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TORSION THEORIES INDUCED BY TILTING MODULES

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Introduction. Let k be a commutative field, and A a finite-dimensional k-algebra. By a module will always be meant a finitely generated right module. Following [**8**], we shall call a module T_A a tilting module if (1) $\mathrm{pd}T_A \leqq 1$, (2) $\mathrm{Ext}^1_A(T, T) = 0$ and (3) there is a short exact sequence

$$0 \to A_A \to T'_A \to T''_A \to 0,$$

with T' and T'' direct sums of direct summands of T. Given a tilting module T_A, the full subcategories

$$\mathscr{F} = \{M_A | \mathrm{Hom}_A(T, M) = 0\} \quad \text{and}$$

$$\mathscr{T} = \{M_A | \mathrm{Ext}^1_A(T, M) = 0\}$$

of the category mod$A$ of A-modules are respectively the torsion-free class and the torsion class of a torsion theory $(\mathscr{T}, \mathscr{F})$ on mod$A$ [**8**]. The aim of the present paper is to find conditions on a torsion theory in order that it be induced by a tilting module. This problem has already been considered by Hoshino [**10**] who proved that if $(\mathscr{T}, \mathscr{F})$ is a torsion theory such that $\mathscr{T}$ contains all injectives, and either $\mathscr{T}$ or $\mathscr{F}$ contains only finitely many non-isomorphic indecomposable modules, then $(\mathscr{T}, \mathscr{F})$ is induced by a tilting module. However, while the first condition is obviously necessary, the second is not as the following example shows: let A be a tame one-relation algebra resulting from the glueing of the preinjective component of a tame hereditary algebra with the preprojective component of another [**12**], then the slice module of a complete slice in the glued component is a tilting module inducing a torsion theory $(\mathscr{T}, \mathscr{F})$ such that both $\mathscr{T}$ and $\mathscr{F}$ contain infinitely many non-isomorphic indecomposable modules. We shall thus start by proving:

THEOREM. *A torsion theory* $(\mathscr{T}, \mathscr{F})$ *on* mod$A$ *is induced by a tilting module if and only if* $\mathscr{T}$ *contains all injective modules, and either* $\mathscr{T}$ is generated, *or* $\mathscr{F}$ *is cogenerated* (*as subcategories of* mod$A$) *by a* (*finitely generated*) *module.*

Section (1) will be devoted to the proof of this theorem. In Section (2), we shall study the case where $(\mathscr{T}, \mathscr{F})$ is a splitting torsion theory induced

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by a tilting module. It is then possible to give an explicit description of such a tilting module, provided the algebra A has a preprojective component containing all projectives, and the torsion-free modules are preprojective. We shall then apply our results in Section (3), to give a sufficient condition for such an algebra to be a tilted algebra.

Throughout this paper, we shall freely use the properties of the Auslander-Reiten translations $\tau = DTr$ and $\tau^{-1} = TrD$, as in [**2**]. For tilting modules and their properties, we shall refer to [7] and [**8**].

1. Tilting torsion theories.

Definition (1.1). A torsion theory $(\mathcal{T}, \mathcal{F})$ on mod$A$ will be called a *tilting torsion theory* if there exists a tilting module T_A such that

$$\mathcal{T} = \mathcal{T}(T_A) = \{M_A | \mathrm{Ext}_A^1(T, M) = 0\}$$

and

$$\mathcal{F} = \mathcal{F}(T_A) = \{M_A | \mathrm{Hom}_A(T, M) = 0\}.$$

Our objective will be to prove the following:

THEOREM (1.2). *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion theory on* mod$A$, *the following assertions are equivalent*:

(i) $(\mathcal{T}, \mathcal{F})$ *is a tilting torsion theory*,

(ii) $\mathcal{T}$ *is generated by a (finitely generated) faithful module*,

(iii) $\mathcal{F}$ *is cogenerated by a (finitely generated) module N with no injective summands and such that* $\mathrm{pd}(\tau^{-1}N) \leqq 1$.

Remarks (1.3). (1) Assume that $\mathcal{T}$ is generated by the module M_A. Then M_A is faithful if and only if there exists an epimorphism

$$M^{(t)} \to D({}_AA) \to 0 \quad \text{for some } t \in \mathbf{N}.$$

Thus, we may replace in (ii) the condition that M is faithful by the condition that $\mathcal{T}$ contains the minimal injective cogenerator DA, or, equivalently, all injective A-modules. Again, if $\mathcal{F}$ satisfies (iii), $\mathrm{pd}(\tau^{-1}N) \leqq 1$ implies that

$$\mathrm{Hom}_A(I, N) = 0$$

for any injective A-module I_A, by Lemma (2.2) of [7], and hence injectives are torsion. We may therefore reformulate Theorem (1.2) as follows: $(\mathcal{T}, \mathcal{F})$ is a tilting torsion theory if and only if $\mathcal{T}$ contains all injective A-modules, and either $\mathcal{T}$ is generated, or $\mathcal{F}$ is cogenerated by a module.

