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ON THE CONSTRUCTION OF CONVERGENT ITERATIVE SEQUENCES OF POLYNOMIALS

QIU WEIYUAN

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Abstract

We answer two conjectures suggested by Zalman Rubinstein. We prove his Conjecture 1, that is, we construct convergent iterative sequences for $f_m^{-1}(z)$ with an arbitrary initial point, where $f_m(z) = z + z^m$ with $m \geq 2$. We also show by several counterexamples that Rubinstein's Conjecture 2 is generally false.

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1. Introduction

Zalman Rubinstein constructed convergent iterative sequences for the polynomials $f(z) = z + z^m$, $m \geq 2$, with initial point in the lemniscate $\{z \mid |f'(z)| \leq 1\}$ by variational methods. His main results showed that for every point $z_0 \in \{z \mid |f'(z)| \leq 1\}$, the iterative sequence $z_{n+1} = f(z_n)$, $n = 0, 1, \ldots$, converges to 0 as $n \to \infty$. In the particular case $m = 2$, convergent iterative sequences were constructed also for $f^{-1}(z)$ with an arbitrary initial point. For the case $m > 2$, and more generally, for polynomials with positive real coefficients, the following two conjectures were mentioned in [1].

CONJECTURE 1. *Let $f(z) = z + z^m$, $m \geq 2$. There exists a determination of $f^{-1}(z)$ such that for every $z_0 \in \mathbb{C}$ the sequence $z_n = f^{-1}(z_{n-1})$ tends to zero as $n \to \infty$.*

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CONJECTURE 2. *Let* $f(z) = z + a_2 z^2 + \cdots + a_m z^m$ *be of degree* $m \geq 2$, *and assume that* $a_k \geq 0$ *for all* k. *Then for every* z_0 *such that* $|f'(z_0)| \leq 1$, *the sequence* $z_{n+1} = f(z_n)$ *converges.*

In this paper, we will discuss the above two problems. We will show that Conjecture 1 is true, while Conjecture 2 is generally false, by way of several counterexamples.

2. Definitions and lemmas

We need some results of the Fatou and Julia theory of iteration ([3], [4] and [5]; also see [2]). Let $f(z)$ be a polynomial. Denote $f^n = f \circ f \circ \cdots \circ f$ as the nth order iteration of f. The *Fatou set* F of f is the maximal open set in which $\{f^n\}$ is a normal family. The *Julia set* J of f is the complement of F. The point z is called an *nth order periodic point* if $f^n(z) = z$ and $f^k(z) \neq z$ for all $0 < k < n$. Such an nth order periodic point z is called *attractive* (*repulsive* or *rationally indifferent* respectively) if $|(f^n)'(z)| < 1$ ($|(f^n)'(z)| > 1$ or $(f^n)'(z)$ is a root of unity respectively). We also call $\{f^n(z)\}$ a *forward orbit* of f at z, and denote by $f^{-n}(z)$ the inverse images of f^n at z, for $n = 1, 2, \ldots$. Every branch of $f^{-n}(z)$ on a domain is denoted by $f_j^{-n}(z)$.

The following results of Fatou and Julia will be used.

(1) F is open. J is perfect and non-empty. F and J are completely invariant under f, that is, $f(F) = f^{-1}(F) = F$, etc.

(2) The Julia set coincides with the closure of the set of repulsive periodic points.

(3) Every attractive periodic point is in F and every repulsive or rational indifferent periodic point in J.

(4) If f is a polynomial, then the unbounded component $A(\infty)$ of F is exactly the set of all points whose iterative sequences tend to infinity.

(5) If z_0 is not a limit point of the forward orbit of some point $z \notin J$, then every accumulation point of $\{f^{-n}(z)\}$ belongs to J.

(6) Let $\{f_j^{-n}(z)\}_{j,n}$ be any infinite set of inverse branches which are holomorphic in a domain D, and suppose that there exists an open subset of D containing no limit points of the forward orbit of any point $z \notin J$. Then $\{f_j^{-n}(z)\}$ is normal in D and every convergent subsequence tends to a constant.

