JFP **14** (4): 379–427, 2004. © 2004 Cambridge University Press
DOI: 10.1017/S0956796803004945 Printed in the United Kingdom

Transformation techniques for context-sensitive rewrite systems[*]

JÜRGEN GIESL[†]
LuFG Informatik II, RWTH Aachen, Ahornstr. 55, 52074 Aachen, Germany
(*e-mail:* `giesl@informatik.rwth-aachen.de`)

AART MIDDELDORP[‡]
Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba 305-8573, Japan
(*e-mail:* `ami@is.tsukuba.ac.jp`)

Abstract

Context-sensitive rewriting is a computational restriction of term rewriting used to model non-strict (lazy) evaluation in functional programming. The goal of this paper is the study and development of techniques to analyze the termination behavior of context-sensitive rewrite systems. For that purpose, several methods have been proposed in the literature which transform context-sensitive rewrite systems into ordinary rewrite systems such that termination of the transformed ordinary system implies termination of the original context-sensitive system. In this way, the huge variety of existing techniques for termination analysis of ordinary rewriting can be used for context-sensitive rewriting, too. We analyze the existing transformation techniques for proving termination of context-sensitive rewriting and we suggest two new transformations. Our first method is simple, sound, and more powerful than the previously proposed transformations. However, it is not complete, i.e., there are terminating context-sensitive rewrite systems that are transformed into non-terminating term rewrite systems. The second method that we present in this paper is both sound and complete. All these observations also hold for rewriting modulo associativity and commutativity.

Capsule Review

This paper presents two algorithms for translating an arbitrary Context-Sensitive Rewrite System (CSRS) into a regular Term Rewrite System (TRS). The idea is to translate each CSRS into a TRS so that one can reason about its termination property by looking at its TRS counterpart. The first algorithm improves on previous work, in that it translates more terminating CSRSes into terminating TRSes. The second algorithm is not only more powerful but also "complete", in that it translates every terminating CSRS into a terminating TRS; but checking the termination behavior for the resulting TRS can be more easily automated for the first algorithm.

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[*] A preliminary version of this paper appeared in the *Proceedings of the 10th International Conference on Rewriting Techniques and Applications, Lecture Notes in Computer Science 1631*, pp. 271–285, 1999.

[†] Supported by the DFG under grant GI 274/4-1.

[‡] Currently at the Institute of Computer Science, University of Innsbruck (e-mail: aart.middeldorp@uibk.ac.at).

1 Introduction

In the presence of infinite reductions in term rewriting, the search for normal forms is usually guided by adopting a suitable reduction strategy. Consider the following term rewrite system:

$$\begin{array}{rcl} \mathsf{nats} \to \mathsf{adx}(\mathsf{zeros}) & \qquad & \mathsf{adx}(x : y) \to \mathsf{incr}(x : \mathsf{adx}(y)) \\ \mathsf{zeros} \to \mathsf{0} : \mathsf{zeros} & & \mathsf{hd}(x : y) \to x \\ \mathsf{incr}(x : y) \to \mathsf{s}(x) : \mathsf{incr}(y) & & \mathsf{tl}(x : y) \to y \end{array}$$

The function symbol zeros is used to generate the infinite list of 0's. The function $\mathsf{incr}(x)$ increments all elements in the list x by one and adx applied to a list $[x_1, x_2, x_3, \ldots]$ adds the index i to each element x_i, i.e., it generates the list $[x_1 + 1, x_2 + 2, x_3 + 3, \ldots]$. The name adx is therefore an abbreviation for "<u>ad</u>d inde<u>x</u>". Hence, nats reduces to the infinite list of positive integers.

A term like $\mathsf{hd}(\mathsf{tl}(\mathsf{tl}(\mathsf{nats})))$ admits a finite reduction to the normal form $\mathsf{s}^3(\mathsf{0})$ (the third positive integer) as well as infinite reductions. The infinite reductions can for instance be avoided by always contracting the outermost redex. Context-sensitive rewriting (Lucas, 1996, 1998) provides an alternative way of solving the non-termination problem. Rather than specifying which redexes may be contracted, in context-sensitive rewriting for every function symbol one indicates which arguments may not be evaluated and a contraction of a redex is allowed only if it does not take place in a forbidden argument of a function symbol somewhere above it. For instance, by forbidding all contractions in the argument t of a term of the form $s : t$, infinite reductions are no longer possible while normal forms can still be computed. (See Lucas (2002b) for the relationship between normalization under ordinary rewriting and under context-sensitive rewriting.)

Term rewriting is a basic computational paradigm with important applications in the design, analysis, verification, and implementation of functional programs (e.g., see Plasmeijer & van Eekelen (1993)). The above example illustrates that the restriction of context-sensitive rewriting has strong connections with lazy evaluation strategies used in functional programming languages, because it allows us to deal with non-terminating programs and infinite data structures (cf. Lucas, 1998).

A central problem in the development of correct and reliable software is to verify the termination of programs. Moreover, techniques for termination analysis can also be helpful for program transformation, e.g. to guarantee termination of partial evaluation (e.g., see Jones & Glenstrup, 2002). Of course, sometimes algorithms may be formulated in primitive recursive form, thereby guaranteeing their termination. But for many algorithms, the natural formulation is not primitive recursive. Converting such an algorithm into primitive recursive form is not easy and can hardly be done automatically in general.

In the area of term rewriting, methods for (automated) termination proofs have been studied for decades (Knuth & Bendix, 1970; Lankford, 1979; Dershowitz, 1987; Bellegarde & Lescanne, 1990; Dershowitz & Hoot, 1995; Steinbach, 1995; Zantema, 1995; Arts & Giesl, 2000; Borralleras *et al.*, 2000). With these methods, termination of many algorithms in different areas of computer science can easily be proved automatically (e.g., Ackermann's function, arithmetical algorithms like division or Euclid's greatest common divisor algorithm, sorting algorithms, graph algorithms, etc.).

In this paper, we are concerned with the problem of proving termination of context-sensitive rewriting. More precisely, we consider transformations from context-sensitive rewrite systems to ordinary term rewrite systems that are *sound* with respect to termination: termination of the transformed term rewrite system implies termination of the original context-sensitive rewrite system. The main advantage of this transformational approach is that all termination techniques for ordinary term rewriting including future developments can be used to infer termination of context-sensitive systems.

Three sound transformations are reported in the literature, by Lucas (1996), Zantema (1997), and by Ferreira & Ribeiro (1999). We add two more. Our first transformation is simple, its soundness is easily established, and it improves upon the transformations of Lucas (1996), Zantema (1997), and Ferreira & Ribeiro (1999). To be precise, we prove that the class of terminating context-sensitive rewrite systems for which our transformation succeeds is larger than that of Lucas', Zantema's, and Ferreira & Ribeiro's transformation. However, none of these four transformations succeeds in transforming every terminating context-sensitive rewrite system into a terminating term rewrite system. In other words, they all lack *completeness*. We analyze the failure of completeness for our first transformation, resulting in a second transformation which is both sound and complete. However, one should remark that the development of our second transformation does not render our first transformation superfluous, since in practical examples, termination of the system resulting from the second transformation can be harder to prove than termination of the one resulting from the first transformation. Similar statements hold for the transformations of Lucas, Zantema, and Ferreira & Ribeiro.

The rest of the paper is organized as follows. In the next section we recall the definition of context-sensitive rewriting and illustrate its connection with functional programming. In particular, we show how our results on termination analysis of context-sensitive rewriting can be used in order to investigate the termination behavior of (lazy) functional programs. Section 3 recapitulates the transformations of Lucas, Zantema, and Ferreira & Ribeiro. Moreover, we analyze the relationship between these transformations. In Section 4 we present our first transformation and prove that it is sound. Despite being incomplete, we prove that it can handle more systems than the transformations of Lucas, Zantema, and Ferreira & Ribeiro. In Section 5 we refine our first transformation into a sound and complete one. The bulk of this section is devoted to the completeness proof. Section 6 shows that similar to the transformation of Ferreira and Ribeiro, both our transformations easily extend to rewriting modulo associativity and commutativity axioms. In Section 7 we investigate how the transformed system changes when modifying the set of argument positions where reductions are allowed. It turns out that in contrast to all previous transformations, both our transformations have a very natural behavior. We make some concluding remarks in Section 8. Those proof details which are not presented in the main text are given in the appendix.

2 Context-sensitive rewriting

Some familiarity with term rewriting (Baader & Nipkow, 1998) is assumed. We briefly recall some basic definitions. A *signature* is a set $\mathscr{F}$ of function symbols equipped with a mapping "arity : $\mathscr{F} \to \mathbb{N}$", where $\mathbb{N}$ is the set of natural numbers. We always require that every signature contains at least one *constant* (i.e., a function symbol f with $\text{arity}(f) = 0$). We assume the existence of a countably infinite set $\mathscr{V}$ of variables, disjoint from $\mathscr{F}$. The set of terms built from $\mathscr{F}$ and $\mathscr{V}$ is denoted by $\mathscr{T}(\mathscr{F}, \mathscr{V})$. The set of variables contained in a term t is denoted by $\mathscr{V}\text{ar}(t)$. A *linear* term does not contain multiple occurrences of the same variable and a *ground* term does not contain any variables. To denote the set of ground terms, we often write $\mathscr{T}(\mathscr{F})$ instead of $\mathscr{T}(\mathscr{F}, \varnothing)$. A *position* is a sequence of positive integers identifying a subterm occurrence in a term. The empty sequence is denoted by ϵ and called the *root position.* The set $\mathscr{P}\text{os}(t)$ of positions in a term t is inductively defined as follows: $\mathscr{P}\text{os}(t) = \{\epsilon\}$ if $t \in \mathscr{V}$ and $\mathscr{P}\text{os}(t) = \{\epsilon\} \cup \{i\pi \mid 1 \leqslant i \leqslant n, \pi \in \mathscr{P}\text{os}(t_i)\}$ if $t = f(t_1, \ldots, t_n)$. If $\pi \in \mathscr{P}\text{os}(t)$ then $t|_\pi$ denotes the subterm of t at position π and $t(\pi)$ denotes the function symbol or variable occurring at position π. We write $\text{root}(t)$ for $t(\epsilon)$; this is called the *root symbol* of t. Furthermore, $t[u]_\pi$ denotes the term that is obtained from t by replacing the subterm at position π by the term u. The set $\mathscr{P}\text{os}(t)$ is partitioned into $\mathscr{P}\text{os}_\mathscr{V}(t) = \{\pi \in \mathscr{P}\text{os}(t) \mid t|_\pi \in \mathscr{V}\}$ and $\mathscr{P}\text{os}_\mathscr{F}(t) = \mathscr{P}\text{os}(t) \setminus \mathscr{P}\text{os}_\mathscr{V}(t)$. A substitution σ is a mapping from $\mathscr{V}$ to $\mathscr{T}(\mathscr{F}, \mathscr{V})$ such that its domain $\{x \in \mathscr{V} \mid \sigma(x) \neq x\}$ is finite. The result of applying σ to a term t is denoted by $t\sigma$.

A *term rewrite system* (TRS for short) $\mathscr{R}$ over a signature $\mathscr{F}$ is a set of rewrite rules $l \to r$ with $l, r \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ such that $l \notin \mathscr{V}$ and $\mathscr{V}\text{ar}(r) \subseteq \mathscr{V}\text{ar}(l)$. A TRS is *left-linear* if the left-hand sides of all rewrite rules are linear terms. The binary relation $\to_\mathscr{R}$ on $\mathscr{T}(\mathscr{F}, \mathscr{V})$ is defined as follows: $s \to_\mathscr{R} t$ if and only if there exist a rewrite rule $l \to r \in \mathscr{R}$, a substitution σ, and a position $\pi \in \mathscr{P}\text{os}(s)$ such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. We say that s *reduces (in one step)* to t by contracting the *redex* $l\sigma$ at position π. The root symbols of left-hand sides of rewrite rules are called *defined*, whereas all other function symbols are *constructors*. For the signature $\mathscr{F}$ of a TRS $\mathscr{R}$ we denote the set of defined symbols by $\mathscr{F}_\mathscr{D}$ and the constructors by $\mathscr{F}_\mathscr{C}$.

Let $\to$ be a binary relation on terms. We say that $\to$ is *closed under contexts* if $s \to t$ implies $u[s]_\pi \to u[t]_\pi$ for all terms u and positions $\pi \in \mathscr{P}\text{os}(u)$. The relation $\to$ is *closed under substitutions* if $s \to t$ implies $s\sigma \to t\sigma$ for all substitutions σ. A relation that is closed under contexts and substitutions is called a *rewrite relation.* The transitive reflexive closure of $\to$ is denoted by $\to^*$. If $s \to^* t$ we say that s *reduces* to t. A term s is a *normal form* if there is no term t with $s \to t$. We write $s \to^! t$ if $s \to^* t$ with t a normal form. Let $s \uparrow t$ denote the existence of a term u such that $u \to^* s$ and $u \to^* t$. We write $s \downarrow t$ if there exists a term u such that $s \to^* u$ and $t \to^* u$. A TRS $\mathscr{R}$ is *terminating* if there are no infinite reductions $t_1 \to_\mathscr{R} t_2 \to_\mathscr{R} \cdots$ and *confluent* if $\uparrow_\mathscr{R} \subseteq \downarrow_\mathscr{R}$. Every term t in a confluent and terminating TRS $\mathscr{R}$ reduces to a unique normal form, denoted by $t\downarrow_\mathscr{R}$.

The following definition introduces context-sensitive rewriting.

Definition 1
Let $\mathscr{F}$ be a signature. A function $\mu : \mathscr{F} \to \mathscr{P}(\mathbb{N})$ is called a replacement map *if $\mu(f)$ is a subset of $\{1, \ldots, \text{arity}(f)\}$ for all $f \in \mathscr{F}$. A* context-sensitive rewrite system

(CSRS for short) is a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ that is equipped with a replacement map μ. The context-sensitive rewrite relation $\rightarrow_{\mathcal{R},\mu}$ is defined as the restriction of the usual rewrite relation $\rightarrow_{\mathcal{R}}$ to contractions of redexes at active *positions. A position π in a term t is (μ-)active if $\pi = \epsilon$ or $t = f(t_1, \ldots, t_n)$, $\pi = i\pi'$, $i \in \mu(f)$, and π' is active in t_i. So $s \rightarrow_{\mathcal{R},\mu} t$ if and only if there exist a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. In the following, we often abbreviate $\rightarrow_{\mathcal{R},\mu}$ to $\rightarrow_\mu$ when $\mathcal{R}$ can be inferred from the context.*

Consider the TRS of the introduction. By taking $\mu(:) = \mu(\mathsf{s}) = \varnothing$ and $\mu(\mathsf{incr}) = \mu(\mathsf{adx}) = \mu(\mathsf{hd}) = \mu(\mathsf{tl}) = \{1\}$, we obtain a terminating CSRS. The term $\mathsf{0} : \mathsf{zeros}$, which has an infinite reduction in the TRS, is a normal form of the CSRS because the reduction step to $\mathsf{0} : (\mathsf{0} : \mathsf{zeros})$ is no longer possible as the contracted redex occurs at an inactive position ($2 \notin \mu(:)$).

Context-sensitive rewriting subsumes ordinary rewriting (when $\mu(f) = \{1, \ldots, n\}$ for every n-ary function symbol f). Context-sensitive rewriting can also be used to model non-strict evaluation in functional programming where one uses a leftmost-outermost strategy. Here, a term s can be evaluated to a term t ($s \overset{\mathsf{ns}}{\rightarrow} t$) if the reduction takes place at the root position. Moreover, s may also be evaluated below the root at a position π if this is necessary in order to find out whether a rule $l \rightarrow r$ might be applicable for a root reduction. In particular, we must have $\mathrm{root}(l) = \mathrm{root}(s)$. This implies that terms with a constructor at their root position cannot be evaluated further (they are in *(weak) head normal form*). In addition, evaluating $s|_\pi$ must be necessary to check whether l matches with s and π is required to be the minimal such position with respect to the lexicographic order on positions. Here, a position $\pi_1 = m_1 \cdots m_k$ is smaller than a position $\pi_2 = n_1 \cdots n_l$ if there is an $i \in \{1, \ldots, \min(k+1, l)\}$ such that $m_j = n_j$ for all $j < i$, and $m_i < n_i$ if $i \leqslant k$. Similar to most functional programming languages, we restrict ourselves to left-linear rules here. Then evaluating $s|_\pi$ is necessary to match l with s if and only if the function symbols $s(\pi)$ and $l(\pi)$ are different. The formal definition of non-strict evaluation is given below.

Definition 2

Let $\mathcal{R}$ be a left-linear TRS. A term s rewrites to a term t with non-strict evaluation *($s \overset{\mathsf{ns}}{\rightarrow}_{\mathcal{R}} t$) if and only if there is a rule $l \rightarrow r \in \mathcal{R}$ such that* $\mathrm{root}(s) = \mathrm{root}(l)$ *and either $s = l\sigma$ and $t = r\sigma$ for some substitution σ or $s|_\pi \overset{\mathsf{ns}}{\rightarrow} t'$ and $t = s[t']_\pi$ for the minimum position $\pi \in \mathcal{P}os_{\mathcal{F}}(l) \cap \mathcal{P}os(s)$ with respect to the lexicographic order on positions such that $s(\pi) \neq l(\pi)$.*

Of course, non-strict evaluation is non-deterministic since there may be several applicable rules $l \rightarrow r$. In functional programming languages, this non-determinism is usually solved by ordering the rules (or equations) from top to bottom and by taking the first applicable rule. As an example regard the following rewrite rules:

$$\mathsf{f}(x) \rightarrow \mathsf{g}(\mathsf{f}(x), \mathsf{b}) \tag{1}$$

$$\mathsf{g}(\mathsf{s}(x), \mathsf{s}(y)) \rightarrow \mathsf{0} \tag{2}$$

$$\mathsf{g}(x,\mathsf{0}) \to \mathsf{0} \tag{3}$$

$$\mathsf{b} \to \mathsf{0} \tag{4}$$

The term $\mathsf{f}(\mathsf{0})$ can be reduced at the root position to $\mathsf{g}(\mathsf{f}(\mathsf{0}),\mathsf{b})$. Now in non-strict evaluation one may try to evaluate this term further with rule (2). The minimum position where the subterm of the left-hand side $\mathsf{g}(\mathsf{s}(x),\mathsf{s}(y))$ does not match the corresponding subterm of $\mathsf{g}(\mathsf{f}(\mathsf{0}),\mathsf{b})$ is 1. Hence, the subterm $\mathsf{f}(\mathsf{0})$ is evaluated further which leads to non-termination. Indeed, such a functional program would be non-terminating. However, if one exchanges rules (2) and (3), then a functional program would first try to reduce the term $\mathsf{g}(\mathsf{f}(\mathsf{0}),\mathsf{b})$ with rule (3) and hence, one would have termination. This cannot be detected with $\overset{\mathsf{ns}}{\to}$, since here any of the applicable rules may be selected. Note that if the order of the rules in the above example would be unchanged, but the arguments of g would be exchanged in all rules, then $\overset{\mathsf{ns}}{\to}$ terminates. Another difference is that non-strict evaluation does not capture sharing whereas in many functional programming languages some common subterms are shared for efficiency reasons (evaluation strategies resulting from non-strict evaluation with sharing are called *lazy evaluation*).

Now we show that non-strict evaluation can be simulated by context-sensitive rewriting. To this end, we use the *canonical* replacement map μ_{c} which is the most restrictive replacement map ensuring that non-variable subterms of left-hand sides of rules are at active positions (Lucas, 1998). In other words, $i \in \mu_{\mathrm{c}}(f)$ if and only if there is a rule $l \to r \in \mathcal{R}$ and a subterm $f(t_1,\ldots,t_n)$ of l such that $t_i \notin \mathcal{V}$. Lucas (2002b) recently proved that termination of $(\mathcal{R},\mu_{\mathrm{c}})$ implies *top-termination* of $\mathcal{R}$, i.e., that there is no $\mathcal{R}$-reduction with infinitely many root reductions. However, this does not yet imply termination of non-strict evaluation as can be seen from the top-terminating system consisting of the two rules $\mathsf{f}(x) \to \mathsf{g}(\mathsf{f}(x))$ and $\mathsf{g}(\mathsf{0}) \to \mathsf{0}$ where non-strict evaluation is not terminating. The following theorem shows the new result that context-sensitive rewriting with the canonical replacement map can also simulate non-strict evaluation. The reason is that μ_{c} only makes those positions inactive where one would never reduce during non-strict evaluation, since evaluation on these positions is not necessary in order to apply rules at higher positions in the term.