(2) Observe also that, by Propositions (4.6) and (4.7) of [**4**], the torsion theory $(\mathcal{T}, \mathcal{F})$ is such that $\mathcal{T}$ is generated (respectively, $\mathcal{F}$ is cogenerated) by a finitely generated module if and only if $\mathcal{T}$ has a finite cover (respectively, $\mathcal{F}$ has a finite cocover), or equivalently, if and only if mod$A$

is functorially finite over $\mathscr{T}$ (respectively, over $\mathscr{F}$). In this case, $\mathscr{T}$ (respectively, $\mathscr{F}$) has relative Auslander-Reiten sequences [**5**].

We shall need the following definitions and results from [**5**] and [**6**]:

Definition. (1.4). Let $\mathscr{C}$ be a full subcategory of mod$A$ closed under extensions. A module M in $\mathscr{C}$ will be called *Ext-projective* (respectively, *Ext-injective*) in $\mathscr{C}$ if

$$\mathrm{Ext}_A^1(M, -)\,|_{\mathscr{C}} = 0$$

(respectively, $\mathrm{Ext}_A^1(-, M)\,|_{\mathscr{C}} = 0$).

THEOREM (1.5). *Let* $(\mathscr{T}, \mathscr{F})$ *be a torsion theory in* mod$A$, *and* t *be the idempotent torsion radical, then*:

(i) *If* $M \in \mathscr{T}$ *is indecomposable, then*:

$$M \text{ is Ext-projective in } \mathscr{T} \Leftrightarrow \tau M \in \mathscr{F}$$

and M *is Ext-injective in* $\mathscr{T} \Leftrightarrow M \tilde{\rightarrow} tI$ *for some indecomposable injective module* $I_A \notin \mathscr{F}$.

(ii) *If, moreover,* $\mathscr{T}$ *is generated by a module* G_A, *then* G *is Ext-projective in* $\mathscr{T}$. *Also, the number of non-isomorphic indecomposable torsion modules which are Ext-projective, and the number of those which are Ext-injective are finite and equal.*

(iii) *Dually, let* $N \in \mathscr{F}$ *be indecomposable, then*:

$$N \text{ is Ext-injective in } \mathscr{F} \Leftrightarrow \tau^{-1}N \in \mathscr{T}$$

and N *is Ext-projective in* $\mathscr{F} \Leftrightarrow N \tilde{\rightarrow} P/tP$ *for some indecomposable projective module* $P_A \notin \mathscr{T}$.

(iv) *If, moreover,* $\mathscr{F}$ *is cogenerated by a module* H_A, *then* H *is Ext-injective in* $\mathscr{F}$. *Also, the number of non-isomorphic indecomposable torsion-free modules which are Ext-projective, and the number of those which are Ext-injective are finite and equal.*

Proof of Theorem (1.2). Assume first that the torsion theory $(\mathscr{T}, \mathscr{F})$ is induced by the tilting module T_A. Then $\mathscr{T}$ is generated by T_A [**8**], which is a faithful module, as can be seen from the short exact sequence:

$$0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$$

(with T' and T'' direct sums of summands of T). On the other hand, the torsion-free class is cogenerated by τT which has no injective summands and which satisfies

$$\mathrm{pd}(\tau^{-1}(\tau T)) \leqq 1.$$

Thus, (i) implies (ii) and (iii).

Let now the torsion theory $(\mathscr{T}, \mathscr{F})$ be such that $\mathscr{T}$ is generated by the (finitely generated) faithful module M_A. We may assume, without loss of generality, that M is the direct sum of non-isomorphic indecomposables. Let $T_1, T_2, \ldots, T_m$ be a complete set of representatives of the isomorphism classes of indecomposable Ext-projective modules in $\mathscr{T}$. We claim that

$$T_A = \bigoplus_{i=1}^{m} T_i$$

is a tilting module.

Let us start by showing that

$$\mathrm{pd}T_i \leqq 1 \quad \text{for } 1 \leqq i \leqq m.$$

We may assume that T_i is not projective. Then $\tau T_i \in \mathscr{F}$ (by (1.5)) and, since injectives are torsion,

$$\mathrm{Hom}_A(I, \tau T_i) = 0$$

for any injective A-module I_A. Therefore $\mathrm{pd}T_i \leqq 1$ [7]. Next,

$$\mathrm{Ext}_A^1(T, T) = 0,$$

because T is Ext-projective in $\mathscr{T}$. There only remains to construct a short exact sequence:

$$0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$$

with T', T'' direct sums of summands of T_A. Let $(f_i)_{1 \leqq i \leqq t}$ be a basis of the k-vector space $\mathrm{Hom}_A(A, M)$, and put

$$f = (f_i)_i : A \rightarrow M^{(t)}.$$

Then f is a monomorphism (for, M being faithful cogenerates A_A). We claim that for $N \in \mathscr{T}$,

$$\mathrm{Hom}_A(f, N) : \mathrm{Hom}_A(M^{(t)}, N) \rightarrow \mathrm{Hom}_A(A, N)$$