Now suppose that $g(z) = z + a_m z^m + \cdots$ is a power series analytic at the origin. For $0 < \theta < \pi/2$ and sufficiently small $\rho > 0$, we define the domain

$$D(j,\theta,\rho) = \left\{ z \mid 0 < |z| < \rho, -\gamma - \frac{(2j-2)\pi}{m-1} - \frac{\pi-\theta}{m-1} < \arg z < -\gamma - \frac{(2j-2)\pi}{m-1} + \frac{\pi-\theta}{m-1} \right\}$$

for $j = 1, 2, \ldots, m-1$ and the "star domain" $D(\theta,\rho) = \bigcup_{j=1}^{m-1} D(j,\theta,\rho)$, where γ is a constant satisfying $-a_m \exp\{-i\gamma(m-1)\} > 0$.

Lemma 1 [6, Lemma 9]. *Let $g(z) = z + a_m z^m + \cdots$ be analytic at the origin. Then for given $0 < \theta < \pi/2$ and sufficiently small $\rho > 0$, we have $g(D(\theta,\rho)) \subset D(\theta,\rho)$ and the iteration $g^n(z)$ converges to zero locally uniformly in $D(\theta,\rho)$.*

Lemma 2. *Let $f(z) = z + z^m$. Then $\{z \mid z^{m-1} \in \mathbf{R}\}$, which we abbreviate to $\{z^{m-1} \in \mathbf{R}\}$, and $\{z^{m-1} > 0\}$ are both invariant under f, and $\{z^{m-1} > 0\} \subset A(\infty)$.*

Proof. If $z = \rho e^{k\pi i/(m-1)} \in \{z^{m-1} \in \mathbf{R}\}$, $0 \le \rho < +\infty$, then

$$f(z) = \rho e^{k\pi i/(m-1)}(1 \pm \rho^{m-1}) \in \{z^{m-1} \in \mathbf{R}\}.$$

If $z = \rho e^{2k\pi i/(m-1)} \in \{z^{m-1} > 0\}$, $0 < \rho < +\infty$, then

$$(f(z))^{m-1} = ((\rho + \rho^m)e^{2k\pi i/(m-1)})^{m-1} = (\rho + \rho^m)^{m-1} > 0.$$

These show that $\{z^{m-1} \in \mathbf{R}\}$ and $\{z^{m-1} > 0\}$ are both invariant under f.

Because $f(z) = (|z| + |z|^m)e^{2k\pi i/(m-1)}$ for $z \in \{z^{m-1} > 0\}$, and also $|f(z)| = |z| + |z|^m \ge |z|$, we have by induction that

$$\begin{aligned} f^n(z) &= |f^{n-1}(z)|(1 + |f^{n-1}(z)|^{m-1})e^{2k\pi i/(m-1)} \\ &= f^{n-1}(z)(1 + |f^{n-1}(z)|^{m-1}) \\ &= z \prod_{k=0}^{n-1}(1 + |f^k(z)|^{m-1}). \end{aligned}$$

Hence

$$|f^n(z)| \ge |z|(1 + |z|^{m-1})^n = \rho(1 + \rho^{m-1})^n,$$

which tends to infinity as $n \to \infty$, that is $\{z^{m-1} > 0\} \subset A(\infty)$, from Result 4 above.