Theorem 3

Let $\mathcal{R}$ be a left-linear TRS. If $(\mathcal{R},\mu_{\mathrm{c}})$ is terminating then non-strict evaluation is terminating.[1]

Proof

Let $s \overset{\mathsf{ns}}{\to}_{\mathcal{R}} t$. We show $s \to_{\mathcal{R},\mu_{\mathrm{c}}} t$ by structural induction on s. If the reduction $s \overset{\mathsf{ns}}{\to}_{\mathcal{R}} t$ takes place at the root position then we obviously have $s \to_{\mathcal{R},\mu_{\mathrm{c}}} t$, too. Otherwise there exists a rule $l \to r$ and a minimum position $\pi \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(l) \cap \mathcal{P}\mathrm{os}(s)$ with respect

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[1] Lucas (2002a) recently proved that under the same conditions as in Theorem 3, termination of context-sensitive rewriting is equivalent to termination of *lazy rewriting* (Fokkink *et al.*, 2000). However, since the leftmost evaluation strategy is not imposed in lazy rewriting, this notion has less connections to lazy functional programming than our notion of non-strict evaluation. In fact, the purpose of lazy rewriting is not to model the evaluation strategy of lazy functional languages, but to extend eager implementations in order to improve their termination behavior and efficiency.

to the lexicographic order on positions such that $s(\pi) \neq l(\pi)$. According to the definition of μ_{c}, π is an active position in l. By minimality of π, the function symbols above π must be the same in l and in s. Thus, π is also an active position in s. We have $s|_\pi \overset{\mathsf{ns}}{\to} t'$ such that $t = s[t']_\pi$. Since $\pi \neq \epsilon$ we can apply the induction hypothesis to conclude $s|_\pi \to_{\mathscr{R},\mu_{\mathrm{c}}} t'$. Since π is active in s, this implies $s \to_{\mathscr{R},\mu_{\mathrm{c}}} t$, as desired. □

The reverse of the above theorem does not hold. In other words, termination of $(\mathscr{R}, \mu_{\mathrm{c}})$ is a sufficient but not a necessary criterion for the termination of non-strict evaluation (and hence of the corresponding functional program). The reason is that context-sensitive rewriting does not capture the fact that in non-strict evaluation subterms of a rule are checked in *leftmost* order. Exchanging the arguments of g in the rules (1)–(4) would affect termination of non-strict evaluation, but not of context-sensitive rewriting. Because of the left-hand side $\mathsf{g}(\mathsf{s}(y), \mathsf{s}(x))$, we have $\mu_{\mathrm{c}}(\mathsf{g}) = \{1, 2\}$ and hence $(\mathscr{R}, \mu_{\mathrm{c}})$ remains non-terminating.

Another problem is that the canonical replacement map makes argument positions of constructors active if constructors occur nested in left-hand sides. However, this problem can be avoided by transforming the rules into a form without nested constructors in left-hand sides. Then one would have $\mu_{\mathrm{c}}(c) = \varnothing$ for all constructors c and thus, all terms with constructors on their root position would be in normal form (in this way, (weak) head normal forms can be simulated by context-sensitive rewriting).

To summarize, if one is interested in termination of (first-order) lazy functional programs, analyzing the termination behavior of $(\mathscr{R}, \mu_{\mathrm{c}})$ is much more accurate than analyzing full termination of $\mathscr{R}$. For example, in the nats-system from the introduction the canonical replacement map makes the arguments of the constructors s and ":" inactive, which results in a terminating CSRS. So developing methods for termination proofs of context-sensitive rewriting is useful for termination analysis of lazy functional programs. The advantage of such an approach is that in this way, the whole variety of techniques developed for termination of term rewriting becomes available for termination proofs of lazy functional languages.

Moreover, context-sensitive rewriting (with other replacement maps) can be applied (Lucas, 2001a; Lucas, 2001b) to study the termination behavior of programming languages like OBJ (Clavel *et al.*, 1996; Diaconescu & Futatsugi, 1998; Goguen *et al.*, 2000) where the user can supply strategy annotations to control the evaluation.

Apart from termination analysis, context-sensitive rewriting can also be used for evaluation of functional programs. Here the interesting case is when $\mathscr{R}$ admits infinite reductions and μ is defined in such a way that $\to_{\mathscr{R},\mu}$ is terminating but still capable of computing all ($\mathscr{R}$-)normal forms. For the latter aspect we refer to Lucas (1998; 2002b); in the remainder of this paper we are only concerned with termination of context-sensitive rewriting.

3 Transforming context-sensitive rewrite systems

In this section we review the existing transformations for termination analysis of context-sensitive rewrite systems. Lucas (1996) presented a simple transformation from CSRSs to TRSs which is sound with respect to termination. The idea of his transformation is to remove the inactive arguments of every function symbol appearing in the rewrite rules of the CSRS.

Definition 4
Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. The TRS $\mathscr{R}^{\mathsf{L}}_\mu$ over the signature $\mathscr{F}_{\mathsf{L}} = \{f_\mu \mid f \in \mathscr{F}\}$ where the arity of f_μ is $|\mu(f)|$ consists of the rules $l{\downarrow_\mathscr{L}} \to r{\downarrow_\mathscr{L}}$ for all $l \to r \in \mathscr{R}$. Here $\mathscr{L}$ is the terminating and confluent TRS consisting of all rules of the form $f(x_1,\ldots,x_n) \to f_\mu(x_{i_1},\ldots,x_{i_k})$ such that $\mu(f) = \{i_1,\ldots,i_k\}$ with $i_1 < \cdots < i_k$. In the following we denote Lucas' transformation $(\mathscr{R},\mu) \mapsto \mathscr{R}^{\mathsf{L}}_\mu$ by Θ_{L} and we abbreviate $\to_{\mathscr{R}^{\mathsf{L}}_\mu}$ to $\to_{\mathsf{L}}$.

The idea is that instead of a context-sensitive reduction of a term t one now regards the reduction of the term $t{\downarrow_\mathscr{L}}$ with respect to the TRS $\mathscr{R}^{\mathsf{L}}_\mu$. As an example, consider the TRS $\mathscr{R}$ of the introduction where μ is again defined as $\mu(:) = \mu(\mathsf{s}) = \varnothing$ and $\mu(\mathsf{incr}) = \mu(\mathsf{adx}) = \mu(\mathsf{hd}) = \mu(\mathsf{tl}) = \{1\}$. Then $\mathscr{R}^{\mathsf{L}}_\mu$ consists of the following rewrite rules:

$$
\begin{aligned}
\mathsf{nats}_\mu &\to \mathsf{adx}_\mu(\mathsf{zeros}_\mu) & \qquad \mathsf{adx}_\mu(:_\mu) &\to \mathsf{incr}_\mu(:_\mu)\\
\mathsf{zeros}_\mu &\to :_\mu & \mathsf{hd}_\mu(:_\mu) &\to x\\
\mathsf{incr}_\mu(:_\mu) &\to :_\mu & \mathsf{tl}_\mu(:_\mu) &\to y
\end{aligned}
$$

Due to the extra variable[2] in the right-hand sides of the rules for hd_μ and tl_μ, $\mathscr{R}^{\mathsf{L}}_\mu$ is not terminating:

$$\mathsf{tl}_\mu(:_\mu) \to_{\mathsf{L}} \mathsf{tl}_\mu(:_\mu) \to_{\mathsf{L}} \cdots$$

Zantema (1997) presented a more complicated transformation in which subterms at inactive positions are marked rather than discarded. The transformed system $\mathscr{R}^{\mathsf{Z}}_\mu$ consists of two parts. The first part results from a translation of the rewrite rules of $\mathscr{R}$, as follows. Every function symbol f occurring in a left or right-hand side is replaced by $\underline{f}$ (a fresh function symbol of the same arity as f) if it occurs in an inactive argument of the function symbol directly above it. These new function symbols are used to block further reductions at this position. In addition, if a variable x occurs in an inactive position in the left-hand side l of a rewrite rule $l \to r$ then all occurrences of x in r are replaced by $\mathsf{a}(x)$. Here a is a new unary function symbol which is used to activate blocked function symbols again. The second part of $\mathscr{R}^{\mathsf{Z}}_\mu$ consists of rewrite rules that are needed for blocking and unblocking function symbols.

Definition 5
Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. The TRS $\mathscr{R}^{\mathsf{Z}}_\mu$ over the signature $\mathscr{F}_{\mathsf{Z}} = \mathscr{F} \cup \{\underline{f} \mid f \in \mathscr{F}\} \cup \{\mathsf{a}\}$ consists of two parts, i.e., $\mathscr{R}^{\mathsf{Z}}_\mu = \mathscr{R}^{\mathsf{Z}_1}_\mu \cup \mathscr{R}^{\mathsf{Z}_2}_\mu$. The first part $\mathscr{R}^{\mathsf{Z}_1}_\mu$ consists of the rules $\mathsf{Z}(l) \to \mathsf{Z}(r)\sigma_l$ for all $l \to r \in \mathscr{R}$. The mappings Z and Z' from

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[2] Extra variables can be instantiated by arbitrary terms. So strictly speaking, $\mathscr{R}^{\mathsf{L}}_\mu$ is not a TRS.

$\mathcal{T}(\mathcal{F},\mathcal{V})$ to $\mathcal{T}(\mathcal{F}_{\mathsf{Z}},\mathcal{V})$ are defined inductively by

$$\begin{aligned} \mathsf{Z}(x) = \mathsf{Z}'(x) &= x \\ \mathsf{Z}(f(t_1,\ldots,t_n)) &= f(u_1,\ldots,u_n) \\ \mathsf{Z}'(f(t_1,\ldots,t_n)) &= \underline{f}(u_1,\ldots,u_n) \end{aligned}$$

with $u_i = \mathsf{Z}(t_i)$ if $i \in \mu(f)$ and $u_i = \mathsf{Z}'(t_i)$ if $i \notin \mu(f)$, for all $1 \leqslant i \leqslant n$, and the substitution σ_l is defined by

$$\sigma_l(x) = \begin{cases} \mathsf{a}(x) & \text{if } x \text{ appears in an inactive position in } l \\ x & \text{otherwise} \end{cases}$$

The second part $\mathcal{R}_\mu^{\mathsf{Z}_2}$ consists of $\mathsf{a}(x) \to x$ together with

$$\begin{aligned} f(x_1,\ldots,x_n) &\to \underline{f}(x_1,\ldots,x_n) \\ \mathsf{a}(\underline{f}(x_1,\ldots,x_n)) &\to f(x_1,\ldots,x_n) \end{aligned}$$

for every n-ary f for which $\underline{f}$ appears in $\mathcal{R}_\mu^{\mathsf{Z}_1}$. We denote Zantema's transformation $(\mathcal{R},\mu) \mapsto \mathcal{R}_\mu^{\mathsf{Z}}$ by Θ_{Z} and we abbreviate $\to_{\mathcal{R}_\mu^{\mathsf{Z}}}$ to $\to_{\mathsf{Z}}$. Moreover, $\mathcal{F}_\mu^{\mathsf{Z}}$ denotes the subsignature of $\mathcal{F}_{\mathsf{Z}}$ which consists of the function symbols of $\mathcal{R}_\mu^{\mathsf{Z}}$.

In the approach of Zantema, the aim is to translate the context-sensitive reduction of a term t into an $\mathcal{R}_\mu^{\mathsf{Z}}$-reduction of the term $\mathsf{Z}(t)$. The example CSRS $(\mathcal{R},\mu)$ is transformed into

$$\begin{array}{rclcrclcrcl}
\mathsf{nats} & \to & \mathsf{adx}(\mathsf{zeros}) & \quad & \mathsf{0} & \to & \underline{\mathsf{0}} & \quad & \mathsf{a}(\underline{\mathsf{0}}) & \to & \mathsf{0} \\
\mathsf{zeros} & \to & \underline{\mathsf{0}} : \underline{\mathsf{zeros}} & & \mathsf{s}(x) & \to & \underline{\mathsf{s}}(x) & & \mathsf{a}(\underline{\mathsf{s}}(x)) & \to & \mathsf{s}(x) \\
\mathsf{incr}(x : y) & \to & \underline{\mathsf{s}}(\mathsf{a}(x)) : \underline{\mathsf{incr}}(\mathsf{a}(y)) & & \mathsf{zeros} & \to & \underline{\mathsf{zeros}} & & \mathsf{a}(\underline{\mathsf{zeros}}) & \to & \mathsf{zeros} \\
\mathsf{adx}(x : y) & \to & \mathsf{incr}(\mathsf{a}(x) : \underline{\mathsf{adx}}(\mathsf{a}(y))) & & \mathsf{incr}(x) & \to & \underline{\mathsf{incr}}(x) & & \mathsf{a}(\underline{\mathsf{incr}}(x)) & \to & \mathsf{incr}(x) \\
\mathsf{hd}(x : y) & \to & \mathsf{a}(x) & & \mathsf{adx}(x) & \to & \underline{\mathsf{adx}}(x) & & \mathsf{a}(\underline{\mathsf{adx}}(x)) & \to & \mathsf{adx}(x) \\
\mathsf{tl}(x : y) & \to & \mathsf{a}(y) & & \mathsf{a}(x) & \to & x & & & &
\end{array}$$

Zantema's transformation is sound but not complete as we have the infinite reduction

$$\begin{aligned} \mathsf{adx}(\mathsf{zeros}) &\to_{\mathsf{Z}} \mathsf{adx}(\underline{\mathsf{0}} : \underline{\mathsf{zeros}}) \to_{\mathsf{Z}} \mathsf{incr}(\mathsf{a}(\underline{\mathsf{0}}) : \underline{\mathsf{adx}}(\mathsf{a}(\underline{\mathsf{zeros}}))) \to_{\mathsf{Z}}^{+} \mathsf{incr}(\mathsf{0} : \underline{\mathsf{adx}}(\mathsf{zeros})) \\ &\to_{\mathsf{Z}} \underline{\mathsf{s}}(\mathsf{a}(\mathsf{0})) : \underline{\mathsf{incr}}(\mathsf{a}(\underline{\mathsf{adx}}(\mathsf{zeros}))) \to_{\mathsf{Z}}^{+} \underline{\mathsf{s}}(\mathsf{0}) : \underline{\mathsf{incr}}(\mathsf{adx}(\mathsf{zeros})) \to_{\mathsf{Z}} \cdots \end{aligned}$$

Zantema's method appears to be more powerful than Lucas' transformation since already the rule $\mathsf{tl}(x : y) \to y$ is transformed into a non-terminating rule by Θ_{L} whereas it remains terminating under the transformation Θ_{Z}. However, the two methods are incomparable.

Example 6

Consider the CSRS $(\mathcal{R},\mu)$ consisting of the rules $\mathsf{c} \to \mathsf{f}(\mathsf{g}(\mathsf{c}))$ and $\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(x)$ with $\mu(\mathsf{f}) = \mu(\mathsf{g}) = \varnothing$. Lucas' transformation yields the terminating TRS $\mathcal{R}_\mu^{\mathsf{L}} =$

$\{\mathsf{c}_\mu \to \mathsf{f}_\mu,\ \mathsf{f}_\mu \to \mathsf{g}_\mu\}$ *whereas* $\mathcal{R}_\mu^{\mathsf{Z}}$

$$\begin{array}{ccc}
\mathsf{c} \to \mathsf{f}(\underline{\mathsf{g}}(\underline{\mathsf{c}})) & \mathsf{f}(x) \to \underline{\mathsf{f}}(x) & \mathsf{a}(\underline{\mathsf{f}}(x)) \to \mathsf{f}(x) \\
\mathsf{f}(\underline{\mathsf{g}}(x)) \to \mathsf{g}(\mathsf{a}(x)) & \mathsf{g}(x) \to \underline{\mathsf{g}}(x) & \mathsf{a}(\underline{\mathsf{g}}(x)) \to \mathsf{g}(x) \\
 & \mathsf{c} \to \underline{\mathsf{c}} & \mathsf{a}(\underline{\mathsf{c}}) \to \mathsf{c}
\end{array}$$

does not terminate: $\mathsf{c} \to_{\mathsf{Z}} \mathsf{f}(\underline{\mathsf{g}}(\underline{\mathsf{c}})) \to_{\mathsf{Z}} \mathsf{g}(\mathsf{a}(\underline{\mathsf{c}})) \to_{\mathsf{Z}} \mathsf{g}(\mathsf{c}) \to_{\mathsf{Z}} \cdots$

Ferreira & Ribeiro (1999) refined Zantema's transformation further. The first part of their transformed system $\mathcal{R}_\mu^{\mathsf{FR}}$ results from the first part of $\mathcal{R}_\mu^{\mathsf{Z}}$ by underlining all function symbols (except a) which occur below an underlined symbol. So for example, if $2 \notin \mu(:)$ then a term $x : \underline{\mathsf{f}}(\mathsf{g}(y))$ in Zantema's transformation would now be replaced by $x : \underline{\mathsf{f}}(\underline{\mathsf{g}}(y))$. Thus, in Ferreira & Ribeiro's transformation *all* function symbols of terms occurring in inactive arguments are underlined (instead of just the root symbols of such terms as in $\mathcal{R}_\mu^{\mathsf{Z}}$).

Definition 7
Let $(\mathcal{R}, \mu)$ *be a CSRS over a signature* $\mathcal{F}$. *The TRS* $\mathcal{R}_\mu^{\mathsf{FR}}$ *over the signature* $\mathcal{F}_{\mathsf{Z}}$ *consists of two parts, i.e.,* $\mathcal{R}_\mu^{\mathsf{FR}} = \mathcal{R}_\mu^{\mathsf{FR}_1} \cup \mathcal{R}_\mu^{\mathsf{FR}_2}$. *The first part* $\mathcal{R}_\mu^{\mathsf{FR}_1}$ *consists of the rules* $\mathsf{FR}(l) \to \mathsf{FR}(r)\sigma_l$ *for all* $l \to r \in \mathcal{R}$. *The mappings* FR *and* FR' *from* $\mathcal{T}(\mathcal{F},\mathcal{V})$ *to* $\mathcal{T}(\mathcal{F}_{\mathsf{Z}},\mathcal{V})$ *are defined inductively by*

$$\begin{aligned}
\mathsf{FR}(x) = \mathsf{FR}'(x) &= x \\
\mathsf{FR}(f(t_1,\ldots,t_n)) &= f(u_1,\ldots,u_n) \\
\mathsf{FR}'(f(t_1,\ldots,t_n)) &= \underline{f}(\mathsf{FR}'(t_1),\ldots,\mathsf{FR}'(t_n))
\end{aligned}$$

with $u_i = \mathsf{FR}(t_i)$ *if* $i \in \mu(f)$ *and* $u_i = \mathsf{FR}'(t_i)$ *if* $i \notin \mu(f)$, *for all* $1 \leqslant i \leqslant n$. *The substitution* σ_l *is defined as in Zantema's transformation (Definition 5). The second part* $\mathcal{R}_\mu^{\mathsf{FR}_2}$ *consists of* $\mathsf{a}(x) \to x$ *together with*

$$\begin{aligned}
f(x_1,\ldots,x_n) &\to \underline{f}(x_1,\ldots,x_n) \\
\mathsf{a}(\underline{f}(x_1,\ldots,x_n)) &\to f\left([\![x_1]\!]_1^f,\ldots,[\![x_n]\!]_n^f\right)
\end{aligned}$$

for every n*-ary* f *for which* $\underline{f}$ *appears in* $\mathcal{R}_\mu^{\mathsf{FR}_1}$, *and*

$$\mathsf{a}(f(x_1,\ldots,x_n)) \to f\left([\![x_1]\!]_1^f,\ldots,[\![x_n]\!]_n^f\right)$$

for every n*-ary* f *for which* $\underline{f}$ *does not appear in* $\mathcal{R}_\mu^{\mathsf{FR}_1}$. *Here* $[\![t]\!]_i^f = \mathsf{a}(t)$ *if* $i \in \mu(f)$ *and* $[\![t]\!]_i^f = t$ *otherwise. We denote Ferreira & Ribeiro's transformation* $(\mathcal{R},\mu) \mapsto \mathcal{R}_\mu^{\mathsf{FR}}$ *by* Θ_{FR} *and we abbreviate* $\to_{\mathcal{R}_\mu^{\mathsf{FR}}}$ *to* $\to_{\mathsf{FR}}$ *and* $\to_{\mathcal{R}_\mu^{\mathsf{FR}_i}}$ *to* $\to_{\mathsf{FR}_i}$. *We add a prime (′) for the transformation which excludes the rules* $\mathsf{a}(f(x_1,\ldots,x_n)) \to f([\![x_1]\!]_1^f,\ldots,[\![x_n]\!]_n^f)$. *The sub-signature of* $\mathcal{F}_{\mathsf{Z}}$ *which consists of the function symbols of* $\mathcal{R}_\mu^{\mathsf{FR}}$ *is denoted by* $\mathcal{F}_\mu^{\mathsf{FR}}$.