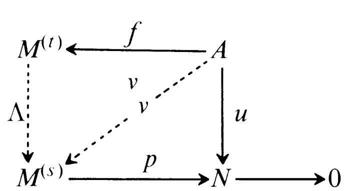
is an epimorphism. Since $N \in \mathscr{T}$, there exists an epimorphism

$$p : M^{(s)} \rightarrow N \quad \text{for some } s \in \mathbf{N}.$$

Since A_A is projective, to any morphism $u : A \rightarrow N$ corresponds

$$v = (v_j)_{1 \leqq j \leqq s} : A \rightarrow M^{(s)}$$

such that $u = pv$:



Now $v_j:A \to M$ can be written as

$$v_j = \sum_{i=1}^{t} \lambda_j^i f_i \quad \text{for some } \lambda_j^i \in k.$$

Therefore $v = \Lambda f$, where $\Lambda = [\lambda_j^i]$ is an $s \times t$ scalar matrix, hence an A-linear map from $M^{(t)}$ to $M^{(s)}$. We thus have:

$$u = pv = (p\Lambda)f = \mathrm{Hom}_A(f, N)(p\Lambda)$$

which proves our claim.

Setting $C = \mathrm{Coker} f$, we have a short exact sequence:

$$0 \to A_A \xrightarrow{f} M^{(t)} \to C \to 0.$$

Since M is Ext-projective in $\mathscr{T}$ (by (1.5)), $M^{(t)}$ is a direct sum of summands of T. Applying the functor $\mathrm{Hom}_A(-, N)$, with $N \in \mathscr{T}$, to the previous sequence, we obtain an exact sequence:

$$0 \to \mathrm{Hom}_A(C, N) \to \mathrm{Hom}_A(M^{(t)}, N) \xrightarrow{\mathrm{Hom}_A(f, N)} \mathrm{Hom}_A(A, N)$$
$$\to \mathrm{Ext}_A^1(C, N) \to \mathrm{Ext}_A^1(M^{(t)}, N) \to \mathrm{Ext}_A^1(A, N) = 0.$$

Since $\mathrm{Hom}_A(f, N)$ is an epimorphism, we deduce that:

$$\mathrm{Ext}_A^1(C, N) \xrightarrow{\sim} \mathrm{Ext}_A^1(M^{(t)}, N) = 0$$

because M is Ext-projective. Hence C is also Ext-projective, thus is a direct sum of summands of T. This completes the proof that T_A is a tilting module. Finally, $\mathscr{T}(T_A) = \mathscr{T}$: since M_A is a summand of T, $\mathscr{T} \subseteq \mathscr{T}(T_A)$, and since T is generated by M, we have $\mathscr{T} = \mathscr{T}(T_A)$ which implies that $\mathscr{F} = \mathscr{F}(T_A)$, and $(\mathscr{T}, \mathscr{F})$ is a tilting torsion theory. We have thus proved that (ii) implies (i).

Assume now that $\mathscr{F}$ is cogenerated by the module N_A with no injective summands and such that $\mathrm{pd}(\tau^{-1}N) \leqq 1$. As already observed, this implies that injectives are torsion. Let $N_1, N_2, \dots, N_r$ be a complete set of representatives of the isomorphism classes of the indecomposable Ext-injectives in $\mathscr{F}$. Since no N_i is injective, $\tau^{-1}N_i \in \mathscr{T}$ is not zero and, since

$$\tau(\tau^{-1}N_i) \xrightarrow{\sim} N_i \in \mathscr{F},$$

$\tau^{-1}N_i$ is in fact Ext-projective in $\mathscr{T}$. On the other hand, let $P_1, P_2, \dots, P_s$ be the non-isomorphic indecomposable projective torsion modules, and put:

$$T_A = \left(\bigoplus_{i=1}^{r} \tau^{-1}N_i\right) \oplus \left(\bigoplus_{j=1}^{s} P_j\right).$$

Obviously, T_A is Ext-projective. We claim that in fact T_A is the direct sum of all non-isomorphic indecomposable Ext-projectives in $\mathcal{T}$. Indeed, if M_A is an indecomposable Ext-projective torsion module, then either M is projective, in which case $M \tilde{\to} P_j$ for some $1 \leqq j \leqq s$, or else $\tau M \neq 0$, and then $\tau M \in \mathcal{F}$. Since

$$\tau^{-1}(\tau M) \tilde{\to} M \in \mathcal{T},$$

τM is indecomposable Ext-injective in $\mathcal{F}$, hence

$$\tau M \tilde{\to} N_i \quad \text{for some } 1 \leqq i \leqq r$$

and consequently

$$M \tilde{\to} \tau^{-1} N_i.$$

Our claim follows.