Lemma 3. *Let $l_k = \{z \mid z = \rho e^{(2k+1)\pi i/(m-1)}, -\infty < \rho < +\infty\}$, $k = 1, 2, \ldots, m-1$, be a straight line in $\{z^{m-1} \in \mathbf{R}\}$, and let h_k be the subset of l_k,*

$$h_k = \{z \mid z = \rho e^{(2k+1)\pi i/(m-1)}, \rho > \rho_0\},$$

where $\rho_0 = ((m-1)/m)(1/m)^{1/(m-1)}$. Then if m is even, all m branches of $f^{-1}(h_k)$ are disjoint from $\{z^{m-1} \in \mathbf{R}\}$. If m is odd, there is a branch of $f^{-1}(h_k)$:

$$\{z | z = re^{(2k+1)\pi i/(m-1)}, -\infty < r < r_0 < -(1/m)^{1/(m-1)}\},$$

which is contained in l_k. The other $m-1$ branches of $f^{-1}(h_k)$ are disjoint from $\{z^{m-1} \in \mathbf{R}\}$.

PROOF. We first prove that $f^{-1}(h_k) \cap \{z^{m-1} \in \mathbf{R}\} \subset l_k$. In fact, if $z = re^{i\theta} \in f^{-1}(h_k) \cap \{z^{m-1} \in \mathbf{R}\}$, we have $e^{i(m-1)\theta} = \pm 1$ and there is $\rho > \rho_0$ such that $z^m + z = \rho e^{(2k+1)\pi i/(m-1)}$, that is,

$$re^{i\theta}(1 \pm r^{m-1}) = \rho e^{(2k+1)\pi i/(m-1)}.$$

Now $\rho \neq 0$ implies $r \neq 0$ and $(1 \pm r^{m-1}) \neq 0$. Thus, the above equality shows that $z = re^{i\theta}$ and $f(z) = \rho e^{(2k+1)\pi i/(m-1)}$ lie on the same straight line l_k.

However,if $z = re^{(2k+1)\pi i/(m-1)} \in l_k$ with r real and $z \in f^{-1}(h_k)$, then we have

$$r - r^m = \rho \quad \text{where } \rho > \rho_0$$

or

$$\varphi_\rho(r) = r^m - r + \rho = 0.$$

It is easy to check that when m is even and $\rho > \rho_0$, the equation has no real root, so $f^{-1}(h_k) \cap \{z^{m-1} \in \mathbf{R}\} = \varnothing$.

If m is odd, there is a unique real root r_ρ of equation $\varphi_\rho(r) = 0$ and r_ρ belongs to the interval $(-\infty, r_1)$, where $r_1 = -(1/m)^{1/(m-1)}$. We now want to prove that the real root r_ρ is a one-to-one continuous function of ρ when $\rho > \rho_0$. Suppose $\rho_0 < \rho, \rho'$. Then $r_\rho - r_\rho^m = \rho$ and $r_{\rho'} - r_{\rho'}^m = \rho'$. We have

$$r_{\rho'} - r_\rho - (r_{\rho'}^m - r_\rho^m) = \rho' - \rho,$$

or

$$(r_{\rho'} - r_\rho)\left(1 - \sum_{k=0}^{m-1} r_{\rho'}^k r_\rho^{m-1-k}\right) = \rho' - \rho.$$

Since r_ρ, $r_{\rho'}$ are both less than $r_1 = -(1/m)^{1/(m-1)}$, we have that $r_{\rho'}^k r_\rho^{m-1-k}$ is more than $1/m$ for $k = 0, 1, \ldots, m-1$. This means that $\sum_{k=0}^{m-1} r_{\rho'}^k r_\rho^{m-1-k} > 1$ or $1 - \sum_{k=0}^{m-1} r_{\rho'}^k r_\rho^{m-1-k} < 0$. Hence $\rho' > \rho$ implies $r_{\rho'} < r_\rho$. We have thus shown that r_ρ is a strictly monotone function for $\rho > \rho_0$. If we fix $\rho > \rho_0$ and let ρ' be sufficiently close to ρ, we can be sure that r_ρ and $r_{\rho'}$ are all less than a constant $c < r_1$. Then $\sum_{k=0}^{m-1} r_{\rho'}^k r_\rho^{m-1-k} - 1$ will be greater than a positive constant δ (dependent only on ρ). Hence, from the equality

