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A NOTE ON SUMS OF PRIMES

BY

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ABSTRACT. Under the assumption of the prime k-tuplets conjecture we show that it is possible to construct an infinite sequence of integers, such that the average of any two is prime.

Recently Pomerance, Sárközy and Stewart [2] constructed sets of integers A and B for which every element of $A + B$ is prime; and a set of odd integers A for which $\frac{1}{2}(a + a')$ is prime for any $a \neq a'$ in A. These sets were chosen from $\{1, 2, \ldots, N\}$ so as to make them as large as possible. A natural question that arises is whether we can construct *infinite* sets with these properties, and we do this here under the assumption of Hardy and Littlewood's

PRIME k-TUPLETS CONJECTURE. *Suppose that $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ are integers such that each $(a_j, b_j) = 1$ and, for each prime $p \leq k$, there exists an integer x for which none of $a_1x + b_1, \ldots, a_kx + b_k$ are divisible by p. Then there are arbitrarily large integers x for which each of $a_1x + b_1, \ldots, a_kx + b_k$ is prime.*

Actually we will prove a considerable (though technical) generalization of the above questions:

THEOREM. *Let $c_1, c_2, \ldots, c_N$ be given positive integers and suppose that the prime k–tuplets conjecture is true. We can construct infinite sets $A_1, A_2, \ldots, A_N$ of distinct odd prime numbers such that every element of the set $\frac{1}{d}\{c_1A_1 + \ldots + c_NA_N\}$ is prime, where $g = gcd(c_1, c_2, \ldots, c_N)$ and $d = gcd(2g, c_1 + c_2 + \ldots + c_N)$.*

REMARK: The set $\{c_1A_1 + \ldots + c_NA_N\}$ is defined to be the set whose elements are the sum of any c_1 elements of A_1, any c_2 elements of $A_2, \ldots$, and any c_N elements of A_N. Note that the element $c_1a_1 + \ldots + c_Na_N$ of $c_1A_1 + \ldots + c_NA_N$ must be divisible by d.

By taking $N = 2, c_1 = 1, c_2 = 2, A = A_1$ and $B = \{2a : a \in A_2\}$ in the Theorem above, we have constructed infinite sets of integers A and B for which every element of $A + B$ is prime. By taking $N = 1$ and $c_1 = 2$ we have constructed an infinite set of integers A for which $\frac{1}{2}(a + a')$ is prime for any a, $a' \in A$.

Before the Theorem we prove

LEMMA. *For any given $B > 0$ we can find, under the same hypothesis as in the Theorem, distinct primes $a_1, \ldots, a_N$, each greater than B, such that $\frac{1}{d}(c_1a_1 + \ldots + c_Na_N)$ is prime.*

PROOF. Without loss of generality we may assume that $g = 1$. Let D be the product of the odd primes dividing $c_1c_2 \ldots c_N$. We shall choose integers $r_1, \ldots, r_N$ which satisfy the following congruences: $r_i \equiv 1 \pmod p$ for each prime $p = q$ dividing D, and also

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for $p = 4$ and $q = 2$, for each i, *unless*
p divides $c_1 + c_2 + \ldots + c_N$ and i is the smallest index for which q does not divide c_i, in which case we take $r_i \equiv -1 \pmod{p}$. Such integers exist because of the Chinese Remainder Theorem.

We now choose $a_2, \ldots, a_N$ to be distinct primes, greater than B, with each $a_i \equiv r_i \pmod{4D}$, which is certainly possible by Dirichlet's Theorem for primes in arithmetic progressions. We are thus left with having to find an arbitrarily large integer x such that both $a_1 = 4Dx + r_1$ and $2Dc_0x + e$ are prime, where $c_0 = 2c_1/d$ and $e = (c_1r_1 + c_2a_2 + \ldots + c_Na_N)/d$. However, by the choice of the r_i's, we know that both r_1 and e are coprime with $2Dc_0$ and so, by the prime k–tuplets conjecture, arbitrarily large such x exist.

PROOF OF THE THEOREM: By construction. The first elements of each set are given by taking $B = e^{c_1+c_2+\ldots+c_N}$ in the Lemma. We then continue to construct the sets by adding one new prime to each set in turn, starting $A_1, A_2, \ldots, A_N, A_1, \ldots$ etc.

We now show how to select a suitable prime p to add to the set F_j, once we have already constructed the subsets $F_1, F_2, \ldots F_N$ of $A_1, A_2, \ldots, A_N$ in this way:

Suppose that q was the last prime added to F_j and let m be the product of the primes $< q/2$. We shall be choosing p of the form $p = q + mx$ for some sufficiently large choice of x, so that p is larger than any prime previously chosen and also larger than $3c_j \prod_{i=1}^n (a_i + 1)^{c_i}$, where a_i is the cardinality of F_i.

Let $G_j = F_j \cup \{p\}$ and $G_i = F_i$ otherwise. An element of $\frac{1}{d}\{c_1G_1 + \ldots + c_NG_N\}$ that contains p in its sum can be seen to be equal to an element of $\frac{1}{d}\{c_1F_1 + \ldots + c_NF_N\}$ that contains q in its sum plus some integer multiple of mx/d: Therefore it can be written in the form

$$(*) \qquad r + tmx/d$$

for some t in the range $1 \leq t \leq c_j$, where r is a prime, with $r > q/2$. By definition $(r, tm/d) = 1$, and so each such integer (*) is free of prime factors less than $q/2$.

There are $k - 1 \leq c_j \prod_{i=1}^n a_i^{c_i}$ $(< q/3)$ such elements (*), and none of them is divisible by any prime $\leq k$ when we take $x = 0$. Therefore, by the prime k–tuplets conjecture, there are infinitely many integers x such that $q + mx$ and all of the integers (*) are prime. Hence, by choosing a sufficiently large such integer x, we can ensure that p is a suitable element to be added to F_j.

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