We now show that T_A is a tilting module. First, it is evident that

$$\mathrm{Ext}_A^1(T, T) = 0.$$

On the other hand, for every injective module I_A,

$$\mathrm{Hom}_A(I, \tau T) = \mathrm{Hom}_A(I, \bigoplus_{i=1}^{r} N_i) = 0,$$

and therefore $\mathrm{pd}\,T_A \leqq 1$. There only remains to prove that $r + s = n$, where n is the number of non-isomorphic simple A-modules [7]. By (1.5), r equals the number of non-isomorphic indecomposable Ext-projective torsion-free modules, and $L \in \mathcal{F}$ is indecomposable Ext-projective if and only if $L \tilde{\to} P/tP$ for some indecomposable projective module $P_A \notin \mathcal{T}$ (where t denotes the idempotent torsion radical). Therefore $r = n - s$, and T_A is indeed a tilting module.

Finally, since N is Ext-injective in $\mathcal{F}$, its indecomposable summands are summands of

$$\bigoplus_{i=1}^{r} N_i = \tau T,$$

hence $\mathcal{F} \subseteq \mathcal{F}(T_A)$. Since $\tau T \in \mathcal{F}$, we deduce that $\mathcal{F} = \mathcal{F}(T_A)$, and therefore $(\mathcal{T}, \mathcal{F})$ is induced by the tilting module T_A. This completes the proof of the theorem.

COROLLARY. (1.6). *Let A be a representation-finite algebra. A torsion theory $(\mathcal{T}, \mathcal{F})$ on* mod$A$ *is a tilting torsion theory if and only if all injectives are torsion.*

COROLLARY (1.7). *Let A be a finite-dimensional algebra, and M_A a faithful module such that* $\mathrm{Hom}_A(M, \tau M) = 0$. *Then there exists a module X_A such that $T = M \oplus X$ is a tilting module.*

Proof. Let $\mathscr{T}$ be the subcategory of mod$A$ generated by M. Since

$$\mathrm{Hom}_A(M, \tau M) = 0,$$

$\mathscr{T}$ is closed under extensions [**5**] and is therefore, since M is faithful, the torsion class of a tilting torsion theory $(\mathscr{T}, \mathscr{F})$. Moreover, the direct sum T'_A of all non-isomorphic indecomposable Ext-projective torsion modules is a tilting module inducing $(\mathscr{T}, \mathscr{F})$. However, M_A is itself Ext-projective in $\mathscr{T}$. Therefore its indecomposable summands are also summands of T', and the corollary follows.

2. Splitting tilting torsion theories.

Definition (2.1). A tilting module T_A will be called *separating* if the torsion theory $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ in mod$A$ is splitting. In other words, if any indecomposable module M_A is such that either

$$\mathrm{Hom}_A(T, M) = 0 \quad \text{or} \quad \mathrm{Ext}_A^1(T, M) = 0.$$

The following are examples of separating tilting modules: the APR tilts studied in [**3**], the slice modules of complete slices in tilted algebras [**8**], the tilting modules used in the proof of the sufficiency part of the main theorem in [**1**]. If T_A is a tilting module and $B = \mathrm{End}T_A$, it follows directly from the Brenner-Butler theorem [**8**] that T_A is separating if and only if ${}_BT$ is splitting in the sense of [**1**]. Thus, an algebra A is iterated tilted of type Δ if and only if there exists a sequence of algebras $A_0 = A, A_1, \ldots, A_m$ with A_m hereditary of type Δ, and a sequence of separating tilting modules $T^{(i)}_{A_i}$ $(0 \leqq i < m)$ such that $\mathrm{End}T^{(i)}_{A_i} = A_{i+1}$.

We shall need the following result from [**9**]:

LEMMA (2.2). *Let T_A be a tilting module. The following assertions are equivalent*:

(i) *T_A is a separating tilting module.*

(ii) *If $M_A \in \mathscr{F}(T_A)$, then $\tau M \in \mathscr{F}(T_A)$.*

(iii) *If $N_A \in \mathscr{T}(T_A)$, then $\tau^{-1}N \in \mathscr{T}(T_A)$.*

PROPOSITION (2.3). *Let T_A be a separating tilting module, then*:

(i) *Two non-isomorphic indecomposable summands of T_A lie in distinct τ-orbits of the Auslander-Reiten quiver of A.*

(ii) *Let T_0 and T_1 be indecomposable summands of T_A such that there exist $s, t \geqq 0$ and an irreducible map $\tau^{-s}T_0 \to \tau^t T_1$. Then, if T_1 is not projective, both s and t equal zero.*

Proof. (i) Let T_0 and $T_1 = \tau^{-t}T_0$ $(t \geqq 0)$ be two indecomposable summands of T lying in the same τ-orbit. Since $\mathrm{Ext}_A^1(T, T_0) = 0$, (2.2) implies that

$$\mathrm{Ext}_A^1(T, \tau^{-i}T_0) = 0 \quad \text{for every } i \geqq 0.$$

In particular, if $t \neq 0$,

$$\mathrm{Ext}_A^1(T, \tau^{-(t-1)}T_0) = 0.$$

But

$$\tau^{-(t-1)}T_0 \tilde{\rightarrow} \tau T_1 \quad \text{and} \quad \mathrm{Ext}_A^1(T_1, \tau T_1) \neq 0.$$

Therefore $t = 0$ and $T_0 \tilde{\rightarrow} T_1$.