$$|r_{\rho'} - r_\rho| = \frac{|\rho' - \rho|}{\left|1 - \sum_{k=0}^{m-1} r_{\rho'}^k r_\rho^{m-1-k}\right|}$$

it follows that r_ρ is continuous for $\rho > \rho_0$. We now know that the ray line $\{r_\rho e^{(2k+1)\pi i/(m-1)} | \rho > \rho_0\}$ is a branch of $f^{-1}(h_k)$ contained in $\{z | z = re^{(2k+1)\pi i/(m-1)}, -\infty < r < r_1\}$. By the monotonicity and continuity of r_ρ, that branch is

$$\{z | z = re^{(2k+1)\pi i/(m-1)}, -\infty < r < r_0 < r_1\},$$

with endpoint r_0, the negative root of the equation $r - r^m = \rho_0$. And the other branches of $f^{-1}(h_k)$ are disjoint from $\{z^{m-1} \in \mathbf{R}\}$.

LEMMA 4. *The figure of $f^{-1}(h_k)$ is symmetrical about the straight line l_k:*

PROOF. Let $z_1 = re^{(((2k+1)\pi/(m-1))+\theta)i} \in f^{-1}(h_k)$. We will prove that $z_2 = re^{(((2k+1)\pi/(m-1))-\theta)i} \in f^{-1}(h_k)$, where r, θ are real. In fact, there is $\rho > \rho_0$ such that

$$z_1^m + z_1 = re^{(((2k+1)\pi/(m-1))+\theta)i}(1 + r^{m-1}e^{(m-1)\theta i}) = \rho e^{(2k+1)\pi i/(m-1)}.$$

That is

$$(r\cos\theta + r^m \cos m\theta) + i(r\sin\theta + r^m \sin m\theta) = \rho,$$

or

$$r\sin\theta + r^m \sin m\theta = 0.$$

Hence

$$\begin{aligned} z_2^m + z_2 &= re^{(((2k+1)\pi/(m-1))-\theta)i}(1 + r^m e^{-(m-1)\theta i}) \\ &= ((r\cos\theta + r^m\cos m\theta) - i(r\sin\theta + r^m \sin m\theta))e^{(2k+1)\pi i/(m-1)} \\ &= \rho e^{(2k+1)\pi i/(m-1)} = z_1^m + z_1. \end{aligned}$$

This shows the symmetry of the figure of $f^{-1}(h_k)$.

3. Theorem and its proof

Let $f(z) = z + z^m, m \geq 2$. The critical points (singularities) of $f^{-1}(z)$ are

$$c_k = \rho_0 e^{(2k+1)\pi i/(m-1)}, \qquad k = 1, 2, \ldots, m-1,$$

where $\rho_0 = ((m-1)/m)(1/m)^{1/(m-1)}$, and ∞. Let $L = \{z | z = \rho e^{(2k+1)\pi i/(m-1)}, \rho_0 < \rho < +\infty, k = 1, 2, \ldots, m-1\}$. Then we can choose a single-value analytic branch of f^{-1} on the domain $\mathbb{C}\backslash\overline{L}$. We have

THEOREM. *Let $f(z) = z + z^m$, $m \geq 2$. Then there exists an analytic determination of $f^{-1}(z)$ in $\mathbb{C}\backslash\overline{L}$ which satisfies $f^{-1}(0) = 0$, is continuous to*

$\overline{L}$ one sidedly, and is such that for every $z_0 \in \mathbb{C}$ the sequence $z_n = f^{-1}(z_{n-1})$ tends to zero as $n \to \infty$.

