obviously a positive form $\varphi = \langle a, b, c \rangle$ is decomposable iff φ represents some $t \in \dot{T}$. We therefore say that F satisfies (P_3) if every positive three dimensional form over F represents a $t \in \dot{T}_F$.

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Let us recall some notations of the paper [1]. Arason and Pfister called a torsion form φ of dimension $2n$ (i.e., $\text{sgn}_P\varphi = 0$ for all $P \in X$) strongly balanced if there are two dimensional torsion forms $\varphi_1, \ldots, \varphi_n$ such that $\varphi = \varphi_1 + \cdots + \varphi_n$ where '+' denotes the orthogonal sum. A field F is said to satisfy (A) if every torsion form over F is strongly balanced. Thus F satisfies (A) iff every torsion form over F of dimension greater than two is decomposable. The connection between the properties (A) and (P_3) is given by the following theorem:

THEOREM 1. *Let F be a formally real field. Then the following statements are equivalent: (i) F satisfies* (P_3). *(ii) Every torsion quaternion form* $\langle 1, a, b, ab\rangle$ *with* $a, b \in F^\cdot$ *represents an* $s \in -\dot{T}$. *(iii) F satisfies (A).*

The equivalence of (*ii*) and (*iii*) is given by [1] Satz 4. For the remaining part of the proof we need:

LEMMA 1. *Let* $a, b \in F^\cdot$ *such that* $H(a) \subset H(b)$. *Then there exist* $s, t \in T$ *such that* $b = ta + s$.

PROOF. (see also [5] Lemma 6.3). The forms $\langle 1, a\rangle$ and $\langle b, ba\rangle$ have the same signature values. Thus they are T-isometric in the sense of [7]. By [7] (1.19) we have $b \in D_T\langle 1, a\rangle$. □

PROOF OF THEOREM 1: $(i) \longrightarrow (ii)$: $\langle -1, -a, -b, -ab\rangle$ represents an $s \in \dot{T}$ iff the positive form $\langle -a, -b, -ab\rangle$ does. $(ii) \longrightarrow (i)$: Let $\langle a, b, c\rangle$ be a positive form over F. Then we have $H(-a, -b) = \phi$ and $H(-c) \subset H(a, b) = H(ab)$. Thus there are $t, s \in T$ such that $c = s - abt$. We can assume $t \neq 0$. Now the torsion form $\langle -1, at, bt, -ab\rangle$ represents an element of $\dot{T}$ and so does $\langle t\rangle\langle at, bt, -ab\rangle$. Now if $r = g^2a + h^2b - abtj^2$ for $g, h, j \in F$ and $r \in \dot{T}$, it follows that $r + sj^2 \in \dot{T}$ is represented by $\langle a, b, c\rangle$. □

REMARKS. a) Let K be an algebraic number field. Then by [1] Satz 5 the rational function field $K(x)$ satisfies (P_3). b) Let $W(F)$ be the Witt ring of F and $I(F)$ the augmentation ideal. If $I^3(F)$ is torsion free then F satisfies (P_3).

PROOF. For every torsion quaternion form φ over F we have $2 \times \varphi = 0$ in $W(F)$. Now apply [12] 2.13.14. □

EXAMPLE. 1. We want to construct a field satisfying (P_3) and having arbitrarily high Pythagoras number, u-invariant and stability index. (For the definition of these field invariants see [7], [10], [12].) Let $F_1 = \mathbb{R}((t_1)) \ldots ((t_n))$ and $F_2 = \mathbb{Z}/3\mathbb{Z}((t_1)) \ldots ((t_n))$. By [10] 2.1 there exists a unique ordered field F_3 such that the Pythagoras number of F_3 is 2^n. Now let $s(F_i)$ denote the quadratic form scheme of F_i in the sense of [4]. The property (P_3) is closed under formation of direct sums and group extensions. Now by [6] the direct sum $\oplus_{i=1}^3 s(F_i)$ is a field scheme which satisfies (P_3).

From [3] it follows that fields with $|F^\cdot/F^{\cdot 2}| \leqq 32$ satisfy (P_3). It is unknown whether there exist fields which satisfy SAP (see [7]) but do not satify (P_3).