(ii) Let the indecomposable summands T_0 and T_1 of T be such that there exist $s, t \geqq 0$ and an irreducible map

$$\tau^{-s}T_0 \rightarrow \tau^t T_1.$$

Assume moreover that T_1 is not projective. Then $T_0 \in \mathscr{T}(T_A)$ implies that $\tau^{-s}T_0 \in \mathscr{T}(T_A)$, and therefore

$$\tau^t T_1 \notin \mathscr{F}(T_A).$$

Since $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ is splitting,

$$\tau^t T_1 \in \mathscr{T}(T_A).$$

Hence, if $t \neq 0$,

$$\tau^{-(t-1)}\tau^t T_1 = \tau T_1 \in \mathscr{T}(T_A).$$

But this is impossible, since $\mathrm{Ext}_A^1(T_1, \tau T_1) \neq 0$. Therefore $t = 0$. There remains to show that $s = 0$ as well. If $s \neq 0$, we have a chain of irreducible maps

$$T_0 \rightarrow \ldots \rightarrow \tau^{-(s-1)}T_0 \rightarrow \tau T_1.$$

But $\mathrm{Ext}_A^1(T_1, \tau T_1) \neq 0$ implies that $\tau T_1 \in \mathscr{F}(T_A)$, hence

$$\tau^{-(s-1)}T_0 \in \mathscr{F}(T_A),$$

and we deduce that $T_0 \in \mathscr{F}(T_A)$, an absurdity.

It follows that if, in a τ-orbit of the Auslander-Reiten quiver of A, there exists an indecomposable summand T_i of the separating tilting module T, then an indecomposable M_A in this τ-orbit belongs to $\mathscr{T}(T_A)$ if and only if there exists $t \geqq 0$ such that

$$M \tilde{\rightarrow} \tau^{-t}T_i,$$

and an indecomposable N_A in the same τ-orbit belongs to $\mathscr{F}(T_A)$ if and only if there exists $s > 0$ such that

$$N \tilde{\rightarrow} \tau^s T_i.$$

On the other hand, if there is no indecomposable summand of T in this τ-orbit, it is entirely contained either in $\mathscr{T}(T_A)$, or in $\mathscr{F}(T_A)$. In particular, no indecomposable summand of T_A lies in a periodic τ-orbit. We also have:

COROLLARY (2.4). *Let T_A be a separating tilting module. Then, for any chain of indecomposable modules and irreducible maps of the form*

$$T_0 = M_0 \to M_1 \to \ldots \to M_r \to M_{r+1} = \tau^s T_1$$

with $s, r \geqq 0$, T_0 and T_1 indecomposable summands of T, and no M_i $(1 \leqq i \leqq r)$ an indecomposable summand of T, we must have $s = 0$ and, moreover, if $r \geqq 1$, T_1 must be projective.

Proof. Indeed, in such a chain, all modules lie in $\mathscr{T}(T_A)$, since $T_0 \in \mathscr{T}(T_A)$. In particular,

$$\tau^s T_1 \in \mathscr{T}(T_A).$$

Hence $s = 0$. Let us now suppose that $r \geqq 1$, and that T_1 is not projective. Observe that no M_i is projective, since an indecomposable projective torsion module is a summand of T. Thus, the irreducible map $M_r \to T_1$ induces an irreducible map $\tau M_r \to \tau T_1$. Since $\tau T_1 \in \mathscr{F}(T_A)$, $\tau M_r \in \mathscr{F}(T_A)$ as well. But then M_r is Ext-projective in $\mathscr{T}(T_A)$, that is to say, M_r is an indecomposable summand of T, a contradiction.

Definition (2.5). Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver. A *separating slice* $\Sigma = (\Sigma_0, \Sigma_1)$ of Γ is a full subquiver such that:

(1) Σ_0 contains exactly one representative from each τ-orbit in Γ.

(2) For every sequence of arrows of the form

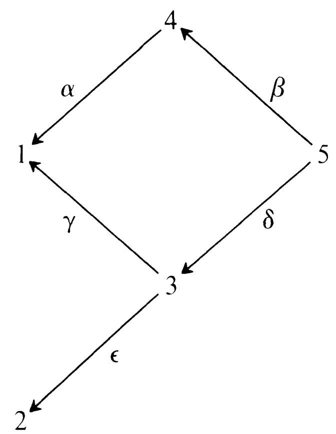
$$x = z_0 \to z_1 \to \ldots \to z_r \to z_{r+1} = \tau^s y$$

with $s, r \geqq 0$, $x, y \in \Sigma_0$ and $z_i \notin \Sigma_0$ for $1 \leqq i \leqq r$, we have that $s = 0$ and moreover, if $r \geqq 1$, y is projective.