Similar to Zantema's approach, here the context-sensitive reduction of a term t is translated into an $\mathcal{R}_\mu^{\mathsf{FR}}$-reduction of the term $\mathsf{FR}(t)$. Note that we always have $\mathcal{F}_\mu^{\mathsf{Z}} \subseteq \mathcal{F}_\mu^{\mathsf{FR}}$. In Theorem 22(b) we will show that the rules $\mathsf{a}(f(x_1,\ldots,x_n)) \to f([\![x_1]\!]_1^f,\ldots,[\![x_n]\!]_n^f)$ are superfluous. In other words, Θ_{FR}' is already a sound transformation.

The example CSRS $(\mathcal{R}, \mu)$ is transformed into

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\mathsf{nats} & \to & \mathsf{adx}(\mathsf{zeros}) & \mathsf{0} & \to & \underline{\mathsf{0}} & \mathsf{a}(\underline{\mathsf{0}}) & \to & \mathsf{0} \\
\mathsf{zeros} & \to & \underline{\mathsf{0}} : \underline{\mathsf{zeros}} & \mathsf{s}(x) & \to & \underline{\mathsf{s}}(x) & \mathsf{a}(\underline{\mathsf{s}}(x)) & \to & \mathsf{s}(x) \\
\mathsf{incr}(x : y) & \to & \underline{\mathsf{s}}(\mathsf{a}(x)) : \underline{\mathsf{incr}}(\mathsf{a}(y)) & \mathsf{zeros} & \to & \underline{\mathsf{zeros}} & \mathsf{a}(\underline{\mathsf{zeros}}) & \to & \mathsf{zeros} \\
\mathsf{adx}(x : y) & \to & \mathsf{incr}(\mathsf{a}(x) : \underline{\mathsf{adx}}(\mathsf{a}(y))) & \mathsf{incr}(x) & \to & \underline{\mathsf{incr}}(x) & \mathsf{a}(\underline{\mathsf{incr}}(x)) & \to & \mathsf{incr}(\mathsf{a}(x)) \\
\mathsf{hd}(x : y) & \to & \mathsf{a}(x) & \mathsf{adx}(x) & \to & \underline{\mathsf{adx}}(x) & \mathsf{a}(\underline{\mathsf{adx}}(x)) & \to & \mathsf{adx}(\mathsf{a}(x)) \\
\mathsf{tl}(x : y) & \to & \mathsf{a}(y) & \mathsf{a}(x) & \to & x & \mathsf{a}(\mathsf{nats}) & \to & \mathsf{nats} \\
 & & & & & & \mathsf{a}(x : y) & \to & x : y \\
 & & & & & & \mathsf{a}(\mathsf{hd}(x)) & \to & \mathsf{hd}(\mathsf{a}(x)) \\
 & & & & & & \mathsf{a}(\mathsf{tl}(x)) & \to & \mathsf{tl}(\mathsf{a}(x))
\end{array}$$

Again, this transformation technique is sound but not complete, because the infinite reduction with $\mathcal{R}^{\mathsf{Z}}_{\mu}$ sketched above is also possible with both $\mathcal{R}^{\mathsf{FR}}_{\mu}$ and $\mathcal{R}^{\mathsf{FR}'}_{\mu}$ (where the reduction from $\underline{\mathsf{s}}(\mathsf{0}) : \underline{\mathsf{incr}}(\mathsf{a}(\underline{\mathsf{adx}}(\mathsf{zeros})))$ to $\underline{\mathsf{s}}(\mathsf{0}) : \underline{\mathsf{incr}}(\mathsf{adx}(\mathsf{zeros}))$ now takes two steps instead of one). Moreover, Ferreira & Ribeiro's method is still incomparable with Lucas' transformation. This can be shown with the same example used above to demonstrate the incomparability of the transformations of Zantema and Lucas (Example 6).

Finally, let us compare Ferreira and Ribeiro's technique with the one of Zantema. As illustrated in (Ferreira & Ribeiro, 1999), there are examples where their technique succeeds, whereas Zantema's fails. For the one-rule TRS $\mathcal{R}$

$$\mathsf{f}(x) \to \mathsf{g}(\mathsf{h}(\mathsf{f}(x)))$$

from Zantema (1997) with $\mu(\mathsf{g}) = \varnothing$ and $\mu(\mathsf{h}) = \mu(\mathsf{f}) = \{1\}$, $\mathcal{R}^{\mathsf{Z}}_{\mu}$ is not terminating since it contains the rule $\mathsf{f}(x) \to \mathsf{g}(\underline{\mathsf{h}}(\mathsf{f}(x)))$. On the other hand, $\mathcal{R}^{\mathsf{FR}}_{\mu}$ is terminating since here one has the rule $\mathsf{f}(x) \to \mathsf{g}(\underline{\mathsf{h}}(\underline{\mathsf{f}}(x)))$ instead. (For example, the recursive path order (Dershowitz, 1982) with precedence $\mathsf{a} \succ \mathsf{f} \succ \underline{\mathsf{f}} \succ \mathsf{g} \succ \mathsf{h} \succ \underline{\mathsf{h}}$ applies.)

Ferreira & Ribeiro (1999) conjectured that their method is more powerful than the one of Zantema. Below we prove this (non-trivial) conjecture. So Ferreira and Ribeiro's transformation proves termination of more CSRSs than Zantema's.

In order to relate the two transformations, we have to show that every reduction between two ground terms s and t in $\mathcal{R}^{\mathsf{FR}}_{\mu}$ corresponds to a similar reduction between related ground terms $\Phi(s)$ and $\Phi(t)$ in $\mathcal{R}^{\mathsf{Z}}_{\mu}$. Here, Φ is a mapping which removes all occurrences of a and all additional underlining that is done in Ferreira & Ribeiro's transformation, but not in Zantema's. In particular, Φ has to remove the underlining from every function symbol $\underline{f}$ that appears in an active argument position of the function symbol directly above it. So in the example above, we would have $\Phi(\mathsf{g}(\underline{\mathsf{h}}(\underline{\mathsf{f}}(x)))) = \mathsf{g}(\underline{\mathsf{h}}(\mathsf{f}(x)))$. Hence, when defining $\Phi(f(t_1, \ldots, t_n))$ or $\Phi(\underline{f}(t_1, \ldots, t_n))$, if i is an active argument of f, then any potential underlining of t_i's root symbol should be removed. Here, the argument position of a is also considered active (e.g., $\Phi(\mathsf{g}(\mathsf{a}(\underline{\mathsf{h}}(x)))) = \mathsf{g}(\mathsf{h}(x))$). Moreover, the underlining is also removed if $\underline{f}$ does not belong to the signature $\mathcal{F}^{\mathsf{Z}}_{\mu}$. So in the above example, all occurrences of $\underline{\mathsf{f}}$ would be replaced by f. For the formal definition of Φ, we define an auxiliary mapping

Φ' which is like Φ except that the underlining from an underlined root symbol is always removed.

Definition 8
Let $(\mathcal{R},\mu)$ be a CSRS over a signature $\mathcal{F}$. We define two mappings Φ and Φ' from $\mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$ to $\mathcal{T}(\mathcal{F}_\mu^{\mathsf{Z}})$ inductively as follows:

$$\Phi(f(t_1,\ldots,t_n)) = \Phi'(f(t_1,\ldots,t_n)) = \Phi'(\underline{f}(t_1,\ldots,t_n)) = f\big(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f\big)$$

$$\Phi(\underline{f}(t_1,\ldots,t_n)) = \begin{cases} \underline{f}\big(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f\big) & \text{if } \underline{f} \in \mathcal{F}_\mu^{\mathsf{Z}} \\ f\big(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f\big) & \text{if } \underline{f} \notin \mathcal{F}_\mu^{\mathsf{Z}} \end{cases}$$

$$\Phi(\mathsf{a}(t)) = \Phi'(\mathsf{a}(t)) = \Phi'(t)$$

with $\langle t\rangle_i^f = \Phi'(t)$ if $i \in \mu(f)$ and $\langle t\rangle_i^f = \Phi(t)$ if $i \notin \mu(f)$, for all $1 \leqslant i \leqslant n$.

The next two lemmata show that every reduction step $s \to_{\mathsf{FR}} t$ corresponds to a reduction from $\Phi(s)$ to $\Phi(t)$ in $\mathcal{R}_\mu^{\mathsf{Z}}$. More precisely, we have the following correspondence.

Lemma 9
For all terms $s, t \in \mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$, if $s \to_{\mathsf{FR}_1} t$ then $\Phi(s) \to_{\mathsf{Z}}^{+} \Phi(t)$.

Lemma 10
For all terms $s, t \in \mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$, if $s \to_{\mathsf{FR}_2} t$ then $\Phi(s) \to_{\mathsf{Z}}^{*} \Phi(t)$.

We refer to Appendix A for the proofs of these two lemmata. With these lemmata we obtain the desired result on the transformations of Zantema and Ferreira & Ribeiro.

Theorem 11
Let $(\mathcal{R},\mu)$ be a CSRS. If $\mathcal{R}_\mu^{\mathsf{Z}}$ is terminating then $\mathcal{R}_\mu^{\mathsf{FR}}$ is terminating.

Proof
Suppose that $\mathcal{R}_\mu^{\mathsf{FR}}$ admits an infinite reduction. Then there also exists an infinite reduction of ground terms:

$$t_1 \to_{\mathsf{FR}} t_2 \to_{\mathsf{FR}} t_3 \to_{\mathsf{FR}} t_4 \to_{\mathsf{FR}} \cdots$$

Since $\mathcal{R}_\mu^{\mathsf{FR}_2}$ is terminating, the reduction must contain infinitely many $\mathcal{R}_\mu^{\mathsf{FR}_1}$-steps. Hence, by applying Lemmata 9 and 10, we obtain an infinite $\mathcal{R}_\mu^{\mathsf{Z}}$-reduction starting from $\Phi(t_1)$. □

To summarize, we have reviewed three transformation techniques from the literature which transform CSRSs into ordinary TRSs and we have investigated the relationship between these three transformations. All three methods are sound, i.e., if the transformed TRS terminates then the original CSRS is also terminating. But none of these three methods is complete, e.g., they all transform the nats example from the introduction into a non-terminating TRS, although the original CSRS is terminating. This already indicates that there are many natural and interesting systems where these techniques are not applicable.

4 A sound transformation

In this section, we present our first transformation from CSRSs to TRSs. The advantage of this transformation is that it is easy and more powerful than the transformations of Lucas, Zantema, and Ferreira & Ribeiro. In the transformation we extend the original signature $\mathscr{F}$ of the TRS by a unary function symbol mark and a function symbol f_{active} of arity n for every n-ary defined function symbol $f \in \mathscr{F}_{\mathscr{D}}$. Essentially, the idea for the transformation is to mark the active positions in a term on the object level, because those positions are the only ones where context-sensitive rewriting may take place. For this purpose we use the function symbols f_{active}. Thus, instead of a rule $f(l_1, \dots, l_n) \to r$ the transformed TRS should contain a rule whose left-hand side is $f_{\mathsf{active}}(l_1, \dots, l_n)$. Now an instance of a left-hand side $f(\cdots)$ can only be rewritten if it exposes the fact that it is at an active position (it does that by being of the form $f_{\mathsf{active}}(\cdots)$). Moreover, after rewriting an instance of l to the corresponding instance of r, we have to mark the new active positions in the resulting term. For that purpose we replace every occurrence of a defined function symbol f at an active position in r by f_{active} and every occurrence of a variable x at an active position by $\mathsf{mark}(x)$. The symbol mark is used to ensure that in instantiations of r, defined function symbols at active positions in the substitution part are marked as well. This is achieved by the rules

$$\begin{array}{ll} \mathsf{mark}(f(x_1, \dots, x_n)) \to f_{\mathsf{active}}\left([x_1]^f_1, \dots, [x_n]^f_n\right) & \text{if } f \in \mathscr{F}_{\mathscr{D}} \\ \mathsf{mark}(f(x_1, \dots, x_n)) \to f\left([x_1]^f_1, \dots, [x_n]^f_n\right) & \text{if } f \in \mathscr{F}_{\mathscr{C}} \end{array}$$

where the form of the argument $[x_i]^f_i$ depends upon whether i is an active argument of f: if $i \in \mu(f)$ then x_i must also be marked active and thus $[x_i]^f_i = \mathsf{mark}(x_i)$, otherwise the ith argument of f is not active and we define $[x_i]^f_i = x_i$. Let $\mathscr{M}$ denote the set of all these mark-rules. Since $\mathscr{M}$ is confluent and terminating, every term t has a unique normal form $t{\downarrow}_{\mathscr{M}}$ with respect to $\mathscr{M}$. It is easy to see that transforming the right-hand side r as described above yields the term $\mathsf{mark}(r){\downarrow}_{\mathscr{M}}$. Finally, we also need rules to deactivate terms. For example, consider the TRS consisting of the following rewrite rules:

$$\mathsf{b} \to \mathsf{f}(\mathsf{c}) \qquad\qquad \mathsf{f}(\mathsf{c}) \to \mathsf{b} \qquad\qquad \mathsf{c} \to \mathsf{d}$$

No matter how the replacement map μ is defined, the resulting CSRS is not terminating. Suppose $\mu(\mathsf{f}) = \{1\}$. In the transformed system we would have the rules

$$\begin{array}{rlrl} \mathsf{b}_{\mathsf{active}} & \to \mathsf{f}_{\mathsf{active}}(\mathsf{c}_{\mathsf{active}}) & \mathsf{mark}(\mathsf{b}) & \to \mathsf{b}_{\mathsf{active}} \\ \mathsf{f}_{\mathsf{active}}(\mathsf{c}) & \to \mathsf{b}_{\mathsf{active}} & \mathsf{mark}(\mathsf{c}) & \to \mathsf{c}_{\mathsf{active}} \\ \mathsf{c}_{\mathsf{active}} & \to \mathsf{d} & \mathsf{mark}(\mathsf{d}) & \to \mathsf{d} \\ & & \mathsf{mark}(\mathsf{f}(x)) & \to \mathsf{f}_{\mathsf{active}}(\mathsf{mark}(x)) \end{array}$$

This TRS is terminating because $\mathsf{b}_{\mathsf{active}}$ rewrites to $\mathsf{f}_{\mathsf{active}}(\mathsf{c}_{\mathsf{active}})$, but if we cannot deactivate the subterm $\mathsf{c}_{\mathsf{active}}$ then the second rule is not applicable. Thus, we have to add the rule $\mathsf{c}_{\mathsf{active}} \to \mathsf{c}$. To summarize, we obtain the following transformation.

Definition 12

Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. The TRS $\mathscr{R}_\mu^1$ over the signature $\mathscr{F}_1 = \mathscr{F} \cup \{f_{\mathsf{active}} \mid f \in \mathscr{F}_\mathscr{D}\} \cup \{\mathsf{mark}\}$ consists of the following rewrite rules:

$$\begin{array}{rll} f_{\mathsf{active}}(l_1,\ldots,l_n) \to \mathsf{mark}(r){\downarrow_\mathscr{M}} & & \text{for all } f(l_1,\ldots,l_n) \to r \in \mathscr{R} \\ \mathsf{mark}(f(x_1,\ldots,x_n)) \to f_{\mathsf{active}}\big([x_1]_1^f,\ldots,[x_n]_n^f\big) & & \text{for all } f \in \mathscr{F}_\mathscr{D} \\ \mathsf{mark}(f(x_1,\ldots,x_n)) \to f\big([x_1]_1^f,\ldots,[x_n]_n^f\big) & & \text{for all } f \in \mathscr{F}_\mathscr{C} \\ f_{\mathsf{active}}(x_1,\ldots,x_n) \to f(x_1,\ldots,x_n) & & \text{for all } f \in \mathscr{F}_\mathscr{D} \end{array}$$

Here $\mathscr{M}$ is the (confluent and terminating) subset of $\mathscr{R}_\mu^1$ consisting of all mark*-rules and $[t]_i^f = \mathsf{mark}(t)$ if $i \in \mu(f)$ and $[t]_i^f = t$ otherwise. We denote the transformation $(\mathscr{R},\mu) \mapsto \mathscr{R}_\mu^1$ by Θ_1 and we abbreviate $\to_{\mathscr{R}_\mu^1}$ to $\to_1$.*

Soundness of our transformation is an easy consequence of the following lemma which shows how context-sensitive reduction steps are simulated in the transformed system. The context-sensitive reduction of a term t is now translated into a reduction of the term $\mathsf{mark}(t){\downarrow_\mathscr{M}}$ in the TRS $\mathscr{R}_\mu^1$.

Lemma 13

Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$ and let $s,t \in \mathscr{T}(\mathscr{F})$. If $s \to_\mu t$ then $\mathsf{mark}(s){\downarrow_\mathscr{M}} \to_1^+ \mathsf{mark}(t){\downarrow_\mathscr{M}}$.

Proof

There is a rewrite rule $l \to r \in \mathscr{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. We prove the lemma by induction on π. If $\pi = \epsilon$ then $s = l\sigma$ and $t = r\sigma$. An easy induction on the structure of $s = f(s_1,\ldots,s_n)$ reveals that $\mathsf{mark}(s){\downarrow_\mathscr{M}} \to_1^* f_{\mathsf{active}}(s_1,\ldots,s_n)$ (one just has to deactivate all inner occurrences of activated function symbols). Since $f_{\mathsf{active}}(s_1,\ldots,s_n) \to \mathsf{mark}(r){\downarrow_\mathscr{M}}\sigma$ is an instance of a rule in $\mathscr{R}_\mu^1$ we obtain $\mathsf{mark}(s){\downarrow_\mathscr{M}} \to_1^* f_{\mathsf{active}}(s_1,\ldots,s_n) \to_1 \mathsf{mark}(r){\downarrow_\mathscr{M}}\sigma \to_1^* \mathsf{mark}(r\sigma){\downarrow_\mathscr{M}} = \mathsf{mark}(t){\downarrow_\mathscr{M}}$. If $\pi = i\pi'$ then we have $s = f(s_1,\ldots, s_i,\ldots,s_n)$ and $t = f(s_1,\ldots,t_i,\ldots,s_n)$ with $s_i \to_\mu t_i$. Note that $i \in \mu(f)$ due to the definition of context-sensitive rewriting. For $1 \leqslant j \leqslant n$ we define $s_j' = \mathsf{mark}(s_j){\downarrow_\mathscr{M}}$ if $j \in \mu(f)$ and $s_j' = s_j$ if $j \notin \mu(f)$. The induction hypothesis yields $s_i' = \mathsf{mark}(s_i){\downarrow_\mathscr{M}} \to_1^+ \mathsf{mark}(t_i){\downarrow_\mathscr{M}}$. Note that $\mathsf{mark}(s){\downarrow_\mathscr{M}}$ is $f_{\mathsf{active}}(s_1',\ldots,s_i',\ldots,s_n')$ if $f \in \mathscr{F}_\mathscr{D}$ and $f(s_1',\ldots,s_i',\ldots,s_n')$ if $f \in \mathscr{F}_\mathscr{C}$. Similarly, $\mathsf{mark}(t){\downarrow_\mathscr{M}}$ is $f_{\mathsf{active}}(s_1',\ldots, \mathsf{mark}(t_i){\downarrow_\mathscr{M}},\ldots,s_n')$ if $f \in \mathscr{F}_\mathscr{D}$ and $f(s_1',\ldots,\mathsf{mark}(t_i){\downarrow_\mathscr{M}},\ldots,s_n')$ if $f \in \mathscr{F}_\mathscr{C}$. Hence, the result follows. □

Theorem 14

Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. If $\mathscr{R}_\mu^1$ is terminating then $(\mathscr{R},\mu)$ is terminating.

Proof

If $(\mathscr{R},\mu)$ is not terminating then there exists an infinite reduction of ground terms. Any such sequence is transformed by the previous lemma into an infinite reduction in $\mathscr{R}_\mu^1$. □

The converse of the above theorem does not hold, i.e., the transformation is incomplete.