EXAMPLE. 2. Using methods of Cassels, Arason and Pfister constructed a torsion quaternion form over $\mathbb{Q}(x, y)$ which is not strongly balanced. From [A, P] Beispiel 1 it follows that the positive form $\rho = \langle -x,\ 1+y^2+3x,\ x+y^2+3x^2\rangle$ over $\mathbb{Q}(x, y)$ is indecomposable.

2. **Indecomposable positive forms**. In this section we want to construct indecomposable positive forms of arbitrary dimension n.

Case A) n is even. Let $F = \mathbb{R}(z)$ and $F(n) := F((t_1)) \ldots ((t_n))$. First, we define for every $r \in \mathbb{N}$ a subset H_r of $H(z)$. Let

$$\begin{aligned} H_r &:= \{P \in H(z) \mid r <_p z <_p r+1\} \\ &= H(\alpha_r) \end{aligned}$$

where $\alpha_r := (z-r)(r+1-z)$. Then we have

$$H(z) \supset \dot{\bigcup_{r \in \mathbb{N}}} H(\alpha_r).$$

For all $i \neq j$ we have $H(\alpha_i) \cap H(\alpha_j) = \phi$. For $n = 0,\ 1,\ \ldots$ we define the form φ_n over $F(n)$ in the following way:

$$\varphi_n = \langle 1, z\rangle + \sum_{i=1}^{n} \langle 1, \alpha_i\rangle t_i.$$

Now dim $\varphi_n = 2n+2$ and the signature values of φ_n are 0, 2 and 4. Assume that φ_n is decomposable: $\varphi_n = \rho_1 + \rho_2$ with $\rho_1,\ \rho_2 \neq 0$ positive. Every decomposition of this kind is compatible with the orthogonal decomposition in residue class forms. Thus we can assume that $\langle 1, z\rangle$ is a summand of ρ_1 and ρ_2 is a sum of the $\langle 1, \alpha_i\rangle t_i$, which is impossible. Therefore φ_n is indecomposable.

Case B) n is odd. We set $F = \mathbb{Q}(x, y, z)$ and $F(n) := F((t_1)) \ldots ((t_n))$. Now, over F we define $\varphi = \langle -x,\ 1+y^2+3x,\ x+y^2x+3x^2+z^2\rangle$. For the embedding $F \hookrightarrow L := \mathbb{Q}(x, y)((z))$ we have $\varphi_L = \rho$ where ρ is defined as in example 1. Thus φ is indecomposable. We also have

$$\text{sgn}_P \varphi = \begin{cases} 3 & \text{if } P \in H(\det\ \varphi), \\ 1 & \text{otherwise.} \end{cases}$$

Let $a := x + y^2x + 3x^2$. Then the determinant of φ is $-1 - (z^2)/a$ mod squares. One sees immediately that for all $r \in \mathbb{N}$ there is an ordering $P \in X_F$ such that $r <_P -(z^2)/a <_P r+1$. Now as above we set

$$\begin{aligned} H_r &= H(\alpha_r), \\ \alpha_r &:= -\left(\frac{z^2}{a} + r\right)\left(r + 1 + \frac{z^2}{a}\right) \end{aligned}$$

and have

$$H(det\ \varphi) \supset \dot{\bigcup_{r\in\mathbb{N}}} H(\alpha_r)$$

and all $H(\alpha_r)$ are non-empty. For $n = 0,\ 1,\ \ldots$ we define the form φ_n over $F(n)$ as

$$\varphi_n = \varphi + \sum_{i=1}^{n} \langle 1, \alpha_i \rangle t_i.$$

Then dim $\varphi_n = 2n + 3$ and the signature values of φ_n are 1, 3 and 5. As above we get that φ_n is indecomposable.

3. **The property** (P). We say that a formally real field F satisfies (P) if every positive form over F represents a nonzero sum of squares. We now want to study this property and soon find a large class of fields which satisfy (P), the pythagorean fields. To show this we use Marshall's language of spaces of orderings (see [8]). The definition of the property (P) carries over to spaces of orderings in the obvious way.