Observe that any algebra having a preprojective component admits separating slices in that component (consider for instance the full subquiver consisting of the projective indecomposables). Also, a path from one connected component of a separating slice Σ to another must factor through a projective, but each connected component is not necessarily path-complete: we may have a path

$$x = z_0 \to z_1 \to \ldots \to z_r \to z_{r+1} = y$$

with x, y in the same connected component of Σ, and $z_i \notin \Sigma_0$ for $1 \leqq i \leqq r$. For instance, in the Auslander-Reiten quiver of the algebra of the quiver (following) bound by $\alpha\beta = \gamma\delta = 0$, the modules $P(4)$, $P(5)$, $\tau^{-1}P(1)$, $\tau^{-1}P(3)$ and $\tau^{-2}P(2)$ define a separating slice which is connected but not path-complete. Finally, the existence of separating slices in a component of the Auslander-Reiten quiver does not imply the absence of oriented cycles in that component.



THEOREM (2.6). *Let A be a finite-dimensional k-algebra having a preprojective component containing all projective A-modules, and T_A be a preprojective module. The following assertions are equivalent*;

(i) *T_A is a separating tilting module.*

(ii) *T_A satisfies the conditions of Proposition* (2.3).

(iii) *The indecomposable summands of T_A form a separating slice in the preprojective component of A.*

Proof. We have already seen that (i) ⇒ (ii) ⇒ (iii). There only remains to prove that (iii) ⇒ (i). Let Σ be a separating slice, and $T_1, T_2, \dots, T_n$ be a complete set of representatives of the isomorphism classes of modules in Σ. Let us put

$$\mathscr{F}_0 = \{\tau^s T_i | s > 0, \ 1 \leqq i \leqq n\} \quad \text{and}$$

$$\mathscr{T}_0 = \text{ind}A \setminus \mathscr{F}_0$$

(where ind$A$ denotes the full subcategory of mod$A$ consisting of a set of representatives of the isomorphism classes of indecomposable A-modules), and let $\mathscr{T}$, $\mathscr{F}$ denote respectively the additive subcategories of mod$A$ generated by $\mathscr{T}_0$, $\mathscr{F}_0$. We claim that $(\mathscr{T}, \mathscr{F})$ is a torsion theory (necessarily splitting) in mod$A$. It suffices in fact to prove that the classes $\mathscr{T}$ and $\mathscr{F}$ are orthogonal, that is to say, that

$$\text{Hom}_A(M, N) = 0 \quad \text{for } M \in \mathscr{T} \text{ and } N \in \mathscr{F},$$

since the maximality of $\mathscr{T}$ and $\mathscr{F}$ follows from the fact that ind$A$ is the disjoint union of $\mathscr{T}_0$ and $\mathscr{F}_0$. Let us thus assume that $M \in \mathscr{T}_0$ and $N \in \mathscr{F}_0$ are such that

$$\text{Hom}_A(M, N) \neq 0.$$

Since N is preprojective, so is M, and there exist $s > 0$, $t \geqq 0$, and $1 \leqq i, j \leqq n$ such that

$$N \xrightarrow{\sim} \tau^s T_i \quad \text{and} \quad M \xrightarrow{\sim} \tau^{-t} T_j.$$

Also, there exists a path in the preprojective component from the vertex corresponding to M to the vertex corresponding to N, and hence a chain of indecomposable modules and irreducible maps:

$$T_j \rightarrow \ldots \rightarrow \tau^{-t} T_j \xrightarrow{\sim} M \rightarrow \ldots \rightarrow N \xrightarrow{\sim} \tau^s T_j.$$

Since T_i, T_j belong to Σ, and T_i is not projective (because $s > 0$), at least one of the modules between T_j and N must belong to Σ. The previous chain may then be substituted by a subchain:

$$T_h = M_0 \rightarrow M_1 \rightarrow \ldots \rightarrow M_r \rightarrow M_{r+1} = \tau^s T_i$$

with T_h, T_i in Σ, and M_i not in Σ for $1 \leqq i \leqq r$. This, however, contradicts the fact that Σ is a separating slice. Therefore

$$\mathrm{Hom}_A(M, N) = 0 \quad \text{for all } M \in \mathscr{T}_0,\ N \in \mathscr{F}_0,$$

and hence for all $M \in \mathscr{T}$ and $N \in \mathscr{F}$.

We now prove that

$$T_A = \bigoplus_{i=1}^{n} T_i$$

is a tilting module inducing the torsion theory $(\mathscr{T}, \mathscr{F})$. Observe that, by definition, all injectives are torsion. Since, on the other hand, $\tau T_i \in \mathscr{F}$ for any non-projective indecomposable summand T_i of T, we have

$$\mathrm{Hom}_A(I, \tau T_i) = 0 \quad \text{for any injective } I_A,$$

hence $\mathrm{pd} T \leqq 1$. Moreover, for any two indecomposable summands T_i and T_j of T, we have

$$\mathrm{Ext}_A^1(T_i, T_j) = D\mathrm{Hom}_A(T_j, \tau T_i) = 0.$$

Finally, the number n of non-isomorphic indecomposable summands of T is equal to the number of τ-orbits in the preprojective component, and therefore to the number of non-isomorphic simple A-modules. Hence T_A is a tilting module. There remains to show that $\mathscr{T} = \mathscr{T}(T_A)$ and $\mathscr{F} = \mathscr{F}(T_A)$. If $M \in \mathscr{T}$,