PROOF. We choose $f^{-1}(z)$ in $\mathbb{C}\backslash\overline{L}$ that is an inverse analytic branch of f satisfying $f^{-1}(0) = 0$, and choose f^{-1} on $\overline{L} = \bigcup_{k=1}^{m-1} \bar{h}_k$ that maps h_k onto one of the inverse branches of h_k ending at $z_k = (1/m)^{1/(m-1)} e^{(2k+1)\pi i/(m-1)}$ and $f^{-1}(c_k) = z_k$ for $k = 1, 2, \ldots, m-1$. Thus $f^{-1}(z)$ is well defined on $\mathbb{C}$. We have $f^{-1}(z)$ is continuous when z tends to $\overline{L}$ from one side of $\bar{h}_k$. In fact, there two inverse branches of $\bar{h}_k$ ending at z_k. By Lemma 3, they do not lie on l_k and are disjoint from $\{z^{m-1} \in \mathbf{R}\}$. By Lemma 4, they are symmetrical about l_k. so they, when z_k is added, form a curve through the point z_k which is symmetrical about l_k and separates $\mathbb{C}$ into two regions. For $k = 1, 2, \ldots, m-1$, there are $m-1$ such curves separating $\mathbb{C}$ into m regions, only one of them containing the origin. Then the region containing the origin is the image domain of $\mathbb{C}\backslash\overline{L}$ under f^{-1} as $f^{-1}(0) = 0$. Also $f^{-1}(z)$ constructed as above is continuous to $\overline{L}$ one sidedly. Moreover, $f^{-1}(L) \cap \{z^{m-1} \in \mathbf{R}\} = \varnothing$.

Obviously, $f^{-1}(z)$ is analytic at $z = 0$ with an expansion

$$f^{-1}(z) = z - z^m + \cdots .$$

Let $G = \mathbb{C}\backslash\{z^{m-1} \in \mathbf{R}\}$, which is such that $G \subset \mathbb{C}\backslash\overline{L}$. By Lemma 2, $f^{-1}(G) \subset G \subset \mathbb{C}\backslash\overline{L}$. Now G is the union of $2(m-1)$ components G_j, $j = 1, 2, \ldots, 2(m-1)$, each G_j being a simply connected unbounded sector. Given G_j for some j, $f^{-n}(z)$ is analytic in G_j for all $n > 0$. Since $f^n(z)$ tends to infinity uniformly for z sufficiently large, there exists a region in G_j containing no limit points of the forward orbit of any $z \in \mathbb{C}$. By Result 6 of Fatou and Julia, $\{f^{-n}\}$ is normal in G_j and every convergent subsequence tends to a constant. By Lemma 1, with $0 < \theta < \pi/2$ and sufficiently small $\rho > 0$, $f^{-n}(z)$ tends to 0 locally uniformly in the domain $D(\theta, \rho) = \bigcup_{j=1}^{m-1} D(j, \theta, \rho)$ where

$$D(j,\theta,\rho) = \left\{ z \mid 0 < |z| < \rho, -\frac{(2j-2)\pi}{m-1} - \frac{\pi - \theta}{m-1} < \arg z < -\frac{(2j-2)\pi}{m-1} + \frac{\pi-\theta}{m-1} \right\}.$$

Since the intersection between G_j and $D(\theta, \rho)$ is nonempty, every convergent subsequence of $\{f^{-n}(z)\}$ tends to zero in $D(\theta,\rho) \cap G_j$ and so tends to zero in G_j for $j = 1, 2, \ldots, 2(m-1)$. This shows that $\{f^{-n}(z)\}$ tends to zero in $G = \bigcup_{j=1}^{2(m-1)} G_j$.

Next, we consider the convergence of $f^{-n}(z)$ in the set $\{z^{m-1} \in \mathbf{R}\}$. If $z \in L$, then $f^{-1}(z) \in G$ from Lemmas 3, 4 and the construction of f^{-1}.

The above discussion shows that $f^{-n}(z)$ tends to zero as $n \to \infty$. If $z = 0$ then $f^{-n}(0) \equiv 0$ for all $n > 0$. We will prove that $\{z^{m-1} \in \mathbf{R}\}\backslash(L \cup \{0\}) = \{z^{m-1} > 0\} \cup \{z^{m-1} < 0\}\backslash L$ lies in the Fatou set of f.

