ON THE CURVATURE TENSOR OF EINSTEIN'S GENERALIZED THEORY OF GRAVITATION

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Introduction. Einstein, in his *Generalized Theory of Gravitation* [**1**], deals with a tensor

$$R^{\alpha}_{\beta\rho\sigma} = - R^{\alpha}_{\beta\sigma\rho},$$

whose ninety-six independent components are complex-valued functions of four real variables x_α. Schouten [**4**, p. 261 (89)] has decomposed the general relative tensor, $v_{\alpha\beta\gamma\rho\sigma\tau}$, antisymmetric in α, β, γ, and in ρ, σ, into five irreducible parts. This decomposition can be applied to the R of Einstein if we define the tensor v by

$$(1) \qquad v_{\alpha\beta\gamma\rho\sigma\tau} = v_{[\alpha\beta\gamma][\rho\sigma]\tau} = \epsilon_{\lambda\alpha\beta\gamma} R^{\lambda}_{\tau\rho\sigma},$$

where ϵ is the usual antisymmetric relative tensor with components ± 1. It is essential that all indices run from 1 to 4. The five parts can be expressed in terms of five tensors: a, b, c, d, each with two indices, and h, with four indices. The components of these tensors are linear combinations of the components of R and are introduced in (3), (4), and (5) of this paper.

If the field equations are satisfied there will be conditions on the five tensors given by the equality and reality relations (13) and (14). The purpose of this paper is the determination of these tensors and the derivation of these relations.

From the invariance and the uniqueness of the decomposition [**4**, p. 255] one would expect that these five tensors would appear in any discussion of the physical meaning of the field equations.

1. Schouten's result. This may be stated in terms of the following operators on $v_{\alpha\beta\gamma\rho\sigma\tau}$:

P_{45}, permuting the indices in places 4 and 5,

$n!\,[ijk \ldots]$, the negative symmetric group on $ijk \ldots$,

$n!\,(ijk \ldots)$, the positive symmetric group on $ijk \ldots$,

where n is the number of symbols $ijk \ldots$. These operators are elements of the group ring formed from the symmetric group on six objects [**2**, p. 72].

Schouten's final result is that v may be uniquely and irreducibly decomposed into the following five parts,

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$$
\begin{aligned}
(2) \qquad v_{\alpha\beta\gamma\rho\sigma\tau} = {} & \tfrac{12}{5}\{[1234][56](15)(26) - P_{45}[1234][56](15)(26)\}v_{\alpha\beta\gamma\rho\sigma\tau} && \begin{array}{l}15\\26\\3\\4\end{array}\\
& + \tfrac{12}{5}\{[1236][45](14)(25)\}v_{\alpha\beta\gamma\rho\sigma\tau} && \begin{array}{l}14\\25\\3\\6\end{array}\\
& + 2\{[1234](156) - P_{45}[1234](156)\}v_{\alpha\beta\gamma\rho\sigma\tau} && \begin{array}{l}156\\2\\3\\4\end{array}\\
& + 2\{[123][456](14)(25)(36)\}v_{\alpha\beta\gamma\rho\sigma\tau} && \begin{array}{l}14\\25\\36\end{array}\\
& + \tfrac{16}{5}\{[123][45](146)(25)\}v_{\alpha\beta\gamma\rho\sigma\tau} && \begin{array}{l}146\\25\\3\end{array}
\end{aligned}
$$

The Young diagrams at the right show how the operators are formed. This result is taken from [**4**, p. 261 (89)]. For a detailed account of Young's work the reader should consult [**3**].

2. Decomposition of the tensor R in terms of ninety-six parameters. After the operations indicated in (2) have been actually carried out, the tensor v is to be expressed in terms of the R's, from (1). Since (2) is homogeneous the decomposition applies to relative tensors with regard to weight. It can then be seen that the five parts may be written

$$
\begin{aligned}
(3) \qquad R^{\alpha}{}_{\beta\rho\sigma} = {} & [a_{\beta\sigma}\delta^{\alpha}_{\rho} - a_{\beta\rho}\delta^{\alpha}_{\sigma}] + [b_{\rho\sigma}\delta^{\alpha}_{\beta}]\\
& + [c_{\beta\sigma}\delta^{\alpha}_{\rho} - c_{\beta\rho}\delta^{\alpha}_{\sigma}] + [\epsilon_{\lambda\beta\rho\sigma}d^{\alpha\lambda}] + [h^{\alpha}{}_{\beta\rho\sigma}],
\end{aligned}
$$

where the ninety-six parameters are described in the following table.

TABLE (4)

Tensor	Weight	Symmetry conditions	Number of independent parameters
$a_{\rho\sigma}$	0	$a_{\rho\sigma} + a_{\sigma\rho} = 0$	6
$b_{\rho\sigma}$	0	$b_{\rho\sigma} + b_{\sigma\rho} = 0$	6
$c_{\rho\sigma}$	0	$c_{\rho\sigma} - c_{\sigma\rho} = 0$	10
$d^{\rho\sigma}$	1	$d^{\rho\sigma} - d^{\sigma\rho} = 0$	10
$h^{\alpha}{}_{\beta\rho\sigma}$	0	$h^{\alpha}{}_{[\beta\rho\sigma]} = h^{\alpha}{}_{\beta\alpha\sigma} = 0$	64
			96

Here $h^{\alpha}{}_{\beta\rho\sigma}$ is antisymmetric in ρ and σ, and the thirty-two conditions on this tensor imply that $h^{\alpha}{}_{\alpha\rho\sigma} = 0$.