Example 15
As an example of a terminating CSRS that is transformed into a non-terminating TRS by our transformation, consider the following variant $\mathcal{R}$ *of a well-known example from Toyama (1987):*

$$\mathsf{f}(\mathsf{b},\mathsf{c},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{d} \to \mathsf{b} \qquad \mathsf{d} \to \mathsf{c}$$

If we define $\mu(\mathsf{f}) = \{3\}$ *then the resulting CSRS is terminating because the usual cyclic reduction from* $\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d})$ *to* $\mathsf{f}(\mathsf{d},\mathsf{d},\mathsf{d})$ *and further to* $\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d})$ *can no longer be done, as one would have to reduce the first and second argument of* f*. However, the transformed TRS* $\mathcal{R}_\mu^1$

$$\begin{array}{rcc}
\mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{c},x) \to \mathsf{f}_{\mathsf{active}}(x,x,\mathsf{mark}(x)) & \mathsf{d}_{\mathsf{active}} \to \mathsf{b} & \mathsf{d}_{\mathsf{active}} \to \mathsf{c} \\
\mathsf{mark}(\mathsf{f}(x,y,z)) \to \mathsf{f}_{\mathsf{active}}(x,y,\mathsf{mark}(z)) & \mathsf{mark}(\mathsf{b}) \to \mathsf{b} & \mathsf{f}_{\mathsf{active}}(x,y,z) \to \mathsf{f}(x,y,z) \\
\mathsf{mark}(\mathsf{d}) \to \mathsf{d}_{\mathsf{active}} & \mathsf{mark}(\mathsf{c}) \to \mathsf{c} & \mathsf{d}_{\mathsf{active}} \to \mathsf{d}
\end{array}$$

is not terminating:

$$\begin{aligned}
\mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{c},\mathsf{d}_{\mathsf{active}}) &\to_1 \mathsf{f}_{\mathsf{active}}(\mathsf{d}_{\mathsf{active}},\mathsf{d}_{\mathsf{active}},\mathsf{mark}(\mathsf{d}_{\mathsf{active}})) \\
&\to_1^+ \mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{c},\mathsf{mark}(\mathsf{d})) \to_1 \mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{c},\mathsf{d}_{\mathsf{active}})
\end{aligned}$$

Note that $\mathcal{R}_\mu^{\mathsf{L}}$

$$\mathsf{f}_\mu(x) \to \mathsf{f}_\mu(x) \qquad \mathsf{d}_\mu \to \mathsf{b}_\mu \qquad \mathsf{d}_\mu \to \mathsf{c}_\mu$$

and $\mathcal{R}_\mu^{\mathsf{Z}}$

$$\begin{array}{cccc}
\mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},x) \to \mathsf{f}(x,x,x) & \mathsf{d} \to \mathsf{c} & \mathsf{a}(\underline{\mathsf{b}}) \to \mathsf{b} & \mathsf{b} \to \underline{\mathsf{b}} \\
\mathsf{d} \to \mathsf{b} & \mathsf{a}(\underline{\mathsf{c}}) \to \mathsf{c} & \mathsf{a}(x) \to x & \mathsf{c} \to \underline{\mathsf{c}}
\end{array}$$

also fail to terminate. For example, $\mathcal{R}_\mu^{\mathsf{Z}}$ *admits the cycle*

$$\mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},\mathsf{d}) \to_{\mathsf{Z}} \mathsf{f}(\mathsf{d},\mathsf{d},\mathsf{d}) \to_{\mathsf{Z}}^+ \mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d}) \to_{\mathsf{Z}}^+ \mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},\mathsf{d})$$

Because $\mathcal{R}_\mu^{\mathsf{FR}} = \mathcal{R}_\mu^{\mathsf{Z}} \cup \{\mathsf{a}(\mathsf{f}(x,y,z)) \to \mathsf{f}(x,y,\mathsf{a}(z)),\ \mathsf{a}(\mathsf{d}) \to \mathsf{d}\}$*,* $\mathcal{R}_\mu^{\mathsf{FR}}$ *admits the same cycle.*

Nevertheless, compared to the transformations of Lucas, Zantema, and Ferreira & Ribeiro, our easy transformation is very powerful. There are numerous CSRSs where our transformation succeeds and which cannot be handled by the other three transformations.

Example 16
As a simple example, consider the terminating CSRS $\mathcal{R}$

$$\mathsf{g}(x) \to \mathsf{h}(x) \qquad \mathsf{c} \to \mathsf{d} \qquad \mathsf{h}(\mathsf{d}) \to \mathsf{g}(\mathsf{c})$$

with $\mu(\mathsf{g}) = \mu(\mathsf{h}) = \varnothing$ *from Zantema (1997). The TRS* $\mathcal{R}_\mu^{\mathsf{L}}$

$$\mathsf{g}_\mu \to \mathsf{h}_\mu \qquad \mathsf{c}_\mu \to \mathsf{d}_\mu \qquad \mathsf{h}_\mu \to \mathsf{g}_\mu$$

is non-terminating as it admits the cycle $\mathsf{g}_\mu \to_{\mathsf{L}} \mathsf{h}_\mu \to_{\mathsf{L}} \mathsf{g}_\mu$*. The TRS* $\mathcal{R}_\mu^{\mathsf{Z}}$

$$\begin{array}{cccc}
\mathsf{g}(x) \to \mathsf{h}(\mathsf{a}(x)) & \mathsf{h}(\underline{\mathsf{d}}) \to \mathsf{g}(\underline{\mathsf{c}}) & \mathsf{a}(\underline{\mathsf{c}}) \to \mathsf{c} & \mathsf{c} \to \underline{\mathsf{c}} \\
\mathsf{c} \to \mathsf{d} & \mathsf{a}(x) \to x & \mathsf{a}(\underline{\mathsf{d}}) \to \mathsf{d} & \mathsf{d} \to \underline{\mathsf{d}}
\end{array}$$

is non-terminating as it admits the cycle

$$\mathsf{g}(\underline{\mathsf{c}}) \to_{\mathsf{Z}} \mathsf{h}(\mathsf{a}(\underline{\mathsf{c}})) \to_{\mathsf{Z}} \mathsf{h}(\mathsf{c}) \to_{\mathsf{Z}} \mathsf{h}(\mathsf{d}) \to_{\mathsf{Z}} \mathsf{h}(\underline{\mathsf{d}}) \to_{\mathsf{Z}} \mathsf{g}(\underline{\mathsf{c}})$$

Because $\mathscr{R}_\mu^{\mathsf{Z}} \subseteq \mathscr{R}_\mu^{\mathsf{FR}}$, $\mathscr{R}_\mu^{\mathsf{FR}}$ *is also non-terminating. In contrast, our transformation generates the TRS* $\mathscr{R}_\mu^1$

$$\begin{array}{rlrlrlrl}
\mathsf{g}_{\mathsf{active}}(x) \to \mathsf{h}_{\mathsf{active}}(x) & & \mathsf{c}_{\mathsf{active}} \to \mathsf{d} & & \mathsf{h}_{\mathsf{active}}(\mathsf{d}) \to \mathsf{g}_{\mathsf{active}}(\mathsf{c}) & & & \\
\mathsf{mark}(\mathsf{g}(x)) \to \mathsf{g}_{\mathsf{active}}(x) & & \mathsf{mark}(\mathsf{c}) \to \mathsf{c}_{\mathsf{active}} & & \mathsf{g}_{\mathsf{active}}(x) \to \mathsf{g}(x) & & \mathsf{c}_{\mathsf{active}} \to \mathsf{c} & \\
\mathsf{mark}(\mathsf{h}(x)) \to \mathsf{h}_{\mathsf{active}}(x) & & \mathsf{mark}(\mathsf{d}) \to \mathsf{d} & & \mathsf{h}_{\mathsf{active}}(x) \to \mathsf{h}(x) & & &
\end{array}$$

which is compatible with the recursive path order for the precedence

$$\mathsf{mark} \succ \mathsf{c}_{\mathsf{active}} \succ \mathsf{d} \succ \mathsf{g}_{\mathsf{active}} \succ \mathsf{h}_{\mathsf{active}} \succ \mathsf{g} \succ \mathsf{h} \succ \mathsf{c}$$

and hence terminating.

Moreover, while the techniques of Lucas, Zantema, and Ferreira & Ribeiro fail for the nats example from the introduction, our transformation generates a TRS that is easily proved to be terminating.

Example 17
With our transformation one obtains the following TRS $\mathscr{R}_\mu^1$

$$\begin{array}{rlcrl}
\mathsf{nats}_{\mathsf{active}} & \to \mathsf{adx}_{\mathsf{active}}(\mathsf{zeros}_{\mathsf{active}}) & & \mathsf{hd}_{\mathsf{active}}(x) & \to \mathsf{hd}(x) \\
\mathsf{zeros}_{\mathsf{active}} & \to 0 : \mathsf{zeros} & & \mathsf{tl}_{\mathsf{active}}(x) & \to \mathsf{tl}(x) \\
\mathsf{incr}_{\mathsf{active}}(x : y) & \to \mathsf{s}(x) : \mathsf{incr}(y) & & \mathsf{mark}(\mathsf{nats}) & \to \mathsf{nats}_{\mathsf{active}} \\
\mathsf{adx}_{\mathsf{active}}(x : y) & \to \mathsf{incr}_{\mathsf{active}}(x : \mathsf{adx}(y)) & & \mathsf{mark}(\mathsf{zeros}) & \to \mathsf{zeros}_{\mathsf{active}} \\
\mathsf{hd}_{\mathsf{active}}(x : y) & \to \mathsf{mark}(x) & & \mathsf{mark}(\mathsf{incr}(x)) & \to \mathsf{incr}_{\mathsf{active}}(\mathsf{mark}(x)) \\
\mathsf{tl}_{\mathsf{active}}(x : y) & \to \mathsf{mark}(y) & & \mathsf{mark}(\mathsf{adx}(x)) & \to \mathsf{adx}_{\mathsf{active}}(\mathsf{mark}(x)) \\
\mathsf{nats}_{\mathsf{active}} & \to \mathsf{nats} & & \mathsf{mark}(\mathsf{hd}(x)) & \to \mathsf{hd}_{\mathsf{active}}(\mathsf{mark}(x)) \\
\mathsf{zeros}_{\mathsf{active}} & \to \mathsf{zeros} & & \mathsf{mark}(\mathsf{tl}(x)) & \to \mathsf{tl}_{\mathsf{active}}(\mathsf{mark}(x)) \\
\mathsf{incr}_{\mathsf{active}}(x) & \to \mathsf{incr}(x) & & \mathsf{mark}(0) & \to 0 \\
\mathsf{adx}_{\mathsf{active}}(x) & \to \mathsf{adx}(x) & & \mathsf{mark}(\mathsf{s}(x)) & \to \mathsf{s}(x) \\
 & & & \mathsf{mark}(x : y) & \to x : y
\end{array}$$

Termination of $\mathscr{R}_\mu^1$ *can be proved by the following polynomial interpretation:*

$$\begin{array}{rlcrl}
[\mathsf{nats}] & = 0 & & [\mathsf{hd}](x) & = 5x + 8 \\
[\mathsf{nats}_{\mathsf{active}}] & = 6 & & [\mathsf{hd}_{\mathsf{active}}](x) & = 5x + 9 \\
[\mathsf{zeros}] & = 0 & & [\mathsf{tl}](x) & = 5x + 8 \\
[\mathsf{zeros}_{\mathsf{active}}] & = 1 & & [\mathsf{tl}_{\mathsf{active}}](x) & = 5x + 9 \\
[\mathsf{incr}](x) & = x + 1 & & [0] & = 0 \\
[\mathsf{incr}_{\mathsf{active}}](x) & = x + 2 & & [\mathsf{s}](x) & = x \\
[\mathsf{adx}](x) & = x + 1 & & [x : y] & = x + y \\
[\mathsf{adx}_{\mathsf{active}}](x) & = x + 4 & & [\mathsf{mark}](x) & = 5x + 7
\end{array}$$

Systems for the automated generation of polynomial orders can for instance be found in (Ben Cherifa & Lescanne, 1987; Steinbach, 1994; Giesl, 1995; Contejean et al.*, 2000). See Hong and Jakuš (1998) for a comparison of some of the underlying methods. The above interpretation is computed by CiME (Contejean* et al.*, 2000).*

In fact, there does not exist any example where the methods of Lucas, Zantema, or Ferreira and Ribeiro work but our method fails. In other words, our transformation is more powerful than all other three approaches. One should remark that this also provides an alternative proof of the soundness of these three approaches. We first prove this for the transformation of Lucas.

Theorem 18
Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$. If $\mathscr{R}^{\mathsf{L}}_\mu$ is terminating then $\mathscr{R}^1_\mu$ is terminating.

Proof
We prove termination of $\mathscr{R}^1_\mu$ using the dependency pair approach (Arts & Giesl, 2000; Giesl *et al.*, 2002). The dependency pairs of $\mathscr{R}^1_\mu$ are

$$F_{\mathsf{active}}(l_1, \ldots, l_n) \to G_{\mathsf{active}}(t_1, \ldots, t_n) \qquad (1)$$

$$F_{\mathsf{active}}(l_1, \ldots, l_n) \to \mathsf{MARK}(x) \qquad (2)$$

for all rewrite rules $f(l_1, \ldots, l_n) \to r \in \mathscr{R}$, active subterms $g(t_1, \ldots, t_n)$ of r with a defined root symbol, and active variables x in r,

$$\mathsf{MARK}(f(x_1, \ldots, x_n)) \to F_{\mathsf{active}}\left([x_1]^f_1, \ldots, [x_n]^f_n\right) \qquad (3)$$

for all $f \in \mathscr{F}_\mathscr{D}$, and

$$\mathsf{MARK}(f(x_1, \ldots, x_n)) \to \mathsf{MARK}(x_i) \qquad (4)$$

for all $f \in \mathscr{F}$ and $i \in \mu(f)$. Every cycle of the dependency graph must contain a dependency pair of type (1), (2), or (4). Thus, it is sufficient if dependency pairs of type (1), (2), and (4) are strictly decreasing, whereas for dependency pairs of type (3) it is enough if they are weakly decreasing. Moreover, all rules of $\mathscr{R}^1_\mu$ should be weakly decreasing. Thus, we have to find a reduction pair $(\gtrsim, >)$ such that

$$\begin{aligned} f_{\mathsf{active}}(l_1, \ldots, l_n) &\gtrsim \mathsf{mark}(r){\downarrow_\mathscr{M}} \\ F_{\mathsf{active}}(l_1, \ldots, l_n) &> G_{\mathsf{active}}(t_1, \ldots, t_n) \\ F_{\mathsf{active}}(l_1, \ldots, l_n) &> \mathsf{MARK}(x) \end{aligned}$$

for all rewrite rules $f(l_1, \ldots, l_n) \to r \in \mathscr{R}$, active subterms $g(t_1, \ldots, t_n)$ of r with a defined root symbol, and active variables x in r, and

$$\begin{aligned} \mathsf{mark}(f(x_1, \ldots, x_n)) &\gtrsim f_{\mathsf{active}}\left([x_1]^f_1, \ldots, [x_n]^f_n\right) && \text{for all } f \in \mathscr{F}_\mathscr{D} \\ \mathsf{mark}(f(x_1, \ldots, x_n)) &\gtrsim f\left([x_1]^f_1, \ldots, [x_n]^f_n\right) && \text{for all } f \in \mathscr{F}_\mathscr{C} \\ f_{\mathsf{active}}(x_1, \ldots, x_n) &\gtrsim f(x_1, \ldots, x_n) && \text{for all } f \in \mathscr{F}_\mathscr{D} \\ \mathsf{MARK}(f(x_1, \ldots, x_n)) &\gtrsim F_{\mathsf{active}}\left([x_1]^f_1, \ldots, [x_n]^f_n\right) && \text{for all } f \in \mathscr{F}_\mathscr{D} \\ \mathsf{MARK}(f(x_1, \ldots, x_n)) &> \mathsf{MARK}(x_i) && \text{for all } f \in \mathscr{F},\, i \in \mu(f) \end{aligned}$$

A suitable reduction pair $(\gtrsim, >)$ can be obtained from the reduction relation $\rightarrow_{\mathsf{L}}$ provided the terms in the above inequalities are first transformed into terms over the signature $\mathscr{F}_{\mathsf{L}}$. To this end, we replace all mark- and MARK-terms by their arguments and we replace all activated function symbols f_{active} and the tuple symbols F_{active} by the original symbols f. Then we proceed as in the transformation of Lucas by eliminating all inactive arguments using the TRS $\mathscr{L}$ (Definition 4). Thus, let $\mathscr{L}'$ be the following terminating and confluent TRS:

$$\begin{aligned}\mathscr{L}' = \mathscr{L} &\cup \{\mathsf{mark}(x) \rightarrow x,\ \mathsf{MARK}(x) \rightarrow x\}\\ &\cup \{f_{\mathsf{active}}(x_1,\ldots,x_n) \rightarrow f(x_1,\ldots,x_n) \mid f \in \mathscr{F}_{\mathscr{D}}\}\\ &\cup \{F_{\mathsf{active}}(x_1,\ldots,x_n) \rightarrow f(x_1,\ldots,x_n) \mid f \in \mathscr{F}_{\mathscr{D}}\}\end{aligned}$$

Now we define $>$ by $s > t$ if and only if $s{\downarrow_{\mathscr{L}'}}\ (\rightarrow_{\mathsf{L}} \cup \rhd)^{+}\ t{\downarrow_{\mathscr{L}'}}$. Here $\rhd$ denotes the proper subterm relation. Moreover, let $\gtrsim$ be the relation where $s \gtrsim t$ if and only if $s{\downarrow_{\mathscr{L}'}} \rightarrow_{\mathsf{L}}^{*} t{\downarrow_{\mathscr{L}'}}$. One easily verifies that $(\gtrsim, >)$ is a reduction pair ($>$ is well founded by the termination of $\mathscr{R}_{\mu}^{\mathsf{L}}$), which satisfies the constraints above. Hence, due to the soundness of the dependency pair approach, the termination of $\mathscr{R}_{\mu}^{1}$ is established. □

Now we show that our transformation is also more powerful than those of Zantema and of Ferreira & Ribeiro. In fact, this already holds if one eliminates the rules

$$\mathsf{a}(f(x_1,\ldots,x_n)) \rightarrow f\left([\![x_1]\!]_1^f,\ldots,[\![x_n]\!]_n^f\right)$$

from $\mathscr{R}_{\mu}^{\mathsf{FR}}$. In other words, these rules are superfluous for a sound transformation technique (this is shown in Theorem 22(b) below). Theorem 22(a) states that the resulting transformation Θ'_{FR} is less powerful than our transformation. Theorem 22(c) states that the same is true for Ferreira and Ribeiro's original transformation Θ_{FR} and Theorem 22(d) states that this holds for Zantema's transformation, too. The proof of Theorem 22(a) has the same structure as the one of Theorem 11.

So in order to relate the two transformations, we have to show that every reduction between two ground terms s and t in $\mathscr{R}_{\mu}^{1}$ corresponds to a similar reduction between related ground terms $\Psi(s)$ and $\Psi(t)$ in $\mathscr{R}_{\mu}^{\mathsf{FR}'}$. Here, Ψ is a mapping which removes all active subscripts and mark symbols. Moreover, Ψ underlines function symbols f at an inactive position, provided that $\underline{f} \in \mathscr{F}_{\mu}^{\mathsf{FR}}$.

In principle, all positions below an inactive position are also inactive. However, in the mapping Ψ, every f with $\underline{f} \notin \mathscr{F}_{\mu}^{\mathsf{FR}}$, every f_{active}, and the symbol mark make their active argument positions "active" again. Thus, if $\mu(:) = \varnothing$, then we obtain $\Psi(\mathsf{0} : \mathsf{adx}(\mathsf{zeros})) = \underline{\mathsf{0}} : \underline{\mathsf{adx}}(\underline{\mathsf{zeros}})$, but $\Psi(\mathsf{0} : \mathsf{mark}(\mathsf{adx}(\mathsf{zeros}))) = \underline{\mathsf{0}} : \mathsf{adx}(\mathsf{zeros})$, $\Psi(\mathsf{0} : \mathsf{adx}_{\mathsf{active}}(\mathsf{zeros})) = \underline{\mathsf{0}} : \mathsf{adx}(\mathsf{zeros})$, and $\Psi(\mathsf{0} : \mathsf{tl}(\mathsf{zeros})) = \underline{\mathsf{0}} : \mathsf{tl}(\mathsf{zeros})$, since $\underline{\mathsf{tl}} \notin \mathscr{F}_{\mu}^{\mathsf{FR}}$. For the definition of Ψ we use another mapping Ψ' which is like Ψ except that in Ψ the root position is considered active and in Ψ' it is considered inactive.