By Lemma 2, $\{z^{m-1} > 0\} \subset A(\infty) \subset F$. Let $R = \{z^{m-1} < 0\}\backslash L = \{z | z = re^{(2k+1)\pi i/(m-1)}, 0 < r \leq \rho_0, k = 1, 2, \dots, m-1\}$. For $z = re^{(2k+1)\pi i/(m-1)} \in \overline{R}$, $f'(z) = 1 + mz^{m-1} = 1 - mr^{m-1}$ so $|f'(z)| < 1$ as $0 < r \leq \rho_0$. This implies that $|f'(z)| < 1$ as $z \in \overline{R}$ except for $z = 0$. By [1, Lemma 2 and Theorem 1], we get $R \subset F$, $\overline{R} \cap J = \{0\}$ and R contains no limit points of forward orbits of points in $\mathbb{C}$. Also $\{z^{m-1} > 0\} \subset A(\infty)$ contains no limit points of forward orbits. From Result 5 above the accumulation points of $\{f^{-n}(z)\}$ belong to the Julia set, for every $z \in \{z^{m-1} \in \mathbf{R}\}\backslash(L \cup \{0\})$. If $f^{-n}(z) \in \{z^{m-1} \in \mathbf{R}\}\backslash L$ for all $n > 0$, $f^{-n}(z) \to 0$ as $n \to \infty$ since $\overline{\{z^{m-1} \in \mathbf{R}\}\backslash L} \cap J = \{0\}$. Otherwise, there exists an integer $n > 0$ such that $w = f^{-n}(z) \notin \{z^{m-1} \in \mathbf{R}\}\backslash L$. But we have shown, for $w \notin \{z^{m-1} \in \mathbf{R}\}\backslash L$, that is, for $w \in G$ or $w \in L$, that $f^{-n}(w)$ tends to zero as $n \to \infty$. Hence $f^{-n}(z)$ also tends to zero as $n \to \infty$.

COROLLARY. *Let $f(z) = z + z^m, m \geq 2$. Then for every $z_0 \in \{z | |f'(z)| \leq 1\}$, there exists a sequence $\{z_n\}$ such that $z_{n+1} = f(z_n)$ and $z_n \to 0$, $z_{-n} \to 0$ as $n \to \infty$.*

PROOF. This is a direct consequence of the above theorem and [1, Theorem 1].

4. Counterexamples

In this section we will give two examples to show that Conjecture 2 is false.

EXAMPLE 1. Let $f(z) = z(1 + az)^2$, $a > 1$, be a polynomial with positive real coefficients. Now $f(-1/a) = 0 \in J$ (since $f(0) = 0$ and $f'(0) = 1$ is a root of unity, from Result 3) and $-1/a \in J$ (since the Julia set is completely invariant, from Result 1). It is easy to see that $f'(-1/a) = 0$, so that $-1/a$ is in D, one of the components of $\{z | |f'(z)| < 1\}$. But J is a perfect set and the repulsive periodic points of f are dense in J from Results 1 and 2. There exists at least one repulsive periodic point $p \in D$ with period not less than 2. Thus $f^n(p)$ does not converge.

Since $-1/2 \leq f'(z) < 1$ when $z \in [-1/a, 0)$, we have $D \supset [-1/a, 0)$. So the origin is a boundary point of D. If we restrict the initial point to be in the component of $\{z | |f'(z)| < 1\}$ with boundary point 0, the result is also not true.

In this example, we showed that for a polynomial with positive real coefficients $f(z)$, the set $\{z | |f'(z)| < 1\}$ may contains some points in J. The next example shows that there exists such a polynomial for which there is a region in $\{z \mid |f'(z)| < 1\} \cap F$ in which iterative sequences of all points are divergent.