The ninety-six equations (3) are solved as follows:

$$
(5)\qquad
\begin{aligned}
a_{\rho\sigma} &= -\tfrac{1}{10} R^{\alpha}{}_{\alpha\rho\sigma} + \tfrac{1}{5} R^{\alpha}{}_{\rho\alpha\sigma} - \tfrac{1}{5} R^{\alpha}{}_{\sigma\alpha\rho}\\
b_{\rho\sigma} &= +\tfrac{3}{10} R^{\alpha}{}_{\alpha\rho\sigma} - \tfrac{1}{10} R^{\alpha}{}_{\rho\alpha\sigma} + \tfrac{1}{10} R^{\alpha}{}_{\sigma\alpha\rho}\\
c_{\rho\sigma} &= \tfrac{1}{6} R^{\alpha}{}_{\rho\alpha\sigma} + \tfrac{1}{6} R^{\alpha}{}_{\sigma\alpha\rho}\\
d^{\rho\sigma} &= \tfrac{1}{12} \epsilon^{\rho\alpha\beta\gamma} R^{\sigma}{}_{\alpha\beta\gamma} + \tfrac{1}{12} \epsilon^{\sigma\alpha\beta\gamma} R^{\rho}{}_{\alpha\beta\gamma}
\end{aligned}
$$

These equations show that a, b, and c are tensors and that d is a tensor density. From (3) and (5) $h^{\alpha}{}_{\beta\rho\sigma}$ may be found and seen to be a tensor.

3. The field equations. Equations in R alone. From [**1**, equations 24, 13, 15, 24c] the field equations may be written

$$(6)\qquad R_{\alpha\beta} \equiv \Gamma^{\lambda}_{\alpha\beta,\lambda} - \Gamma^{\lambda}_{\alpha\mu}\Gamma^{\mu}_{\lambda\beta} - \tfrac{1}{2}\Gamma^{\lambda}_{\alpha\lambda,\beta} - \tfrac{1}{2}\Gamma^{\lambda}_{\beta\lambda,\alpha} + \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\mu}_{\lambda\mu} = 0,$$

$$(7)\qquad g_{\alpha\beta;\sigma} = g_{\alpha\beta,\sigma} - g_{\lambda\beta}\Gamma^{\lambda}_{\alpha\sigma} - g_{\alpha\lambda}\Gamma^{\lambda}_{\sigma\beta} = 0,$$

$$(8)\qquad \Gamma^{\alpha}_{\alpha\beta} - \Gamma^{\alpha}_{\beta\alpha} = 0.$$

Equation (6) is equivalent to Einstein's (24), using (8).

It is necessary to manipulate these equations in order to get equations in the R's alone, namely

$$(9)\qquad R^{\lambda}{}_{\lambda\rho\sigma} = 0,$$

$$(10)\qquad R^{\lambda}{}_{\alpha\beta\lambda} + \bar{R}^{\lambda}{}_{\alpha\beta\lambda} = 0.$$

To get (9), differentiate (7) with respect to x_ρ, interchange ρ and σ, subtract, multiply by $g^{\beta\mu}$ and contract with respect to μ and α, using (8). To get (10), use (6), (8), and the definition of R. Substitution from (3) in (9) and (10) gives

$$(11)\qquad a_{\alpha\beta} + 2b_{\alpha\beta} = 0,$$

$$-3a_{\alpha\beta} - 3\bar{a}_{\alpha\beta} + b_{\alpha\beta} + \bar{b}_{\alpha\beta} - 3c_{\alpha\beta} - 3\bar{c}_{\alpha\beta} = 0,$$

$$(12)\qquad 7(b_{\alpha\beta} + \bar{b}_{\alpha\beta}) = 3(c_{\alpha\beta} + \bar{c}_{\alpha\beta}).$$

From (4), $b_{\alpha\beta}$ is antisymmetric and $c_{\alpha\beta}$ is symmetric, and from (11) and (12) the result follows.

If the Einstein field equations are satisfied and if the tensor R is expressed in the form (3), then

$$(13)\qquad a_{\alpha\beta} + 2b_{\alpha\beta} = 0,$$

$$(14)\qquad a_{\alpha\beta},\ b_{\alpha\beta},\ \text{and}\ c_{\alpha\beta}\ \text{are pure imaginary.}$$

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The question as to the sufficiency of these conditions, or whether an alternative formulation of the theory could be based on (13) and (14), is left open.

Added in proof. Since the above was written, the fourth edition of *The Meaning of Relativity* has appeared. The tensor R_{jk} of this edition can be found in terms of the invariant parts by contracting equation (3) of this paper with respect to α and σ:

$$R_{\underline{jk}} = -3c_{jk} = 0, \qquad 2R_{\underset{\vee}{jk}} = -5a_{jk}$$

References

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