Definition 19

Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$. We define two mappings Ψ and Ψ' from $\mathscr{T}(\mathscr{F}_1)$ to $\mathscr{T}(\mathscr{F}_{\mu}^{\mathsf{FR}})$ inductively as follows:

$$\Psi(f(t_1,\dots,t_n)) = \Psi(f_{\mathsf{active}}(t_1,\dots,t_n)) = \Psi'(f_{\mathsf{active}}(t_1,\dots,t_n)) = f\left(\langle t_1\rangle_1^f,\dots,\langle t_n\rangle_n^f\right)$$

$$\Psi'(f(t_1,\dots,t_n)) = \begin{cases} \underline{f}(\Psi'(t_1),\dots,\Psi'(t_n)) & \textit{if } \underline{f} \in \mathscr{F}_\mu^{\mathsf{FR}} \\ f\left(\langle t_1\rangle_1^f,\dots,\langle t_n\rangle_n^f\right) & \textit{if } \underline{f} \notin \mathscr{F}_\mu^{\mathsf{FR}} \end{cases}$$

$$\Psi(\mathsf{mark}(t)) = \Psi'(\mathsf{mark}(t)) = \Psi(t)$$

with $\langle t\rangle_i^f = \Psi(t)$ *if* $i \in \mu(f)$ *and* $\langle t\rangle_i^f = \Psi'(t)$ *if* $i \notin \mu(f)$*, for all* $1 \leqslant i \leqslant n$.

The aim is to show that every reduction step $s \to_1 t$ corresponds to a reduction from $\Psi(s)$ to $\Psi(t)$ in $\mathscr{R}_\mu^{\mathsf{FR}'}$. In the following, $\mathscr{M}_2$ denotes the subset of $\mathscr{R}_\mu^1$ consisting of all rules in $\mathscr{M}$ together with all rules of the form $f_{\mathsf{active}}(x_1,\dots,x_n) \to f(x_1,\dots,x_n)$ and $\mathscr{M}_1 = \mathscr{R}_\mu^1 \backslash \mathscr{M}_2$. Then we have the following correspondence.

Lemma 20
For all terms $s,t \in \mathscr{T}(\mathscr{F}_1)$*, if* $s \to_{\mathscr{M}_1} t$ *then* $\Psi(s) \to_{\mathsf{FR}'}^{+} \Psi(t)$.

Lemma 21
For all terms $s,t \in \mathscr{T}(\mathscr{F}_1)$*, if* $s \to_{\mathscr{M}_2} t$ *then* $\Psi(s) \to_{\mathsf{FR}'}^{*} \Psi(t)$.

The proofs can be found in Appendix B.

Theorem 22
Let $(\mathscr{R},\mu)$ *be a CSRS over a signature* $\mathscr{F}$.

(a) If $\mathscr{R}_\mu^{\mathsf{FR}'}$ *is terminating then* $\mathscr{R}_\mu^1$ *is terminating.*
(b) If $\mathscr{R}_\mu^{\mathsf{FR}'}$ *is terminating then* $(\mathscr{R},\mu)$ *is terminating.*
(c) If $\mathscr{R}_\mu^{\mathsf{FR}}$ *is terminating then* $\mathscr{R}_\mu^1$ *is terminating.*
(d) If $\mathscr{R}_\mu^{\mathsf{Z}}$ *is terminating then* $\mathscr{R}_\mu^1$ *is terminating.*

Proof
Because $\mathscr{M}_2$ is terminating, every infinite $\mathscr{R}_\mu^1$-reduction of ground terms in $\mathscr{T}(\mathscr{F}_1)$ is transformed into an infinite $\mathscr{R}_\mu^{\mathsf{FR}'}$-reduction as a consequence of Lemmata 21 and 20. This proves (a). Claim (b) is an immediate consequence of (a) and the soundness of our transformation (Theorem 14). Claim (c) follows from (a) since $\mathscr{R}_\mu^{\mathsf{FR}'}$ is a subset of $\mathscr{R}_\mu^{\mathsf{FR}}$. Finally, Claim (d) is implied by (c) and Theorem 11. □

The relationship between the various transformations is illustrated in Figure 1. Here, "*Transformation 1* → *Transformation 2*" means that *Transformation 2* is more powerful than *Transformation 1*, i.e., if Transformation 1 yields a terminating TRS, then so does Transformation 2, but not vice versa. We have proved that the relations between the four transformations Θ_{L}, Θ_{Z}, Θ_{FR}, and Θ_1 depicted in Figure 1 really hold and that these are all relations between these transformations (i.e., Lucas' transformation is incomparable with the ones of Zantema and of Ferreira & Ribeiro). Hence, our transformation Θ_1 is the most powerful one up to now. Still, Θ_1 is incomplete (Example 15) and we will introduce a complete transformation Θ_2 in the next section.

One should note that while Θ_1 is incomplete in general, there do exist some restricted completeness results for Θ_1. Lucas (2002b) recently observed that Θ_1 is complete for such CSRSs $(\mathscr{R},\mu)$ where μ is at least as restrictive as the canonical

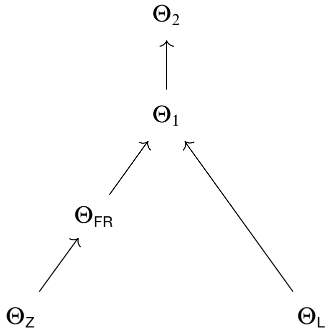

Fig. 1. Relationship between all transformations.

replacement map μ_c associated with $\mathscr{R}$. Moreover, in Giesl & Middeldorp (2003), we investigated the use of Θ_1 for *innermost* termination. It turns out that although termination of $(\mathscr{R}, \mu)$ does not imply termination of $\mathscr{R}^1_\mu$, it at least implies innermost termination of $\mathscr{R}^1_\mu$. An immediate consequence of this result is that Θ_1 is complete for innermost termination of those CSRSs which have the property that innermost termination coincides with termination.[3] The latter is known to be true for orthogonal CSRSs (Giesl & Middeldorp, 2003) and for locally-confluent overlay systems with the additionally property that variables that occur at an active position in a left-hand side l of a rewrite rule $l \rightarrow r$ do not occur at inactive positions in l or r (Gramlich & Lucas, 2002a).

5 A sound and complete transformation

In this section we present a transformation of context-sensitive rewrite systems which is not only sound but also complete with respect to termination.

Let us first investigate why the transformation of Section 4 lacks completeness. Consider again the CSRS $(\mathscr{R}, \mu)$ of Example 15. The reason for the non-termination of $\mathscr{R}^1_\mu$ is that terms may have occurrences of f_{active} symbols at inactive positions, even if we start with a "proper" term (like $\mathsf{f}_{\mathsf{active}}(\mathsf{b}, \mathsf{c}, \mathsf{d}_{\mathsf{active}})$). The "forbidden" occurrences of $\mathsf{d}_{\mathsf{active}}$ in the first two arguments of $\mathsf{f}_{\mathsf{active}}$ (in the term $\mathsf{f}_{\mathsf{active}}(\mathsf{d}_{\mathsf{active}}, \mathsf{d}_{\mathsf{active}}, \mathsf{mark}(\mathsf{d}_{\mathsf{active}})))$) lead to contractions which are impossible in the underlying CSRS. Thus, the key to achieving a complete transformation is to control the number of occurrences of f_{active} symbols. We do this in a rather drastic manner: We will work with a single occurrence of a symbol marked with active. Of course, we cannot forbid the existence of terms with multiple occurrences of f_{active} symbols but we can make sure that no new f_{active} symbols are introduced during the contraction of an active redex.

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[3] These restricted completeness results were originally achieved for a slightly different presentation of our transformation (see Definition 47). However, these results immediately carry over to the current transformation Θ_1.

Instead of having a separate symbol f_{active} for every function symbol f in the signature of the CSRS, we use a new unary function symbol active. Working with a single active occurrence entails that we have to shift it in a non-deterministic fashion downwards to any active position. This is achieved by the rules

$$\mathsf{active}(f(x_1,\dots,x_i,\dots,x_n)) \to f(x_1,\dots,\mathsf{active}(x_i),\dots,x_n)$$

for every $i \in \mu(f)$. By this shifting of the symbol active, our TRS implements an algorithm to search for redexes subject to the constraints of the replacement map μ. Once we have shifted active to the position of the desired redex, we can apply one of the rules

$$\mathsf{active}(l) \to \mathsf{mark}(r)$$

The function symbol mark is used to mark the contractum of the selected redex. In order to continue the reduction it has to be replaced by active again. Since the next reduction step may of course take place at a position above the previously contracted redex, we first have to shift mark upwards through the term, i.e., we use rules of the form

$$f(x_1,\dots,\mathsf{mark}(x_i),\dots,x_n) \to \mathsf{mark}(f(x_1,\dots,x_i,\dots,x_n))$$

for every $i \in \mu(f)$. We want to replace mark by active if we have reached the top of the term. Since it cannot be determined whether mark is on the root position of the term, we introduce a new unary function symbol top to mark the position below which reductions may take place. Thus, the reduction of a term s with respect to a CSRS is modeled by the reduction of the term $\mathsf{top}(\mathsf{active}(s))$ in the transformed TRS. If $\mathsf{top}(\mathsf{active}(s))$ is reduced to a term $\mathsf{top}(\mathsf{mark}(t))$, we are ready to replace mark by active. This suggests adding the rule

$$\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{active}(x))$$

However, as illustrated with the counterexample in Section 4 (Example 15), we have to avoid making infinite reductions with terms which contain inner occurrences of new symbols like active and mark. For that reason we want to make sure that this rule is only applicable to terms that do not contain any other occurrences of the new function symbols. Thus, before reducing $\mathsf{top}(\mathsf{mark}(t))$ to $\mathsf{top}(\mathsf{active}(t))$ we check whether the term t is *proper*, i.e., whether it contains only function symbols from the original signature $\mathscr{F}$. This is easily achieved by new unary function symbols proper and ok. For any ground term $t \in \mathscr{T}(\mathscr{F})$, $\mathsf{proper}(t)$ reduces to $\mathsf{ok}(t)$, but if t contains one of the newly introduced function symbols then the reduction of $\mathsf{proper}(t)$ is blocked. This is done by the rules

$$\mathsf{proper}(c) \to \mathsf{ok}(c)$$

for every constant $c \in \mathscr{F}$ and

$$\begin{aligned} \mathsf{proper}(f(x_1,\dots,x_n)) &\to f(\mathsf{proper}(x_1),\dots,\mathsf{proper}(x_n)) \\ f(\mathsf{ok}(x_1),\dots,\mathsf{ok}(x_n)) &\to \mathsf{ok}(f(x_1,\dots,x_n)) \end{aligned}$$

for every function symbol $f \in \mathscr{F}$ of arity $n > 0$. Then, instead of the rewrite rule $\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{active}(x))$, we take the rules

$$\begin{aligned} \mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{proper}(x)) \\ \mathsf{top}(\mathsf{ok}(x)) &\to \mathsf{top}(\mathsf{active}(x)) \end{aligned}$$

Now the context-sensitive reduction of a term t is translated into a reduction of the term $\mathsf{top}(\mathsf{active}(t))$ with the transformed TRS. This concludes our informal explanation of the new transformation, whose formal definition is summarized below.

Definition 23
Let $(\mathscr{R}, \mu)$ *be a CSRS over a signature* $\mathscr{F}$. *The TRS* $\mathscr{R}^2_\mu$ *over the signature* $\mathscr{F}_2 = \mathscr{F} \cup \{\mathsf{active}, \mathsf{mark}, \mathsf{top}, \mathsf{proper}, \mathsf{ok}\}$ *consists of the following rewrite rules (for all* $l \to r \in \mathscr{R}$, $f \in \mathscr{F}$ *of arity* $n > 0$, $i \in \mu(f)$, *and constants* $c \in \mathscr{F}$*):*

$$\begin{aligned} \mathsf{active}(l) &\to \mathsf{mark}(r) \\ \mathsf{active}(f(x_1,\ldots,x_i,\ldots,x_n)) &\to f(x_1,\ldots,\mathsf{active}(x_i),\ldots,x_n) \\ f(x_1,\ldots,\mathsf{mark}(x_i),\ldots,x_n) &\to \mathsf{mark}(f(x_1,\ldots,x_i,\ldots,x_n)) \\ \mathsf{proper}(c) &\to \mathsf{ok}(c) \\ \mathsf{proper}(f(x_1,\ldots,x_n)) &\to f(\mathsf{proper}(x_1),\ldots,\mathsf{proper}(x_n)) \\ f(\mathsf{ok}(x_1),\ldots,\mathsf{ok}(x_n)) &\to \mathsf{ok}(f(x_1,\ldots,x_n)) \\ \mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{proper}(x)) \\ \mathsf{top}(\mathsf{ok}(x)) &\to \mathsf{top}(\mathsf{active}(x)) \end{aligned}$$

We denote the transformation $(\mathscr{R}, \mu) \mapsto \mathscr{R}^2_\mu$ *by* Θ_2 *and we abbreviate* $\to_{\mathscr{R}^2_\mu}$ *to* $\to_2$.

The following example shows that the rules for proper and ok are essential for completeness.

Example 24
Consider the CSRS $\mathscr{R}$

$$\mathsf{f}(x,\mathsf{g}(x),y) \to \mathsf{f}(y,y,y) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

with $\mu(\mathsf{f}) = \varnothing$ *and* $\mu(\mathsf{g}) = \{1\}$. *This CSRS is clearly terminating. The TRS*

$$\begin{aligned} \mathsf{active}(\mathsf{f}(x,\mathsf{g}(x),y)) &\to \mathsf{mark}(\mathsf{f}(y,y,y)) & \mathsf{active}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{active}(x)) \\ \mathsf{active}(\mathsf{g}(\mathsf{b})) &\to \mathsf{mark}(\mathsf{c}) & \mathsf{g}(\mathsf{mark}(x)) &\to \mathsf{mark}(\mathsf{g}(x)) \\ \mathsf{active}(\mathsf{b}) &\to \mathsf{mark}(\mathsf{c}) & \mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{active}(x)) \end{aligned}$$

that is obtained from $\mathscr{R}^2_\mu$ *by merging the two rules* $\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{proper}(x))$ *and* $\mathsf{top}(\mathsf{ok}(x)) \to \mathsf{top}(\mathsf{active}(x))$ *into* $\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{active}(x))$ *and removing all rules for* proper *and* ok *is non-terminating because* $t = \mathsf{top}(\mathsf{active}(\mathsf{f}(s,s,s)))$ *with* $s = \mathsf{active}(\mathsf{g}(\mathsf{b}))$ *admits the following cycle:*

$$\begin{aligned} t &\to \mathsf{top}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(\mathsf{c}),s,s))) \to \mathsf{top}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(\mathsf{c}),\mathsf{g}(\mathsf{active}(\mathsf{b})),s))) \\ &\to \mathsf{top}(\mathsf{active}(\mathsf{f}(\mathsf{mark}(\mathsf{c}),\mathsf{g}(\mathsf{mark}(\mathsf{c})),s))) \to \mathsf{top}(\mathsf{mark}(\mathsf{f}(s,s,s))) \to t \end{aligned}$$

In the rest of this section we show that our second transformation is both sound and complete. We start with a preliminary lemma, which states that proper has indeed the desired effect.

Lemma 25

Let $(\mathscr{R}, \mu)$ *be a CSRS over a signature* $\mathscr{F}$ *and let* $s, t \in \mathscr{T}(\mathscr{F}_2)$. *We have* $\mathsf{proper}(s) \rightarrow_2^+ \mathsf{ok}(t)$ *if and only if* $s = t$ *and* $s \in \mathscr{T}(\mathscr{F})$.

Proof

The "if" direction is an easy induction proof on the structure of s. The "only if" direction is proved by induction on the length of the reduction. First assume that the first reduction step takes place inside s, so $\mathsf{proper}(s) \rightarrow_2 \mathsf{proper}(s') \rightarrow_2^+ \mathsf{ok}(t)$ for some term s' with $s \rightarrow_2 s'$. The induction hypothesis yields $s' \in \mathscr{T}(\mathscr{F})$. However, an inspection of the rules of $\mathscr{R}_\mu^2$ shows that then $s \rightarrow_2 s'$ is impossible, since terms from $\mathscr{T}(\mathscr{F})$ can never be obtained by $\mathscr{R}_\mu^2$-reductions. So the first reduction step takes place at the root. If s is a constant c, then we obtain $\mathsf{proper}(c) \rightarrow_2 \mathsf{ok}(c)$ and thus $s = c = t \in \mathscr{T}(\mathscr{F})$. Otherwise, a root reduction is only possible if s has the form $f(s_1, \ldots, s_n)$. Then we have $\mathsf{proper}(f(s_1, \ldots, s_n)) \rightarrow_2 f(\mathsf{proper}(s_1), \ldots, \mathsf{proper}(s_n)) \rightarrow_2^+ \mathsf{ok}(t)$. In order to reduce a term $f(\cdots)$ to $\mathsf{ok}(\cdot)$, all arguments of f must reduce to terms with root symbol ok. Hence, we must have $\mathsf{proper}(s_i) \rightarrow_2^+ \mathsf{ok}(t_i)$. The induction hypothesis yields $s_i = t_i \in \mathscr{T}(\mathscr{F})$ and hence $t = f(t_1, \ldots, t_n) = s$, which proves the lemma. □

The next lemma shows how context-sensitive reduction steps are simulated by the second transformation. The "if" part is used in the completeness proof.

Lemma 26

Let $(\mathscr{R}, \mu)$ *be a CSRS over a signature* $\mathscr{F}$ *and let* $s \in \mathscr{T}(\mathscr{F})$. *We have* $s \rightarrow_\mu t$ *if and only if* $\mathsf{active}(s) \rightarrow_2^+ \mathsf{mark}(t)$.

Proof

The "only if" direction is easily proved by induction on the depth of the position of the redex contracted in $s \rightarrow_\mu t$. We prove here the "if" direction by induction on s. There are two possibilities for the rewrite rule of $\mathscr{R}_\mu^2$ that is applied in the first step of the reduction from $\mathsf{active}(s)$ to $\mathsf{mark}(t)$. If a rule of the form $\mathsf{active}(l) \rightarrow \mathsf{mark}(r)$ is used then $s = l\sigma$ for some substitution σ. Since $r\sigma$ contains only symbols from $\mathscr{F}$, $\mathsf{mark}(r\sigma)$ is in normal form and thus $t = r\sigma$. Clearly $s \rightarrow_\mu t$. Otherwise, s must have the form $f(s_1, \ldots, s_i, \ldots, s_n)$ and in the first reduction step $\mathsf{active}(s)$ is reduced to $f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n)$ for some $i \in \mu(f)$. Note that all reductions of the latter term to a term of the form $\mathsf{mark}(t)$ have the form

$$\begin{aligned} f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n) &\rightarrow_2^+ f(s_1, \ldots, \mathsf{mark}(t_i), \ldots, s_n) \\ &\rightarrow_2 \mathsf{mark}(f(s_1, \ldots, t_i, \ldots, s_n)) \end{aligned}$$

Hence $t = f(s_1, \ldots, t_i, \ldots, s_n)$. The induction hypothesis yields $s_i \rightarrow_\mu t_i$ and as $i \in \mu(f)$ we also have $s \rightarrow_\mu t$. □

Soundness of our second transformation is now easily shown.

Theorem 27
Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. If $\mathscr{R}^2_\mu$ is terminating then $(\mathscr{R},\mu)$ is terminating.

Proof
If $(\mathscr{R},\mu)$ is not terminating then there exists an infinite reduction of ground terms in $\mathscr{T}(\mathscr{F})$. Note that $s \to_\mu t$ implies $\mathsf{active}(s) \to_2^+ \mathsf{mark}(t)$ by Lemma 26. Hence it also implies

$$\mathsf{top}(\mathsf{active}(s)) \to_2^+ \mathsf{top}(\mathsf{mark}(t)) \to_2 \mathsf{top}(\mathsf{proper}(t))$$

Moreover, by Lemma 25 we have $\mathsf{proper}(t) \to_2^+ \mathsf{ok}(t)$ and thus

$$\mathsf{top}(\mathsf{proper}(t)) \to_2^+ \mathsf{top}(\mathsf{ok}(t)) \to_2 \mathsf{top}(\mathsf{active}(t))$$

Concatenating these two reductions shows that $\mathsf{top}(\mathsf{active}(s)) \to_2^+ \mathsf{top}(\mathsf{active}(t))$ whenever $s \to_\mu t$. Hence any infinite reduction of ground terms in $(\mathscr{R},\mu)$ is transformed into an infinite reduction in $\mathscr{R}^2_\mu$. □

To prove that the converse of Theorem 27 holds as well, we define $\mathscr{S}^2_\mu$ as the TRS $\mathscr{R}^2_\mu$ without the two rewrite rules for top. The following lemma states that we do not have to worry about $\mathscr{S}^2_\mu$.

Lemma 28
The TRS $\mathscr{S}^2_\mu$ is terminating for any CSRS $(\mathscr{R},\mu)$.

Proof
Let $\mathscr{F}$ be the signature of $(\mathscr{R},\mu)$. The rewrite rules of $\mathscr{S}^2_\mu$ are oriented from left to right by the recursive path order induced by the following precedence on $\mathscr{F}_2$: $\mathsf{active} \succ \mathsf{proper} \succ f \succ \mathsf{ok} \succ \mathsf{mark}$ for every $f \in \mathscr{F}$. It follows that $\mathscr{S}^2_\mu$ is terminating. □

The following lemma implies that the two top-rules must be applied in alternating order.