Let $z_0 \in \mathbb{C}$ be a fixed point of polynomial $f(z)$, and suppose that $\lambda = f'(z_0) = e^{2\pi i\omega}$. Then we have

LEMMA 5 (Siegel [7]). *Let ω be an irrational. Suppose there are positive constants a and b satisfying $|\omega - (m/n)| > a/n^b$ for all integers m, n with $n \geq 1$. Then there exists a neighbourhood U of z_0 and a homeomorphism $\varphi: U \to D_r = \{\zeta \mid |\zeta| < r\}$, $\varphi(z_0) = 0$, such that $\varphi \circ f \circ \varphi^{-1}(\zeta) = e^{2\pi i\omega}\zeta$.*

The set of ω satisfying the condition of Lemma 5 is dense in interval $[0, 1]$.

We will construct a polynomial $f(z)$ satisfying the condition of Conjecture 2, which has a fixed point z_0 different from 0 and is such that $\lambda = f'(z_0) = e^{2\pi i\omega}$, where ω satisfies the condition of Lemma 5. For $g(\zeta) = e^{2\pi i\omega}\zeta: D_r \to D_r$, when $\zeta_1 \in D_r$ and $\zeta_1 \neq 0$, its iterative sequence $\{\zeta_n\}, \zeta_n = g(\zeta_{n-1}) = e^{2n\pi i\omega}\zeta_1$ is dense on circle $\{|\zeta| = |\zeta_1|\}$. Thus ζ_n does not converge as $n \to \infty$, and therefore, for $z_1 = \varphi^{-1}(\zeta_1), z_1 \neq z_0, z_{n+1} = f(z_n)$ is also not convergent as $n \to \infty$. Since $|f'(z_0)| = 1$, we deduce, using the minimum principle, that there is a region V in U disjoint from z_0 such that for all $z \in V$, $|f'(z)| < 1$. This is all we need.

EXAMPLE 2. Choose $\omega \in [0, 1]$, satisfying the condition of Lemma 5. Let $\theta = (2 + \omega)/4$. Then $\pi < 2\pi\theta < 3\pi/2$ or $\cos 2\pi\theta < 0$. Let $r = (|e^{2\pi i\omega} - 1|/|e^{4\pi i\theta} - 1|)^{1/3}$. Let

$$\begin{aligned} f(z) &= z + z^2(z - re^{2\pi i\theta})(z - re^{-2\pi i\theta}) \\ &= z + r^2z^2 - 2r\cos(2\pi\theta)z^3 + z^4. \end{aligned}$$

Then $f(z)$ is a polynomial with positive real coefficients having nonzero fixed point $z_0 = re^{2\pi i\theta}$.

$$\begin{aligned} f'(z_0) &= 1 + 2r^2z_0 - 3r(e^{2\pi i\theta} + e^{-2\pi i\theta})z_0^2 + 4z_0^3 \\ &= 1 + r^3e^{2\pi i\theta}(e^{4\pi i\theta} - 1). \end{aligned}$$

Since $e^{i\alpha} - 1 = |e^{i\alpha} - 1|e^{(\pi+\alpha)i/2}$ for real α,

$$r^3e^{2\pi i\theta}(e^{4\pi i\theta} - 1) = r^3e^{2\pi i\theta}|e^{4\pi i\theta} - 1|e^{(4\pi\theta+\pi)i/2}.$$

when $\theta = (2 + \omega)/4$ and $r = (|e^{2\pi i\omega} - 1|/|e^{4\pi i\theta} - 1|)^{1/3}$, we get

$$\begin{aligned} f'(z_0) &= 1 + |e^{2\pi i\omega} - 1|e^{(8\pi((2+\omega)14)+\pi)i/2} \\ &= 1 + |e^{2\pi i\omega} - 1|e^{(2\pi\omega+\pi)i/2} = e^{2\pi i\omega}. \end{aligned}$$

This completes the construction of our example.

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Institute of Mathematics
Fudan University
Shangai
People's Republic of China