$$\mathrm{Ext}_A^1(T, M) = D\mathrm{Hom}_A(M, \tau T) = 0,$$

since $\tau T \in \mathscr{T}$, hence $M \in \mathscr{T}(T_A)$. Similarly, if $N \in \mathscr{F}$, then

$$\mathrm{Hom}_A(T, N) = 0.$$

This proves that $\mathscr{T} \subseteq \mathscr{T}(T_A)$ and $\mathscr{F} \subseteq \mathscr{F}(T_A)$. Since $(\mathscr{T}, \mathscr{F})$ is splitting, we deduce that

$$\mathscr{T} = \mathscr{T}(T_A) \quad \text{and} \quad \mathscr{F} = \mathscr{F}(T_A),$$

and the proof of the theorem is now complete.

COROLLARY (2.7). *Let A be a finite-dimensional k-algebra having a preprojective component containing all projective A-modules, and $(\mathcal{T}, \mathcal{F})$ a torsion theory in* mod$A$. *The following assertions are equivalent*:

(i) *$(\mathcal{T}, \mathcal{F})$ is induced by a preprojective separating tilting module.*

(ii) *$(\mathcal{T}, \mathcal{F})$ is a splitting tilting torsion theory, with all torsion-free modules preprojective.*

(iii) *$(\mathcal{T}, \mathcal{F})$ is a splitting torsion theory such that $\mathcal{T}$ contains the injectives, and $\mathcal{F}$ contains only finitely many non-isomorphic indecomposable modules.*

Proof. It follows immediately from (2.6) that (i) implies (ii) and (iii). If $(\mathcal{T}, \mathcal{F})$ is a splitting torsion theory, induced by the tilting module T_A, with all torsion-free modules preprojective, then, in particular, τT is preprojective. Since an indecomposable torsion-free module is a predecessor of τT, there are only finitely many non-isomorphic indecomposable torsion-free modules, and (ii) implies (iii).

Let $(\mathcal{T}, \mathcal{F})$ satisfy the hypothesis of (iii), then all torsion-free modules are preprojective. Indeed, if $M_A \in \mathcal{F}$ is not preprojective, there must exist a projective module P_A such that

$$\mathrm{Hom}_A(P, M) \neq 0,$$

and therefore a chain of indecomposable modules and irreducible maps of arbitrary length t:

$$P = N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} N_2 \rightarrow \dots \xrightarrow{f_t} N_t$$

with a map $g:N_t \rightarrow M$ such that $gf_t \dots f_1 \neq 0$ [**11**]. Now, all N_i are preprojective and, since $M \in \mathcal{F}$, all N_i are also torsion-free. But then the arbitrariness of t contradicts the fact that $\mathcal{F}$ contains only finitely many non-isomorphic indecomposables. This proves that any torsion-free module M must be preprojective. Let now T_A be the direct sum of all non-isomorphic indecomposable Ext-projective torsion modules. They are clearly preprojective, since they are either projective, or of the form $\tau^{-1}M$, with $M \in \mathcal{F}$ indecomposable. Since $\mathcal{T}$ contains the injectives, we have again $\mathrm{pd}T \leqq 1$. It is evident that

$$\mathrm{Ext}_A^1(T, T) = 0.$$

Finally, the number of non-isomorphic indecomposable summands of T is equal to the number n of non-isomorphic simple A-modules: indeed, if r is the number of non-isomorphic indecomposable projective torsion modules, the number of non-isomorphic indecomposable Ext-projectives in $\mathcal{F}$, which is equal to the number of non-isomorphic indecomposable Ext-injectives in $\mathcal{F}$, is $n - r$ (by (1.5)). Since an indecomposable summand of T is either projective or of the form $\tau^{-1}N$, with N indecomposable

Ext-injective in $\mathscr{F}$, the number of non-isomorphic summands of T is $r + (n - r) = n$. Therefore T_A is a tilting module, and it is easily seen that $\mathscr{T} = \mathscr{T}(T_A)$, $\mathscr{F} = \mathscr{F}(T_A)$.

3. Application to tilted algebras.

PROPOSITION (3.1). *Let A be a finite-dimensional k-algebra having a preprojective component containing all projectives and define*

$$\mathscr{T} = \{M_A | \mathrm{id} M \leqq 1\},$$

$$\mathscr{F} = \{M_A | \mathrm{id} M > 1\}.$$

If $(\mathscr{T}, \mathscr{F})$ is a torsion theory in mod$A$, *then A is a tilted algebra.*

Proof. If $(\mathscr{T}, \mathscr{F})$ is a torsion theory in mod$A$, it is, by definition, a splitting torsion theory in which all injectives are torsion. Also, an indecomposable module M_A is torsion-free if and only if

$$\mathrm{Hom}_A(\tau^{-1}M, A_A) \neq 0 \quad [7].$$

Let thus $M \in \mathscr{F}$. There exists an indecomposable projective module P_A such that

$$\mathrm{Hom}_A(\tau^{-1}M, P) \neq 0,$$

therefore $\tau^{-1}M$, and consequently M are preprojective, and we have a chain of irreducible maps from M to P. It follows that there are only finitely many non-isomorphic indecomposable torsion-free modules. By (2.7), $(\mathscr{T}, \mathscr{F})$ is induced by the separating tilting module

$$T = \bigoplus_{i=1}^{n} T_i,$$

where $T_1, T_2, \ldots, T_n$ is a complete set of representatives of the isomorphism classes of indecomposable Ext-projective torsion modules. Observe that the T_i are all preprojective.