Lemma 29
Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$ and let $s \in \mathscr{T}(\mathscr{F}_2)$.

- *(a) There is no $t \in \mathscr{T}(\mathscr{F}_2)$ such that $\mathsf{proper}(s) \to_2^+ \mathsf{mark}(t)$.*
- *(b) There is no $t \in \mathscr{T}(\mathscr{F}_2)$ such that $\mathsf{active}(s) \to_2^+ \mathsf{ok}(t)$.*

Proof

(a) We prove the claim by induction on the length of the reduction. If the first reduction step takes place inside s then the claim immediately follows from the induction hypothesis. Otherwise, the first step is a root reduction step. If the first step is $\mathsf{proper}(c) \to_2 \mathsf{ok}(c)$ with $s = c = t$, then the claim is obvious, since the root symbol ok is a constructor which can never be reduced. In the remaining case, we have $s = f(s_1,\ldots,s_n)$ and $\mathsf{proper}(s) \to_2 f(\mathsf{proper}(s_1),\ldots,\mathsf{proper}(s_n))$. In order to rewrite this term to a term with mark as root symbol, one subterm $\mathsf{proper}(s_i)$ must be reduced to $\mathsf{mark}(t_i)$ for some term t_i. However, this contradicts the induction hypothesis.

(b) Again we use induction on the length of the reduction. If the reduction starts inside s, the claim is obvious. If the reduction starts with $\mathsf{active}(s) \to_2 \mathsf{mark}(\cdot)$, then the claim is proved, since mark is a constructor which can never be reduced. The remaining case is $s = f(s_1, \ldots, s_n)$ and $\mathsf{active}(s) \to_2 f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n)$. This term can only be reduced to a term with the root symbol ok if all arguments of f rewrite to ok-terms. In particular, we must have $\mathsf{active}(s_i) \to_2^+ \mathsf{ok}(t_i)$ for some term t_i. This, however, is a contradiction to the induction hypothesis.

□

Now we are ready to present the completeness theorem.

Theorem 30

Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. If $(\mathcal{R}, \mu)$ is terminating then $\mathcal{R}^2_\mu$ is terminating.

Proof

First note that the precedence used in the proof of Lemma 28 cannot be extended to deal with the whole of $\mathcal{R}^2_\mu$ as the rewrite rules for top require $\mathsf{mark} \succ \mathsf{proper}$ and $\mathsf{ok} \succ \mathsf{active}$. Since $\mathcal{R}^2_\mu$ lacks collapsing rules, it is sufficient to prove termination of any *typed* version of $\mathcal{R}^2_\mu$ (Zantema, 1994; Middeldorp & Ohsaki, 2000). Thus, we may assume that the function symbols of $\mathcal{R}^2_\mu$ come from a many-sorted signature, where the only restriction is that the left and right-hand side of any rewrite rule are well typed and of the same type. We use two sorts α and β, with top of type $\alpha \to \beta$ and all other symbols of type $\alpha \times \cdots \times \alpha \to \alpha$. So if $\mathcal{R}^2_\mu$ allows an infinite reduction then there exists an infinite reduction of well-typed terms. Since both types contain a ground term, we may assume for a proof by contradiction that there exists an infinite reduction starting from a well-typed *ground* term t. Terms of type α are terminating by Lemma 28 since they cannot contain the symbol top and thus the only applicable rules stem from $\mathcal{S}^2_\mu$. So t is a ground term of type β, which implies that $t = \mathsf{top}(t')$ with t' of type α. Since t' is terminating, the infinite reduction starting from t must contain a root reduction step. So t' reduces to $\mathsf{mark}(t_1)$ or $\mathsf{ok}(t_0)$ for some terms t_1 or t_0 (of type α).

We first consider the former possibility. The infinite reduction starts with

$$t \to_2^* \mathsf{top}(\mathsf{mark}(t_1)) \to_2 \mathsf{top}(\mathsf{proper}(t_1))$$

Since $\mathsf{proper}(t_1)$ is of type α and thus terminating, after some further reduction steps another step takes place at the root. According to Lemma 29(a), $\mathsf{proper}(t_1)$ cannot reduce to a mark-term. Thus, another root step is only possible if $\mathsf{proper}(t_1)$ reduces to $\mathsf{ok}(t_1')$ for some term t_1'. According to Lemma 25 we must have $t_1 = t_1' \in \mathcal{T}(\mathcal{F})$. Hence the presupposed infinite reduction continues as follows:

$$\mathsf{top}(\mathsf{proper}(t_1)) \to_2^+ \mathsf{top}(\mathsf{ok}(t_1)) \to_2 \mathsf{top}(\mathsf{active}(t_1))$$

Repeating this kind of reasoning reveals that the infinite reduction must be of the following form, where all root reduction steps between $\mathsf{top}(\mathsf{proper}(t_1))$ and

$\mathsf{top}(\mathsf{mark}(t_3))$ are made explicit:

$$\begin{aligned} t &\to_2^+ \mathsf{top}(\mathsf{proper}(t_1)) \to_2^+ \mathsf{top}(\mathsf{ok}(t_1)) \to_2 \mathsf{top}(\mathsf{active}(t_1)) \to_2^+ \mathsf{top}(\mathsf{mark}(t_2)) \\ &\to_2 \mathsf{top}(\mathsf{proper}(t_2)) \to_2^+ \mathsf{top}(\mathsf{ok}(t_2)) \to_2 \mathsf{top}(\mathsf{active}(t_2)) \to_2^+ \mathsf{top}(\mathsf{mark}(t_3)) \\ &\to_2 \cdots \end{aligned}$$

Hence $\mathsf{active}(t_i) \to_2^+ \mathsf{mark}(t_{i+1})$ and $t_i \in \mathscr{T}(\mathscr{F})$ for all $i \geqslant 1$. We obtain

$$t_1 \to_\mu t_2 \to_\mu t_3 \to_\mu \cdots$$

from Lemma 26, contradicting the termination of $(\mathscr{R}, \mu)$.

Next suppose that t' reduces to $\mathsf{ok}(t_0)$ for some term t_0. In this case the infinite reduction starts with $t \to_2^* \mathsf{top}(\mathsf{ok}(t_0)) \to_2 \mathsf{top}(\mathsf{active}(t_0))$. Since $\mathsf{active}(t_0)$ is also of type α and hence terminating, there must be another root reduction step. So $\mathsf{active}(t_0)$ must reduce to $\mathsf{mark}(t_1)$ for some term t_1, since it cannot rewrite to an ok-term by Lemma 29(b). Hence, we end up with $t \to_2^* \mathsf{top}(\mathsf{ok}(t_0)) \to_2 \mathsf{top}(\mathsf{active}(t_0)) \to_2^+ \mathsf{top}(\mathsf{mark}(t_1))$ as in the first case. □

Example 31

To illustrate our new transformation, let us reconsider the CSRS $(\mathscr{R}, \mu)$ *in the counterexample to the completeness of* Θ_1 *(Example 15). Apart from the rules for* proper, ok, *and* top, $\mathscr{R}_\mu^2$ *contains the following rules:*

$$\begin{aligned} \mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},x)) &\to \mathsf{mark}(\mathsf{f}(x,x,x)) & \mathsf{active}(\mathsf{f}(x,y,z)) &\to \mathsf{f}(x,y,\mathsf{active}(z)) \\ \mathsf{active}(\mathsf{d}) &\to \mathsf{mark}(\mathsf{b}) & \mathsf{f}(x,y,\mathsf{mark}(z)) &\to \mathsf{mark}(\mathsf{f}(x,y,z)) \\ \mathsf{active}(\mathsf{d}) &\to \mathsf{mark}(\mathsf{c}) \end{aligned}$$

The term $\mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{c},\mathsf{d}_{\mathsf{active}})$ *admitted an infinite* $\mathscr{R}_\mu^1$*-reduction. In* $\mathscr{R}_\mu^2$*, the corresponding term* $t = \mathsf{top}(\mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{active}(\mathsf{d}))))$ *rewrites to* $\mathsf{top}(\mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}))))$*, but in order to change* mark *back to* active*, all auxiliary symbols below* mark *must be eliminated (this is checked by the rules for* proper *and* ok*). Since this is impossible here, t is terminating. For instance,*

$$\begin{aligned} t &\to \mathsf{top}(\mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d})))) \\ &\to \mathsf{top}(\mathsf{proper}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d})))) \\ &\to \mathsf{top}(\mathsf{f}(\mathsf{proper}(\mathsf{active}(\mathsf{d})),\mathsf{proper}(\mathsf{active}(\mathsf{d})),\mathsf{proper}(\mathsf{active}(\mathsf{d})))) \\ &\to^+ \mathsf{top}(\mathsf{f}(\mathsf{proper}(\mathsf{mark}(\mathsf{b})),\mathsf{proper}(\mathsf{mark}(\mathsf{c})),\mathsf{proper}(\mathsf{mark}(\mathsf{b})))) \end{aligned}$$

6 Context-sensitive rewriting modulo AC

In this section we extend our results to context-sensitive rewriting modulo associativity and commutativity. Operators that are associative and commutative occur frequently in practice. Since the commutativity axiom cannot be oriented into a terminating rewrite rule, one has to work modulo associativity and commutativity to have any hope for terminating computations. (Turning the associativity axiom into a rewrite rule and working modulo commutativity causes non-termination.) Context-sensitive rewriting modulo associativity and commutativity was first studied by Ferreira & Ribeiro (1999). Throughout this section, let $\mathscr{G} \subseteq \mathscr{F}$ be some subset of binary function symbols and let $\mathrm{AC}(\mathscr{G})$ (or just AC if $\mathscr{G}$ can be inferred from the context) consist of the rules

$$f(f(x,y),z) \to f(x,f(y,z))$$
$$f(x,y) \to f(y,x)$$

for all $f \in \mathscr{G}$. As usual, we write $\sim_{\mathrm{AC}}$ for $\leftrightarrow^*_{\mathrm{AC}}$. Then the context-sensitive rewrite relation $\to_{\mu/\mathrm{AC}}$ is defined as follows: $s \to_{\mu/\mathrm{AC}} t$ if and only if there exist terms s' and t' such that $s \sim_{\mathrm{AC}} s' \to_\mu t' \sim_{\mathrm{AC}} t$. Note that a replacement map μ with $\mu(f) = \{1\}$ or $\mu(f) = \{2\}$ for an AC-symbol $f \in \mathscr{G}$ does not make sense, since otherwise associativity and commutativity can be used to bring terms from inactive positions into active ones. Therefore, one demands that the replacement map μ satisfies $\mu(f) = \{1,2\}$ or $\mu(f) = \varnothing$ for all AC-symbols $f \in \mathscr{G}$.[4] In the sequel we tacitly restrict ourselves to replacement maps satisfying this requirement.

Ferreira & Ribeiro (1999) proved that their transformation can also be used in the presence of AC-symbols. More precisely, if $\underline{\mathscr{G}} = \mathscr{G} \cup \{\underline{f} \mid f \in \mathscr{G} \text{ and } \underline{f} \in \mathscr{F}^{\mathsf{FR}}_\mu\}$ then termination of $\mathscr{R}^{\mathsf{FR}}_\mu$ modulo $\mathrm{AC}(\underline{\mathscr{G}})$ implies termination of $(\mathscr{R},\mu)$ modulo $\mathrm{AC}(\mathscr{G})$. Thus, by using any of the methods developed for proving AC-termination (Kapur *et al.*, 1995; Rubio & Nieuwenhuis, 1995; Kapur & Sivakumar, 1997; Marché & Urbain, 1998; Kusakari & Toyama, 2001; Giesl & Kapur, 2001; Rubio, 2002), one can now verify termination of context-sensitive rewriting modulo AC as well.

In this section we prove that analogous statements also hold for our two transformations. Moreover, we show that in the presence of AC-symbols our first transformation is still more powerful than the one of Ferreira & Ribeiro and our second transformation is still complete.

When regarding our first transformation, it is clear that we have to perform a small change in its presentation first. To see this, assume that f is an AC-symbol with replacement map $\mu(\mathsf{f}) = \varnothing$ and consider the TRS $\mathscr{R}$ with the rule $\mathsf{f}(\mathsf{f}(\mathsf{b},\mathsf{c}),\mathsf{d}) \to \mathsf{f}(\mathsf{b},\mathsf{f}(\mathsf{c},\mathsf{d}))$. (Context-sensitive) rewriting modulo AC is obviously not terminating. However, $\mathscr{R}^1_\mu$ would be terminating, since the present rule would be replaced by $\mathsf{f}_{\mathsf{active}}(\mathsf{f}(\mathsf{b},\mathsf{c}),\mathsf{d}) \to \mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{f}(\mathsf{c},\mathsf{d})))\downarrow_{\mathscr{M}} = \mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{f}(\mathsf{c},\mathsf{d}))$. In order to simulate the non-terminating reduction in $\mathscr{R}^1_\mu$ one would need associativity not just for f and $\mathsf{f}_{\mathsf{active}}$, but also for a combination of these two symbols. Hence, in rules of $\mathscr{R}^1_\mu$ of the form $f_{\mathsf{active}}(l_1,\ldots,l_n) \to \mathsf{mark}(r)\downarrow_{\mathscr{M}}$, the rules $\mathsf{mark}(g(\cdots)) \to g_{\mathsf{active}}(\cdots)$ for defined AC-symbols with $\mu(g) = \varnothing$ should not be used to normalize the right-hand sides. This results in a slightly modified transformation Θ'_1.

Definition 32

Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$ and let $\mathscr{G} \subseteq \mathscr{F}$. The TRS $\mathscr{R}^{1'}_\mu$ over the signature $\mathscr{F}_1 = \mathscr{F} \cup \{f_{\mathsf{active}} \mid f \in \mathscr{F}_{\mathscr{D}}\} \cup \{\mathsf{mark}\}$ consists of the following rewrite

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[4] Ferreira & Ribeiro also regard a further restriction of context-sensitive rewriting where one uses a second replacement map in order to restrict those positions where application of AC-axioms is allowed. However, we do not see any motivation for this restriction in practice. Moreover, if one wants to prove termination of the transformed system with existing methods, one can never benefit from this restriction (i.e., one can only prove termination of $\to_{\mu/AC}$ where application of AC-axioms is unrestricted).

rules:

$$\begin{array}{rcll} f_{\mathsf{active}}(l_1,\ldots,l_n) & \to & \mathsf{mark}(r){\downarrow_{\mathcal{M}'}} & \textit{for all } f(l_1,\ldots,l_n)\to r \in \mathcal{R}\\ \mathsf{mark}(f(x_1,\ldots,x_n)) & \to & f_{\mathsf{active}}\big([x_1]^f_1,\ldots,[x_n]^f_n\big) & \textit{for all } f \in \mathcal{F}_\mathcal{D}\\ \mathsf{mark}(f(x_1,\ldots,x_n)) & \to & f\big([x_1]^f_1,\ldots,[x_n]^f_n\big) & \textit{for all } f \in \mathcal{F}_\mathcal{C}\\ f_{\mathsf{active}}(x_1,\ldots,x_n) & \to & f(x_1,\ldots,x_n) & \textit{for all } f \in \mathcal{F}_\mathcal{D} \end{array}$$

Here $\mathcal{M}'$ *is the subset of* $\mathcal{R}^{1'}_\mu$ *consisting of all* mark-*rules except those where* $f \in \mathcal{G}\cap\mathcal{F}_\mathcal{D}$ *and* $\mu(f)=\varnothing$. *Again,* $[t]^f_i = \mathsf{mark}(t)$ *if* $i\in\mu(f)$ *and* $[t]^f_i = t$ *otherwise. We denote the transformation* $(\mathcal{R},\mu)\mapsto\mathcal{R}^{1'}_\mu$ *by* Θ'_1 *and we abbreviate* $\to_{\mathcal{R}^{1'}_\mu}$ *to* $\to_{1'}$.

So in the example above $\mathcal{R}^{1'}_\mu$ differs from $\mathcal{R}^{1}_\mu$ in that the rule $\mathsf{f}_{\mathsf{active}}(\mathsf{f}(\mathsf{b},\mathsf{c}),\mathsf{d}) \to \mathsf{f}_{\mathsf{active}}(\mathsf{b},\mathsf{f}(\mathsf{c},\mathsf{d}))$ is replaced by $\mathsf{f}_{\mathsf{active}}(\mathsf{f}(\mathsf{b},\mathsf{c}),\mathsf{d}) \to \mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{f}(\mathsf{c},\mathsf{d})))$.

Before proving the soundness of the transformation Θ'_1 for termination of context-sensitive rewriting modulo AC, let us first show that in the absence of AC-axioms, $\mathcal{R}^{1'}_\mu$ is really just a slightly different presentation of $\mathcal{R}^{1}_\mu$ (i.e., they do not differ in their termination behavior).

Theorem 33

Let $(\mathcal{R},\mu)$ *be a CSRS over a signature* $\mathcal{F}$. *The TRS* $\mathcal{R}^{1}_\mu$ *is terminating if and only if* $\mathcal{R}^{1'}_\mu$ *is terminating.*

Proof

The “if” direction is trivial, since $\to_1\ \subseteq\ \to^+_{1'}$. For the “only if” direction note that non-termination of $\mathcal{R}^{1'}_\mu$ can only be due to the rules from $\mathcal{R}^{1'}_\mu\setminus\mathcal{M}$. We show that if $s\to_{1'} t$ by application of one of these rules, then we have $s{\downarrow_\mathcal{M}} \to^+_1 t{\downarrow_\mathcal{M}}$. First regard the case where $s|_\pi = f_{\mathsf{active}}(l_1,\ldots,l_n)\sigma$ and $t = s[\mathsf{mark}(r){\downarrow_{\mathcal{M}'}}\sigma]_\pi$ for some rule $l\to r\in\mathcal{R}$. Let $\sigma'(x)=\sigma(x){\downarrow_\mathcal{M}}$ for all variables x. Then we obtain $s{\downarrow_\mathcal{M}} = s{\downarrow_\mathcal{M}}[f_{\mathsf{active}}(l_1,\ldots,l_n)\sigma']_{\pi'} \to_1 s{\downarrow_\mathcal{M}}[\mathsf{mark}(r){\downarrow_\mathcal{M}}\sigma']_{\pi'} \to^*_1 t{\downarrow_\mathcal{M}}$. Next let $s|_\pi = f_{\mathsf{active}}(s_1,\ldots,s_n)$ and $t = s[f(s_1,\ldots,s_n)]_\pi$. Then we obtain $s{\downarrow_\mathcal{M}} = s{\downarrow_\mathcal{M}}[f_{\mathsf{active}}(s_1{\downarrow_\mathcal{M}},\ldots,s_n{\downarrow_\mathcal{M}})]_{\pi'} \to_1 s{\downarrow_\mathcal{M}}[f(s_1{\downarrow_\mathcal{M}},\ldots,s_n{\downarrow_\mathcal{M}})]_{\pi'} \to^*_1 t{\downarrow_\mathcal{M}}$. $\square$

Now we show that transformation Θ'_1 remains sound in the presence of AC-axioms.

Theorem 34

Let $(\mathcal{R},\mu)$ *be a CSRS over a signature* $\mathcal{F}$ *and let* $\mathcal{G}' = \mathcal{G}\cup\{f_{\mathsf{active}} \mid f\in\mathcal{G}\cap\mathcal{F}_\mathcal{D}\}$. *If* $\mathcal{R}^{1'}_\mu$ *is terminating modulo* $\mathrm{AC}(\mathcal{G}')$ *then* $(\mathcal{R},\mu)$ *is terminating modulo* $\mathrm{AC}(\mathcal{G})$.