THEOREM 2. *Let* (X, G) *be a space of orderings. Then* (X, G) *satisfies* (P).

PROOF. Let φ be positive over (X, G). To show that $\varphi + \langle -1 \rangle$ is isotropic we apply the isotropy theorem [9] (1.4) and can therefore assume that X is finite. The property (P) is closed under formation of direct sums. We can therefore assume (X, G) to be a group extension of (X', G') and that (X', G') satisfies (P). But if φ is positive so is the first residue class form of φ. Hence φ represents 1. □

The example $\langle t, -2t \rangle$ over $\mathbb{Q}((t))$ motivates a necessary condition for F to satisfy (P). Let v be a valuation of F, $R = R_v$ the valuation ring, $\Gamma = \Gamma_v$ the value group, $k = k_v$ the residue class field and $\pi = \pi_v : F \longrightarrow k$ the projection. We say that F satisfies (PYT) if for every valuation v of F one of the following conditions is satisfied: (i) Γ_v is 2-divisible; (ii) k_v is pythagorean of not formally real.

THEOREM 3. *(a) if F satisfies (P) then F satisfies (PYT). (b) Assume that F is a SAP-field. Then the following statements are equivalent: (i) F satisfies (P). (ii) F satisfies (PYT). (iii) F satisfies (ED).*

REMARK. The properties *SAP* and (ED) are defined in [7] and [11] resp. The next example shows that in general (PYT) does not imply (P).

PROOF. (a): Let v be a valuation of $\dot{F}$ such that $\Gamma/2\Gamma \neq 0$ and k is formally real and non pythagorean. Then there exist a sum of squares $\Sigma g_i^2 \in k^{\cdot} - k^{\cdot 2}$ and a $d \in F^{\cdot}$ such that $v(d) \neq 0$ in $\Gamma/2\Gamma$. Choose $f_i \in F^{\cdot}$ such that $\pi(f_i) = g_i$ and a $d \in F^{\cdot}$ such that $v(d) \neq 0$ in $\Gamma/2\Gamma$. Choose $f_i \in f^{\cdot}$ such that $\pi(f_i) = g_i$ and set $\varphi = \langle d \rangle \langle 1, -\sum f_i^2 \rangle$. Then φ does not represent a nonzero sum of squares. (b): $(ii) \longrightarrow (iii)$: if F satisfies (PYT) and *SAP* then it follows from the characterisation theorem in [11] that F satisfies (ED). $(iii) \longrightarrow (i)$ is trivial. □

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EXAMPLE 3. Let k be an algebraic number field with two orderings $P_1 \neq P_2$. Let R_1 and R_2 be real closures for P_1 and P_2 in some algebraic closure of k. We set $K = R_1 \cap R_2$ and $F = K(x)$. Then every finite formally real extension of K is pythagorean (see [2]) and hence F satisfies (PYT). The stability index of F is two and the following lemma shows that F does not satisfy (P).

LEMMA 2. *Let K be a field such that there exist two different orderings P_1, P_2 of K. Let $a \in K$ such that $a \in -P_1$ and $a \in -P_2$. Let $F = K(x)$ and $\psi = \langle a, -a(x^2+1)\rangle$. Then ψ does not represent a nonzero sum of squares.*

PROOF. Assume there exist polynomials $g(x)$, $h(x)$, $t(x) \in K[x]$ such that $t(x) = a(h(x)^2 - (x^2+1)g(x)^2)$ and $t(x) \in \dot{T}_F$. We can assume $(g, h) = 1$. Let deg $h = m$. Then we get deg $g = m - 1$. Hence deg g or deg h is odd. Assume deg g is odd. Let g_1 by an irreducible divisor of g with odd degree. Then we have $ah(x)^2 = t(x)$ mod $g_1(x)$ where $h(x) \neq 0$ and $t(x)$ is a sum of squares. But P_2 has an extension to $K[x]/g_1(x)$. The same argument applies to the case where deg h is odd. □

As in the case of property (P_3), it seems difficult to characterize those fields satisfying (P). Next we give two statements equivalent to (P). For a quadratic form φ we set $D(\varphi) = \{a \in F^{\cdot} \mid \varphi = \langle a, \dots \rangle\}$.