We claim that the set $\mathscr{S} = \{T_i | 1 \leqq i \leqq n\}$ forms a complete slice in the preprojective component. Since T is a separating tilting module, $\mathscr{S}$ contains exactly one representative from each τ-orbit. Thus we only have to show that if

$$T_0 \rightarrow M_0 \rightarrow \ldots \rightarrow M_t \rightarrow T_1$$

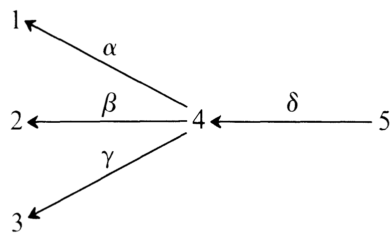
is a chain of indecomposable modules and irreducible maps with $T_0, T_1 \in \mathscr{S}$, then all the M_i belong to $\mathscr{S}$. Suppose indeed that this is not the case. We may assume that T_0 and T_1 are chosen so that no M_i belong to $\mathscr{S}$. Then, by the properties of separating tilting modules, T_1 is projective. Observe that, since $T_0 \in \mathscr{T}$ and $(\mathscr{T}, \mathscr{F})$ is splitting, all the M_i are torsion.

Also, M_t cannot be projective (for, a projective torsion module is a summand of T), hence $\tau M_t \neq 0$. Since M_t does not belong to $\mathscr{S}$, $\tau M_t \notin \mathscr{F}$, therefore $\tau M_t \in \mathscr{T}$. But, on the other hand,

$$\mathrm{Hom}_A(\tau^{-1}(\tau M_t), A_A) \xrightarrow{\sim} \mathrm{Hom}_A(M_t, A) \neq 0$$

because $\mathrm{Hom}_A(M_t, T_1) \neq 0$ and T_1 is projective. We have thus reached a contradiction which completes the proof of our claim, and hence of the proposition.

Remark (3.2). The converse of this proposition is not true, indeed the algebra A of the quiver:



bound by $\alpha\delta = \beta\delta = 0$, is tilted of type $\mathbf{D}_5$, but $(\mathscr{T}, \mathscr{F})$ does not form a torsion theory. In fact, the projective module $P(4)$ has injective dimension 2, while its submodule $P(3)$ has injective dimension 1. On the other hand, its opposite algebra A^{op} satisfies the assumptions of (3.1). Such tilted algebras have an interesting property:

PROPOSITION (3.3). *Let A be a tilted algebra such that its opposite algebra A^{op} satisfies the assumptions of* (3.1), *then, for any tilting module M_A*, $\mathrm{End} M_A$ *is also a tilted algebra.*

Proof. By hypothesis, A has a complete slice $\mathscr{S}$ in its preinjective component such that all modules on the right of $\mathscr{S}$ have projective dimension 2. Let $(T_i)_{1 \leqq i \leqq n}$ be the non-isomorphic indecomposable A-modules in $\mathscr{S}$, and

$$T_A = \bigoplus_{i=1}^{n} T_i$$

be the slice module. Then $B = \mathrm{End} T_A$ is hereditary, $T'_B = D({}_B T)$ is a tilting module and it follows from the Brenner-Butler theorem [**8**] that

$$\mathrm{Tor}_1^A(N, T') = 0$$

if and only if N_A does not lie on the right of $\mathscr{S}$, while

$$N \underset{A}{\otimes} T' = 0$$

if and only if N lies on the right of $\mathscr{S}$. Let now

$$M_A = \bigoplus_j M_j$$

be an arbitrary tilting module, with the M_j indecomposable. Since $\mathrm{pd}M_j \leqq 1$ for all j, no M_j lies on the right of $\mathscr{S}$. Therefore, for each j, there exists a B-module M'_j such that

$$M_j = \mathrm{Hom}_B(T', M'_j).$$

We claim that $M'_B = \bigoplus_j M'_j$ is a tilting module. Since B is hereditary, this follows from the fact that, for any two summands M_j and M_h of M_A:

$$\mathrm{Ext}^1_B(M'_j, M'_h) \xrightarrow{\sim} \mathrm{Ext}^1_A(M_j, M_h) = 0.$$

Since clearly $\mathrm{End}M'_B \xrightarrow{\sim} \mathrm{End}M_A$, we infer that $\mathrm{End}M_A$ is a tilted algebra.

Note. After the completion of this paper, the author has learned that S. O. Smalø had also characterised the tilting torsion theories in **[13]**, obtaining a result equivalent to Theorem (1.2).

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