Proof

As in Lemma 13 and Theorem 14, it is enough to show that for all ground terms $s,t\in\mathcal{T}(\mathcal{F})$, if $s \sim_{\mathrm{AC}(\mathcal{G})} s' \to_\mu t' \sim_{\mathrm{AC}(\mathcal{G})} t$ then

$$\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'} \sim_{\mathrm{AC}(\mathscr{G}')} \mathsf{mark}(s')\!\downarrow_{\mathscr{M}'} \to^{+}_{1'} \mathsf{mark}(t')\!\downarrow_{\mathscr{M}'} \sim_{\mathrm{AC}(\mathscr{G}')} \mathsf{mark}(t)\!\downarrow_{\mathscr{M}'}$$

Similar to the proof of Lemma 13 one shows that $s' \to_{\mu} t'$ implies $\mathsf{mark}(s')\!\downarrow_{\mathscr{M}'} \to^{+}_{1'} \mathsf{mark}(t')\!\downarrow_{\mathscr{M}'}$. So it remains to show that $s \sim_{\mathrm{AC}(\mathscr{G})} s'$ implies $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'} \sim_{\mathrm{AC}(\mathscr{G}')} \mathsf{mark}(s')\!\downarrow_{\mathscr{M}'}$. Using induction on the number of AC-steps, it is sufficient to show that $s \to_{\mathrm{AC}(\mathscr{G})} s'$ implies $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'} \to_{\mathrm{AC}(\mathscr{G}')} \mathsf{mark}(s')\!\downarrow_{\mathscr{M}'}$. Let us regard the case where associativity is applied, i.e., $s|_{\pi} = f(f(s_1,s_2),s_3)$ and $s' = s[f(s_1,f(s_2,s_3))]_{\pi}$ for $f \in \mathscr{G}$, some position π, and some terms s_1, s_2, and s_3. (The case where the commutativity rule is applied is completely analogous.)

First, let π be an active position in s and let $\mu(f) = \{1,2\}$ or $f \in \mathscr{F}_{\mathscr{C}}$. Then $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}|_{\pi} = f'(f'(s'_1,s'_2),s'_3)$ and $\mathsf{mark}(s')\!\downarrow_{\mathscr{M}'} = \mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}[f'(s'_1,f'(s'_2,s'_3))]_{\pi}$ for some terms s'_1, s'_2, and s'_3, where $f' = f_{\mathsf{active}}$ if $f \in \mathscr{F}_{\mathscr{D}}$ and $f' = f$ if $f \in \mathscr{F}_{\mathscr{C}}$. If π is active, $\mu(f) = \varnothing$, and $f \in \mathscr{F}_{\mathscr{D}}$ then $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}|_{\pi} = \mathsf{mark}(f(f(s_1,s_2),s_3))$ and $\mathsf{mark}(s')\!\downarrow_{\mathscr{M}'} = \mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}[\mathsf{mark}(f(s_1,f(s_2,s_3)))]_{\pi}$. If π is an inactive position in s then $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'} = \mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}[f(f(s_1,s_2),s_3)]_{\pi'}$ and $\mathsf{mark}(s')\!\downarrow_{\mathscr{M}'} = \mathsf{mark}(s)\!\downarrow_{\mathscr{M}'}[f(s_1, f(s_2,s_3))]_{\pi'}$ for some position π'. In all cases we clearly have $\mathsf{mark}(s)\!\downarrow_{\mathscr{M}'} \to_{\mathrm{AC}(\mathscr{G}')} \mathsf{mark}(s')\!\downarrow_{\mathscr{M}'}$. ☐

Finally, we compare our transformation Θ'_1 with the one of Ferreira & Ribeiro (1999) when using it for context-sensitive rewriting modulo AC. First, note that Ferreira and Ribeiro's transformation can only be used if the replacement map μ satisfies $\mu(f) = \{1,2\}$ for all AC-symbols f. Otherwise, their transformation is unsound. To illustrate this, consider the CSRS

$$\mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{f}(\mathsf{b},\mathsf{b})) \qquad\qquad \mathsf{f}(\mathsf{f}(\mathsf{c},\mathsf{b}),\mathsf{b}) \to \mathsf{f}(\mathsf{c},\mathsf{c})$$

with $\mu(\mathsf{f}) = \varnothing$ and f an AC-symbol. Clearly, $(\mathscr{R},\mu)$ is not terminating modulo AC. However, $\mathscr{R}^{\mathsf{FR}}_{\mu}$

$$\begin{array}{ccc}
\mathsf{f}(\underline{\mathsf{c}},\underline{\mathsf{c}}) \to \mathsf{f}(\underline{\mathsf{c}},\underline{\mathsf{f}}(\underline{\mathsf{b}},\underline{\mathsf{b}})) & \mathsf{a}(\underline{\mathsf{f}}(x_1,x_2)) \to \mathsf{f}(x_1,x_2) & \mathsf{f}(x_1,x_2) \to \underline{\mathsf{f}}(x_1,x_2) \\
\mathsf{f}(\underline{\mathsf{f}}(\underline{\mathsf{c}},\underline{\mathsf{b}}),\underline{\mathsf{b}}) \to \mathsf{f}(\underline{\mathsf{c}},\underline{\mathsf{c}}) & \mathsf{a}(\underline{\mathsf{b}}) \to \mathsf{b} & \mathsf{b} \to \underline{\mathsf{b}} \\
\mathsf{a}(x) \to x & \mathsf{a}(\underline{\mathsf{c}}) \to \mathsf{c} & \mathsf{c} \to \underline{\mathsf{c}}
\end{array}$$

is terminating modulo $\mathrm{AC}(\{\mathsf{f},\underline{\mathsf{f}}\})$. The problem is that for the desired step from $\mathsf{f}(\underline{\mathsf{c}},\underline{\mathsf{f}}(\underline{\mathsf{b}},\underline{\mathsf{b}}))$ to $\mathsf{f}(\underline{\mathsf{f}}(\underline{\mathsf{c}},\underline{\mathsf{b}}),\underline{\mathsf{b}})$ we need the rule $\mathsf{f}(x,\underline{\mathsf{f}}(y,z)) \to \mathsf{f}(\underline{\mathsf{f}}(x,y),z)$, which is not an associativity axiom.

Thus, Θ'_1 is more widely applicable since our transformation is sound for any replacement map μ (where $\mu(f) = \{1,2\}$ or $\mu(f) = \varnothing$ for AC-symbols f). Moreover, even in the case where $\mu(f) = \{1,2\}$ for all AC-symbols f, our transformation Θ'_1 is still more powerful than the one of Ferreira & Ribeiro. This is shown in the following theorem. Again, $\mathscr{G}$ is a subset of the binary function symbols in $\mathscr{F}$, $\mathscr{G}' = \mathscr{G} \cup \{f_{\mathsf{active}} \mid f \in \mathscr{G} \cap \mathscr{F}_{\mathscr{D}}\}$, and $\underline{\mathscr{G}} = \mathscr{G} \cup \{\underline{f} \mid f \in \mathscr{G} \text{ and } \underline{f} \in \mathscr{F}^{\mathsf{FR}}_{\mu}\}$.

Theorem 35

Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. Let $\mu(f) = \{1,2\}$ for all $f \in \mathscr{G}$. If $\mathscr{R}^{\mathsf{FR}'}_{\mu}$ is terminating modulo $\mathrm{AC}(\underline{\mathscr{G}})$ then $\mathscr{R}^{1'}_{\mu}$ is terminating modulo $\mathrm{AC}(\mathscr{G}')$.

Proof

Similar to the proof of Theorem 22(a), it suffices to show that for all terms from $\mathscr{T}(\mathscr{F}_1)$, $s \sim_{\mathrm{AC}(\mathscr{G}')} s' \rightarrow_{1'} t' \sim_{\mathrm{AC}(\mathscr{G}')} t$ implies

$$\Psi(s) \sim_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s') \rightarrow^{*}_{\mathsf{FR}'} \Psi(t') \sim_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(t)$$

where we have $\rightarrow^{+}_{\mathsf{FR}'}$ instead of $\rightarrow^{*}_{\mathsf{FR}'}$ whenever a rule $f_{\mathsf{active}}(l_1,\ldots,l_n) \rightarrow \mathsf{mark}(r){\downarrow_{\mathscr{M}'}}$ is applied to rewrite s' to t'. Similar to Lemmata 20 and 21 one can show that $s' \rightarrow_{1'} t'$ implies $\Psi(s') \rightarrow^{*}_{\mathsf{FR}'} \Psi(t')$ and if a rule $f_{\mathsf{active}}(l_1,\ldots,l_n) \rightarrow \mathsf{mark}(r){\downarrow_{\mathscr{M}'}}$ is applied in the step from s' to t' then at least one rule of $\mathscr{R}^{\mathsf{FR}'}_{\mu}$ is needed to reduce $\Psi(s')$ to $\Psi(t')$. Hence, it remains to show that if $s \sim_{\mathrm{AC}(\mathscr{G}')} s'$ then $\Psi(s) \sim_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$. Using induction on the number of AC-steps, it is sufficient to show $\Psi(s) \rightarrow_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$ for $s \rightarrow_{\mathrm{AC}(\mathscr{G}')} s'$. We only regard the application of an associativity rule; the proof for commutativity is completely analogous. We consider two cases:

(i) $s|_\pi = f(f(s_1,s_2),s_3)$ and $s' = s[f(s_1,f(s_2,s_3))]_\pi$,
(ii) $s|_\pi = f_{\mathsf{active}}(f_{\mathsf{active}}(s_1,s_2),s_3)$ and $s' = s[f_{\mathsf{active}}(s_1,f_{\mathsf{active}}(s_2,s_3))]_\pi$ with $f \in \mathscr{F}_{\mathscr{D}}$

for some position π, terms s_1, s_2, s_3, and $f \in \mathscr{G}$.

(i) When computing $\Psi(s)$ and $\Psi(s')$, either Ψ or Ψ' is propagated to the subterms $s|_\pi$ and $s'|_\pi$. In the former case we have

$$\begin{aligned}\Psi(s) &= \Psi(s)[\Psi(s|_\pi)]_{\pi'}\\ &= \Psi(s)[\Psi(f(f(s_1,s_2),s_3))]_{\pi'}\\ &= \Psi(s)[f(f(\Psi(s_1),\Psi(s_2)),\Psi(s_3))]_{\pi'}\end{aligned}$$

where the last equality follows from $\mu(f) = \{1,2\}$, and likewise

$$\Psi(s') = \Psi(s)[f(\Psi(s_1),f(\Psi(s_2),\Psi(s_3)))]_{\pi'}$$

for some position π'. Hence $\Psi(s) \rightarrow_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$ by applying the associativity rule for f. In the latter case, we need to distinguish whether or not $\underline{f} \in \mathscr{F}^{\mathsf{FR}}_{\mu}$. If $\underline{f} \notin \mathscr{F}^{\mathsf{FR}}_{\mu}$ then we obtain $\Psi(s) \rightarrow_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$ exactly as before. If $\underline{f} \in \mathscr{F}^{\mathsf{FR}}_{\mu}$ then

$$\begin{aligned}\Psi(s) &= \Psi(s)[\Psi'(s|_\pi)]_{\pi'}\\ &= \Psi(s)[\underline{f}(\underline{f}(\Psi'(s_1),\Psi'(s_2)),\Psi'(s_3))]_{\pi'}\end{aligned}$$

and $\Psi(s') = \Psi(s)[\underline{f}(\Psi'(s_1),\underline{f}(\Psi'(s_2),\Psi'(s_3)))]_{\pi'}$. Because $\underline{f} \in \mathscr{F}^{\mathsf{FR}}_{\mu}$, $\mathrm{AC}(\underline{\mathscr{G}})$ contains the associativity rule for $\underline{f}$ and thus $\Psi(s) \rightarrow_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$.

(ii) We have

$$\begin{aligned}\Psi(s) &= \Psi(s[f_{\mathsf{active}}(f_{\mathsf{active}}(s_1,s_2),s_3)]_\pi)\\ &= \Psi(s)[f(f(\Psi(s_1),\Psi(s_2)),\Psi(s_3))]_{\pi'}\end{aligned}$$

and likewise $\Psi(s') = \Psi(s)[f(\Psi(s_1),f(\Psi(s_2),\Psi(s_3)))]_{\pi'}$, for some position π'. Using the associativity rule for f, we obtain $\Psi(s) \rightarrow_{\mathrm{AC}(\underline{\mathscr{G}})} \Psi(s')$, as desired.

□

Now we prove that soundness and completeness of our second transformation also hold in the presence of AC-axioms.

Theorem 36

Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$. If $\mathscr{R}_\mu^2$ is terminating modulo AC($\mathscr{G}$) *then $(\mathscr{R}, \mu)$ is terminating modulo* AC($\mathscr{G}$).

Proof

We show that for ground terms $s, t \in \mathscr{T}(\mathscr{F})$, $s \rightarrow_{\mu/\mathrm{AC}} t$ implies $\mathsf{top}(\mathsf{active}(s)) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{top}(\mathsf{active}(t))$. By definition, there exist s' and t' such that $s \sim_{\mathrm{AC}} s' \rightarrow_\mu t' \sim_{\mathrm{AC}} t$. As in the proof of Theorem 27, we obtain $\mathsf{top}(\mathsf{active}(s')) \rightarrow^{+}_{2} \mathsf{top}(\mathsf{active}(t'))$ from Lemmata 25 and 26. Clearly $\mathsf{top}(\mathsf{active}(s)) \sim_{\mathrm{AC}} \mathsf{top}(\mathsf{active}(s'))$ and $\mathsf{top}(\mathsf{active}(t')) \sim_{\mathrm{AC}} \mathsf{top}(\mathsf{active}(t))$, and hence the claim is proved. □

To prove completeness, we first extend Lemma 25 about the effect of proper to the AC-case.

Lemma 37

Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$ and let $s, t \in \mathscr{T}(\mathscr{F}_2)$. We have $\mathsf{proper}(s) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{ok}(t)$ *if and only if $s \sim_{\mathrm{AC}} t$ and $s \in \mathscr{T}(\mathscr{F})$.*

Proof

The "if" direction follows from Lemma 25: $s \in \mathscr{T}(\mathscr{F})$ implies that $\mathsf{proper}(s) \rightarrow^{+}_{2} \mathsf{ok}(s)$ and since $\mathsf{ok}(s) \sim_{\mathrm{AC}} \mathsf{ok}(t)$ we obtain $\mathsf{proper}(s) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{ok}(t)$. The proof of the "only if" direction is completely analogous to the corresponding proof in Lemma 25 by using an induction on the length of the $\rightarrow_{2/\mathrm{AC}}$-reduction. □

The next lemma shows that similar to Lemma 26, context-sensitive reduction steps modulo AC can still be simulated by the second transformation.

Lemma 38

Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$ and let $s \in \mathscr{T}(\mathscr{F})$. We have $s \rightarrow_{\mu/\mathrm{AC}} t$ if and only if $\mathsf{active}(s) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{mark}(t)$.

Proof

For the "if" direction we observe that the reduction $\mathsf{active}(s) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{mark}(t)$ can be rearranged into $\mathsf{active}(s) \sim_{\mathrm{AC}} \mathsf{active}(s') \rightarrow^{+}_{2} \mathsf{mark}(t') \sim_{\mathrm{AC}} \mathsf{mark}(t)$. Since $s' \in \mathscr{T}(\mathscr{F})$, we can apply Lemma 26. This yields $s' \rightarrow_\mu t'$ and thus $s \rightarrow_{\mu/\mathrm{AC}} t$ as desired. For the "only if" direction we reason as follows. By definition, there exist terms s' and t' such that $s \sim_{\mathrm{AC}} s' \rightarrow_\mu t' \sim_{\mathrm{AC}} t$. Lemma 26 yields $\mathsf{active}(s') \rightarrow^{+}_{2} \mathsf{mark}(t')$. Clearly $\mathsf{active}(s) \sim_{\mathrm{AC}} \mathsf{active}(s')$ and $\mathsf{mark}(t') \sim_{\mathrm{AC}} \mathsf{mark}(t)$, and therefore $\mathsf{active}(s) \rightarrow^{+}_{2/\mathrm{AC}} \mathsf{mark}(t)$. □

Recall that $\mathscr{S}_\mu^2$ is the TRS $\mathscr{R}_\mu^2$ without the two rewrite rules for top.

Lemma 39

The TRS $\mathscr{S}_\mu^2$ is terminating modulo AC for any CSRS $(\mathscr{R}, \mu)$.

Proof

The rewrite rules of $\mathscr{S}_\mu^2$ are oriented from left to right for example by the AC-extension of the recursive path order from (Kapur *et al.*, 1995), where the precedence is as in Lemma 28. Hence, $\mathscr{S}_\mu^2$ is terminating modulo AC. □

In the AC-case, the two top-rules must also be applied in alternating order.

Lemma 40

Let $(\mathscr{R}, \mu)$ *be a CSRS over a signature* $\mathscr{F}$ *and let* $s \in \mathscr{T}(\mathscr{F}_2)$.

(a) There is no $t \in \mathscr{T}(\mathscr{F}_2)$ *such that* $\mathsf{proper}(s) \to_{2/\mathrm{AC}}^{+} \mathsf{mark}(t)$.

(b) There is no $t \in \mathscr{T}(\mathscr{F}_2)$ *such that* $\mathsf{active}(s) \to_{2/\mathrm{AC}}^{+} \mathsf{ok}(t)$.

Proof

The proof is analogous to the proof of Lemma 29, using induction on the length of the $\to_{2/\mathrm{AC}}$-reduction. The only difference is in part (b), when $s = f(s_1, \ldots, s_n)$ and the reduction starts with $\mathsf{active}(s) \to_{2/\mathrm{AC}} f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n)$. This term can only be reduced to a term with the root symbol ok if $f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n) \to_{2/\mathrm{AC}}^{*} f(\mathsf{ok}(t_1), \ldots, \mathsf{ok}(t_n))$. Since f could be associative, this does not imply that each argument of f must reduce to an ok-term. However, let T consist of all maximal subterms of $f(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n)$ with a root symbol different from f. Then it is easy to show that in order to reduce the whole term to an ok-term, all $t \in T$ must reduce to an ok-term. Since $\mathsf{active}(s_i) \in T$, we must also have $\mathsf{active}(s_i) \to_{2/\mathrm{AC}}^{+} \mathsf{ok}(\cdot)$ which contradicts the induction hypothesis. □

Now we can finally prove the completeness of our second transformation for context-sensitive rewriting modulo AC.

Theorem 41

Let $(\mathscr{R}, \mu)$ *be a CSRS over a signature* $\mathscr{F}$. *If* $(\mathscr{R}, \mu)$ *is terminating modulo AC then* $\mathscr{R}_\mu^2$ *is terminating modulo AC.*

Proof

The proof is very similar to the proof of Theorem 30. Since AC only contains non-collapsing and variable preserving equations, it is again sufficient to prove that a suitably *typed* version of $\mathscr{R}_\mu^2$ is terminating modulo AC (Middeldorp & Ohsaki, 2000). The typing is done as in Theorem 30, i.e., top is of type $\alpha \to \beta$ and all other symbols are of type $\alpha \times \cdots \times \alpha \to \alpha$. By Lemma 39, any term t that is non-terminating modulo AC must be of type β, which implies that $t = \mathsf{top}(t')$ with t' of type α. Since t' is terminating modulo AC and top is not an AC-symbol, the infinite reduction starting from t must contain a root reduction step. So t' reduces to $\mathsf{mark}(t_1')$ or $\mathsf{ok}(t_0)$ for some terms t_1' or t_0 (of type α).

We first consider the former possibility. The infinite reduction starts with

$$t \to_{2/\mathrm{AC}}^{*} \mathsf{top}(\mathsf{mark}(t_1')) \to_{2/\mathrm{AC}} \mathsf{top}(\mathsf{proper}(t_1''))$$

where $t_1' \sim_{\mathrm{AC}} t_1''$. Since $\mathsf{proper}(t_1'')$ is of type α and thus terminating modulo AC, after some further reduction steps another step takes place at the root. According to Lemma 40(a) this is only possible if $\mathsf{proper}(t_1'')$ reduces modulo AC to $\mathsf{ok}(t_1''')$ for some term t_1'''. According to Lemma 37 we must have $t_1'' \sim_{\mathrm{AC}} t_1''' \in \mathscr{T}(\mathscr{F})$. Hence the presupposed infinite reduction continues as follows:

$$\mathsf{top}(\mathsf{proper}(t_1'')) \to^+_{2/\mathrm{AC}} \mathsf{top}(\mathsf{ok}(t_1''')) \to_{2/\mathrm{AC}} \mathsf{top}(\mathsf{active}(t_1))$$

where $t_1 \sim_{\mathrm{AC}} t_1'''$. Thus, by rearranging the AC-steps, we obtain

$$t \to^+_{2/\mathrm{AC}} \mathsf{top}(\mathsf{proper}(t_1)) \to^+_2 \mathsf{top}(\mathsf{ok}(t_1)) \to_2 \mathsf{top}(\mathsf{active}(t_1))$$

Repeating this kind of reasoning reveals that the infinite reduction can be rearranged into the following form, where all root reduction steps between the terms $\mathsf{top}(\mathsf{proper}(t_1))$ and $\mathsf{top}(\mathsf{mark}(t_3))$ are made explicit:

$$\begin{array}{ll} t \to^+_{2/\mathrm{AC}} & \mathsf{top}(\mathsf{proper}(t_1)) \to^+_2 \mathsf{top}(\mathsf{ok}(t_1)) \to_2 \mathsf{top}(\mathsf{active}(t_1)) \to^+_{2/\mathrm{AC}} \mathsf{top}(\mathsf{mark}(t_2)) \\ \to_2 & \mathsf{top}(\mathsf{proper}(t_2)) \to^+_2 \mathsf{top}(\mathsf{ok}(t_2)) \to_2 \mathsf{top}(\mathsf{active}(t_2)) \to^+_{2/\mathrm{AC}} \mathsf{top}(\mathsf{mark}(t_3)) \\ \to_2 & \cdots \end{array}$$

Hence $\mathsf{active}(t_i) \to^+_{2/\mathrm{AC}} \mathsf{mark}(t_{i+1})$ and $t_i \in \mathscr{T}(\mathscr{F})$ for all $i \geqslant 1$. We obtain

$$t_1 \to_{\mu/\mathrm{AC}} t_2 \to_{\mu/\mathrm{AC}} t_3 \to_{\mu/\mathrm{AC}} \cdots$$

from Lemma 38, contradicting the termination of $(\mathscr{R}, \mu)$ modulo AC.