THEOREM 4. *Let F be formally real. Then the following statements are equivalent: (i) F satisfies (P). (ii) Every two dimensional torsion form over F represents a nonzero sum of squares. (iii) For all $a_1, \dots, a_n \in F^{\cdot}$, $t \in \dot{T}$ we have: If $D(\langle a_1, \dots, a_n\rangle) \cap \dot{T} \neq \phi$ then $D(\langle ta_1, a_2, \dots, a_n\rangle) \cap \dot{T} \neq \phi$.*

PROOF. $(i) \longrightarrow (ii)$ is trivial. $(ii) \longrightarrow (iii)$: Assume $s = \sum_{i=1}^n a_i b_i^2$ for $s \in \dot{T}$, $a_i \in F^{\cdot}$, $b_i \in F$. By hypothesis there exist $h \in F$ and $w \in -\dot{T}$ such that $a_1(1 - h^2 t) = w$. Then $s - wb_1^2 \in \dot{T}$ and

$$s - wb_1^2 = a_1 t(hb_1)^2 + \sum_{i=2}^n a_i b_i^2.$$

$(iii) \longrightarrow (i)$ follows from [7] (1.28): Let φ be a positive form over F. By theorem 2 there is a ψ such that $D(\psi) \cap \dot{T} \neq \phi$, dim φ = dim ψ and $\text{sgn}_P\varphi = \text{sgn}_P\psi$ for all $P \in X_F$. Now ψ can be changed to φ by a finite sequence of transformations. Hence by hypothesis $D(\varphi) \cap \dot{T} \neq \phi$. □

REMARK. The field defined in example 1 also satisfies (P) since the property (P) is closed under formation of direct sums.

REMARK. Let F be formally real. Then every positive form φ over F with dim $\varphi \leqq 5$ is a P-form (see [5]).

PROOF. By [5] 3.1 we can assume φ is defined over the space of orderings $(X_F, \dot{F}/\dot{T})$. By theorem 2 a 5-dimensional form is decomposable. Hence we can assume $\varphi =$

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$\langle 1, x, y, xyd \rangle$ where d is the determinant of φ. But the form $\varphi' = \langle d, x, y, xyd \rangle$ is also positive and hence a quaternion form over X_F. We also have $\varphi = \varphi' + \langle 1, -d \rangle$ in $W(F)$. □

Note that by [5] 8.1 there exists an 8-dimensional positive form which is not a P-form. It is still open whether there exist such forms of dimension 6 or 7.

References

1. J. K. Arason, A. Pfister, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math, Z. **153** (1977), 289–296.
2. L. Bröcker, *Characterization of fans and hereditarily pythagorean fields*, Math. Z. **151** (1976), 149–163.
3. A. B. Carson, M. Marshall, *Decomposition of Witt rings*, Can. J. Math. **34** (1986), 1276–1302.
4. C. Cordes, *Quadratic forms over nonformally real fields with a finite number of quaternion algebras*, Pac. J. Math. **63** (1976), 357–365.
5. M. Krüskemper, W. Scharlau, *On positive quadratic forms*, Bull. Soc. Math. Belg. 40 ser. A, (1988), 255–280.
6. M. Kula, *Fields and quadratic form schemes*, Pr. Nauk Uniw. Slask. Katowicach 693, Ann. Math. Sil. **1** (1985), 7–22.
7. T. Y. Lam, *Orderings, valuations and quadratic forms*, Am. Math. Soc. Regional Conf. Series Math. No 52. 1983.
8. M. Marshall, *Classification of finite spaces of orderings*, Canad. J. Math. **31** (1979), 320–330.
9. ———, *Spaces of orderings IV*, Canad. J. Math. **32** (1980), 603–627.
10. A. Prestel, *Remarks on the Pythogoras and Hasse number of real fields*, J. Reine Angew. Math. **303/304** (1978), 284–294.
11. A. Prestel, R. Ware, *Almost isotropic quadratic forms*, J. London Math. Soc. **19** (1979), 241–244.
12. W. Scharlau, *Quadratic and Hermitian forms*, Berlin, Heldelberg, New York, Springer 1985.

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