Next suppose that t' reduces to $\mathsf{ok}(t_0)$ for some term t_0. In this case the infinite reduction starts with $t \to^*_{2/\mathrm{AC}} \mathsf{top}(\mathsf{ok}(t_0)) \to_{2/\mathrm{AC}} \mathsf{top}(\mathsf{active}(t_0'))$ where $t_0 \sim_{\mathrm{AC}} t_0'$. Since $\mathsf{active}(t_0')$ is also of type α and hence terminating modulo AC, there must be another root reduction step. So by Lemma 40(b), $\mathsf{active}(t_0')$ must reduce modulo AC to $\mathsf{mark}(t_1')$ for some term t_1'. Hence, we end up with $t \to^*_{2/\mathrm{AC}} \mathsf{top}(\mathsf{ok}(t_0)) \to_{2/\mathrm{AC}} \mathsf{top}(\mathsf{active}(t_0')) \to^+_{2/\mathrm{AC}} \mathsf{top}(\mathsf{mark}(t_1'))$ as in the first case. □

7 Incrementality

It is natural to expect that termination of a CSRS becomes easier to prove when restricting the associated replacement map. In this section we investigate this issue for the five transformations discussed in this paper.

Definition 42

We call a transformation Θ from CSRSs to TRSs incremental *if $\Theta(\mathscr{R}, \nu)$ is terminating for all those TRSs $\mathscr{R}$ and replacement maps μ, ν where $\Theta(\mathscr{R}, \mu)$ is terminating and where ν is a restriction of μ, i.e., $\nu(f) \subseteq \mu(f)$ for all function symbols f.*

Lucas' transformation is not incremental. Consider the TRS $\mathscr{R}$

$$\mathsf{f}(\mathsf{b}, x) \to \mathsf{f}(\mathsf{c}, x)$$

and replacement maps $\mu(\mathsf{f}) = \{1, 2\}$ and $\nu(\mathsf{f}) = \{2\}$. One easily verifies that $\mathscr{R}^{\mathsf{L}}_\mu$ is terminating and that $\mathscr{R}^{\mathsf{L}}_\nu$ lacks termination. (In particular, this example shows that Lucas' transformation lacks incrementality even in examples where the transformed system is still a proper TRS, i.e., where all variables in right-hand sides of rules occur in the corresponding left-hand sides as well.)

We do not know whether Zantema's transformation is incremental. However, restricting the replacement map may make the task of proving termination of the transformed system more difficult. In particular, there are examples where termination of $\mathscr{R}_\mu^{\mathsf{Z}}$ can be proved by the recursive path order, but termination of $\mathscr{R}_\nu^{\mathsf{Z}}$ cannot be proved by *any* recursive path order. For example, consider the one-rule TRS $\mathscr{R}$

$$\mathsf{f}(x) \to \mathsf{g}(\mathsf{f}(x))$$

and replacement maps μ and ν defined by $\mu(\mathsf{g}) = \nu(\mathsf{g}) = \varnothing$, $\mu(\mathsf{f}) = \{1\}$, and $\nu(\mathsf{f}) = \varnothing$. Termination of the TRS $\mathscr{R}_\mu^{\mathsf{Z}}$

$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(\underline{\mathsf{f}}(x)) & \qquad \mathsf{f}(x) &\to \underline{\mathsf{f}}(x) \\ \mathsf{a}(\underline{\mathsf{f}}(x)) &\to \mathsf{f}(x) & \mathsf{a}(x) &\to x \end{aligned}$$

can be proved by the recursive path order with precedence $\mathsf{a} \succ \mathsf{f} \succ \mathsf{g} \succ \underline{\mathsf{f}}$. The TRS $\mathscr{R}_\nu^{\mathsf{Z}}$

$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(\underline{\mathsf{f}}(\mathsf{a}(x))) & \qquad \mathsf{f}(x) &\to \underline{\mathsf{f}}(x) \\ \mathsf{a}(\underline{\mathsf{f}}(x)) &\to \mathsf{f}(x) & \mathsf{a}(x) &\to x \end{aligned}$$

is terminating but this cannot be proved by any recursive path order since the rule $\mathsf{f}(x) \to \mathsf{g}(\underline{\mathsf{f}}(\mathsf{a}(x)))$ requires both $\mathsf{f} \succ \underline{\mathsf{f}}$ and $\mathsf{f} \succ \mathsf{a}$, whereas the rule $\mathsf{a}(\underline{\mathsf{f}}(x)) \to \mathsf{f}(x)$ requires either $\underline{\mathsf{f}} \succ \mathsf{f}$ or $\mathsf{a} \succ \mathsf{f}$.

Concerning incrementality, the results for Ferreira and Ribeiro's transformation are analogous to the ones for Zantema's transformation. Again, restricting the replacement map can make the termination proof of the transformed system harder. For the previous TRS $\mathscr{R}$, $\mathscr{R}_\mu^{\mathsf{FR}}$ only differs from $\mathscr{R}_\mu^{\mathsf{Z}}$ in that $\mathsf{a}(\underline{\mathsf{f}}(x)) \to \mathsf{f}(x)$ is replaced by the rules $\mathsf{a}(\underline{\mathsf{f}}(x)) \to \mathsf{f}(\mathsf{a}(x))$ and $\mathsf{a}(\mathsf{g}(x)) \to \mathsf{g}(x)$. Its termination proof succeeds with the same recursive path order used for $\mathscr{R}_\mu^{\mathsf{Z}}$. But again, since $\mathscr{R}_\nu^{\mathsf{Z}} \subseteq \mathscr{R}_\nu^{\mathsf{FR}}$, termination of $\mathscr{R}_\nu^{\mathsf{FR}}$ cannot be proved by any recursive path order.

In the rest of this section, we show that the two transformations introduced in this paper are incremental. The following two lemmata are needed in the incrementality proof of Θ_1 to simulate reductions of $\mathscr{R}_\nu^1$ by $\mathscr{R}_\mu^1$ if the replacement map ν is a restriction of the replacement map μ.

Lemma 43

Let $(\mathscr{R}, \mu)$ be a CSRS over a signature $\mathscr{F}$. For all terms $t \in \mathscr{T}(\mathscr{F}_1)$ we have $\mathsf{mark}(t) \to_1^+ t$.

Proof

The lemma is proved by induction on the structure of t. We distinguish three cases. First let $t = \mathsf{mark}(t')$. We obtain

$$\mathsf{mark}(t) = \mathsf{mark}(\mathsf{mark}(t')) \to_1^+ \mathsf{mark}(t') = t$$

by the induction hypothesis. Next let $t = f_{\mathsf{active}}(t_1, \ldots, t_n)$. We obtain

$$\mathsf{mark}(f_{\mathsf{active}}(t_1, \ldots, t_n)) \to_1 \mathsf{mark}(f(t_1, \ldots, t_n)) \to_1 f_{\mathsf{active}}\left([t_1]_1^f, \ldots, [t_n]_n^f\right)$$

If $i \in \mu(f)$ then $[t_i]^f_i = \mathsf{mark}(t_i) \to^+_1 t_i$ by the induction hypothesis. Otherwise, $i \notin \mu(f)$ and we directly obtain $[t_i]^f_i = t_i$. Hence the above reduction continues with $f_{\mathsf{active}}([t_1]^f_1, \dots, [t_n]^f_n) \to^*_1 f_{\mathsf{active}}(t_1, \dots, t_n) = t$. Finally, if $t = f(t_1, \dots, t_n)$ with $f \in \mathcal{F}$ then $\mathsf{mark}(t)$ reduces to $f(t_1, \dots, t_n)$ if $f \in \mathcal{F}_{\mathscr{C}}$ and to $f_{\mathsf{active}}(t_1, \dots, t_n)$ if $f \in \mathcal{F}_{\mathscr{D}}$ as in the previous case. Since $f_{\mathsf{active}}(t_1, \dots, t_n) \to_1 f(t_1, \dots, t_n) = t$, the claim is proved. □

Lemma 44

Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. For all terms $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and substitutions σ such that $t\sigma \in \mathcal{T}(\mathcal{F}_1)$ we have $\mathsf{mark}(t){\downarrow_{\mathcal{M}}}\sigma \to^*_1 t\sigma$.

Proof

We use induction on the structure of t. If t is a variable then $\mathsf{mark}(t){\downarrow_{\mathcal{M}}}\sigma = \mathsf{mark}(t\sigma) \to^+_1 t\sigma$ by Lemma 43. If $t = f(t_1, \dots, t_n)$ then

$$\begin{aligned} &\mathsf{mark}(f(t_1, \dots, t_n)){\downarrow_{\mathcal{M}}}\sigma = f(u_1\sigma, \dots, u_n\sigma) && \text{if } f \in \mathcal{F}_{\mathscr{C}} \\ &\mathsf{mark}(f(t_1, \dots, t_n)){\downarrow_{\mathcal{M}}}\sigma = f_{\mathsf{active}}(u_1\sigma, \dots, u_n\sigma) \to_1 f(u_1\sigma, \dots, u_n\sigma) && \text{if } f \in \mathcal{F}_{\mathscr{D}} \end{aligned}$$

Here, $u_i = \mathsf{mark}(t_i){\downarrow_{\mathcal{M}}}$ if $i \in \mu(f)$ and $u_i = t_i$ if $i \notin \mu(f)$. If $i \in \mu(f)$ then we obtain $u_i\sigma \to^*_1 t_i\sigma$ from the induction hypothesis. Hence $f(u_1\sigma, \dots, u_n\sigma) \to^*_1 t\sigma$. □

Now we are in a position to prove the incrementality of our first transformation.

Theorem 45

The transformation Θ_1 is incremental.

Proof

Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ with replacement maps μ and ν such that $\mathcal{R}^1_\mu$ is terminating and ν is a restriction of μ. It suffices to show that $s \to_{1_\nu} t$ implies $s \to^+_{1_\mu} t$ for all ground terms s and t. Without loss of generality we assume that $\mu \neq \nu$ and that the difference between them is minimal, i.e., $\mu(f) \setminus \nu(f) = \{i\}$ for some function symbol f and $1 \leqslant i \leqslant \mathrm{arity}(f)$, and $\mu(g) = \nu(g)$ for all other function symbols g. The difference between $\mathcal{R}^1_\mu$ and $\mathcal{R}^1_\nu$ is twofold. First of all, in $\mathcal{R}^1_\mu$ we have

$$\mathsf{mark}(f(x_1, \dots, x_n)) \to f'\big([x_1]^{f,\mu}_1, \dots, [x_n]^{f,\mu}_n\big)$$

with $[x_i]^{f,\mu}_i = \mathsf{mark}(x_i)$ and in $\mathcal{R}^1_\nu$ we have

$$\mathsf{mark}(f(x_1, \dots, x_n)) \to f'\big([x_1]^{f,\nu}_1, \dots, [x_n]^{f,\nu}_n\big)$$

with $[x_i]^{f,\nu}_i = x_i$ and $[x_j]^{f,\nu}_j = [x_j]^{f,\mu}_j$ for all other argument positions j. Here, $f' = f_{\mathsf{active}}$ if $f \in \mathcal{F}_{\mathscr{D}}$ and $f' = f$ if $f \in \mathcal{F}_{\mathscr{C}}$. If the reduction $s \to_{1_\nu} t$ was performed with this last rule then there is a position π in s such that $s|_\pi = \mathsf{mark}(f(t_1, \dots, t_n))$ and $t = s[f'([t_1]^{f,\nu}_1, \dots, t_i, \dots, [t_n]^{f,\nu}_n)]_\pi$. Note that $[t_i]^{f,\mu}_i = \mathsf{mark}(t_i) \to^+_{1_\mu} t_i$ by Lemma 43. Hence

$$\begin{aligned} s &\to_{1_\mu} s\big[f'\big([t_1]^{f,\mu}_1, \dots, \mathsf{mark}(t_i), \dots, [t_n]^{f,\mu}_n\big)\big]_\pi \\ &\to^+_{1_\mu} s\big[f'\big([t_1]^{f,\nu}_1, \dots, t_i, \dots, [t_n]^{f,\nu}_n\big)\big]_\pi = t \end{aligned}$$

The second difference between $\mathcal{R}^1_\mu$ and $\mathcal{R}^1_\nu$ is in the translation of the rules of $\mathcal{R}$:

$$g_{\mathsf{active}}(l_1, \dots, l_n) \to \mathsf{mark}(r){\downarrow_{\mathcal{M}_\mu}} = r_\mu$$

in $\mathscr{R}^1_\mu$ and

$$g_{\mathsf{active}}(l_1, \ldots, l_n) \to \mathsf{mark}(r){\downarrow_{\mathscr{M}_\nu}} = r_\nu$$

in $\mathscr{R}^1_\nu$. Suppose the reduction $s \to_{1_\nu} t$ was performed using one of the latter rules. So $s|_\pi = g_{\mathsf{active}}(l_1, \ldots, l_n)\sigma$ and $t = s[r_\nu\sigma]_\pi$ for some position π in s. We have $s \to_{1_\mu} s[r_\mu\sigma]_\pi$, so it suffices to show that $r_\mu\sigma \to^*_{1_\mu} r_\nu\sigma$. We do this by induction on r. If r is a variable then $r_\mu\sigma = r_\nu\sigma$. For the induction step we consider two cases. If $r = h(r_1, \ldots, r_m)$ with $f \neq h$ then $r_\mu\sigma = h'(s_1, \ldots, s_m)$ and $r_\nu\sigma = h'(t_1, \ldots, t_m)$ with $s_j = r_j\sigma = t_j$ if $j \notin \mu(h)$ and $s_j = \mathsf{mark}(r_j){\downarrow_{\mathscr{M}_\mu}}\sigma$ and $t_j = \mathsf{mark}(r_j){\downarrow_{\mathscr{M}_\nu}}\sigma$ if $j \in \mu(h)$. Moreover $h' = h_{\mathsf{active}}$ if $h \in \mathscr{F}_\mathscr{D}$ and $h' = h$ if $h \in \mathscr{F}_\mathscr{C}$. The induction hypothesis yields $s_j \to^*_{1_\mu} t_j$ for $j \in \mu(h)$ and thus $r_\mu\sigma \to^*_{1_\mu} r_\nu\sigma$. Finally, if $r = f(r_1, \ldots, r_n)$ then $r_\mu\sigma = f'(s_1, \ldots, s_n)$ and $r_\nu\sigma = f'(t_1, \ldots, t_n)$ with $s_j = r_j\sigma = t_j$ if $j \notin \mu(f)$, $s_j = \mathsf{mark}(r_j){\downarrow_{\mathscr{M}_\mu}}\sigma$ and $t_j = \mathsf{mark}(r_j){\downarrow_{\mathscr{M}_\nu}}\sigma$ if $j \in \mu(f)\backslash\{i\}$, and $s_i = \mathsf{mark}(r_i){\downarrow_{\mathscr{M}_\mu}}\sigma$ and $t_i = r_i\sigma$. The induction hypothesis yields $s_j \to^*_{1_\mu} t_j$ for $j \in \mu(f)\backslash\{i\}$ and Lemma 44 yields $s_i \to^*_{1_\mu} t_i$. Hence also in this case we obtain the desired $r_\mu\sigma \to^*_{1_\mu} r_\nu\sigma$. □

Incrementality of Θ_2 is an immediate consequence of the following, more general, result.

Theorem 46
Any sound and complete transformation from CSRSs to TRSs is incremental.

Proof
Let Θ be a sound and complete transformation from CSRSs to TRSs. Let $\mathscr{R}$ be a TRS over a signature $\mathscr{F}$ with replacement maps μ and ν such that $\Theta(\mathscr{R}, \mu)$ is terminating and ν is a restriction of μ. Soundness of Θ implies that $(\mathscr{R}, \mu)$ is a terminating CSRS. Since $\to_\nu$ is a restriction of $\to_\mu$, the CSRS $(\mathscr{R}, \nu)$ inherits termination from $(\mathscr{R}, \mu)$. Completeness of Θ yields the termination of $\Theta(\mathscr{R}, \nu)$. □

The results presented in this section also extend to termination modulo AC, i.e., both Θ'_1 and Θ_2 are incremental modulo AC. For Θ_2, the reason is that Theorem 46 carries over to context-sensitive rewriting modulo AC. For Θ'_1, the proof of Theorem 45 cannot be re-used directly. The problem is that we might have a restriction ν of the replacement map μ where $\nu(\mathsf{f}) = \varnothing$ and $\mu(\mathsf{f}) = \{1, 2\}$ for a defined AC-symbol f. Recall that in the transformation Θ'_1 not all mark-rules are used to normalize right-hand sides (one may not use $\mathsf{mark}(g(\cdots))$-rules for defined AC-symbols g with inactive arguments). For example, if we have a rule $\mathsf{a} \to \mathsf{f}(\mathsf{a}, \mathsf{a})$ in $\mathscr{R}$, then $\mathscr{R}^1_\nu$ would contain the rule $\mathsf{a}_{\mathsf{active}} \to \mathsf{mark}(\mathsf{f}(\mathsf{a}, \mathsf{a})){\downarrow_{\mathscr{M}'_\nu}} = \mathsf{mark}(\mathsf{f}(\mathsf{a}, \mathsf{a}))$ and $\mathscr{R}^1_\mu$ would contain $\mathsf{a}_{\mathsf{active}} \to \mathsf{mark}(\mathsf{f}(\mathsf{a}, \mathsf{a})){\downarrow_{\mathscr{M}'_\mu}} = \mathsf{f}_{\mathsf{active}}(\mathsf{a}_{\mathsf{active}}, \mathsf{a}_{\mathsf{active}})$. Thus, $s \to_{1'_\nu} t$ does not imply $s \to^+_{1'_\mu} t$. Instead it can be shown[5] that $s \to_{1'_\nu} t$ implies $s{\downarrow_{\mathscr{M}'_\mu}} \to^+_{1'_\mu} t{\downarrow_{\mathscr{M}'_\mu}}$ for all ground terms s and t.

A natural question is whether termination of $\Theta(\mathscr{R}, \mu)$ is equivalent to termination of $\mathscr{R}$ for the replacement map μ with $\mu(f) = \{1, \ldots, n\}$ for all n-ary function symbols f. For the five transformations studied in this paper this is indeed the case. Because of Figure 1 we only need to show this for Θ_L and Θ_Z. For Lucas' transformation

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[5] The proof can be found in Giesl & Middeldorp (2002).

this is trivial as $\Theta_{\mathsf{L}}(\mathscr{R},\mu) = \mathscr{R}$. We have $\Theta_{\mathsf{Z}}(\mathscr{R},\mu) = \mathscr{R} \cup \{\mathsf{a}(x) \to x\}$. Since a does not appear in $\mathscr{R}$, $\Theta_{\mathsf{Z}}(\mathscr{R},\mu)$ inherits termination from $\mathscr{R}$. For instance, Theorem 6 in (Middeldorp *et al.*, 1996) applies.

8 Conclusion

In this paper, we presented two new transformations from CSRSs to TRSs whose purpose is to reduce the problem of proving termination of CSRSs to the problem of proving termination of TRSs. So in particular, techniques for termination proofs of TRSs can now also be used to analyze the termination behavior of lazy functional programs which may be modeled by CSRSs. Our first transformation Θ_1 is simple, sound, and more powerful than all other transformations suggested in the literature. Our second transformation Θ_2 is not only sound but also complete, so it transforms every terminating CSRS into a terminating TRS.

Nevertheless, Θ_2 does not render the other (incomplete) transformations useless, since termination of $\mathscr{R}_\mu^2$ is often more difficult to prove than termination of the TRSs resulting from the other transformations. For instance, while Θ_1 transforms the CSRS in Example 16 into a TRS whose termination can easily be proved by the recursive path order, no recursive path order can prove termination of the TRS resulting from this CSRS by transformation Θ_2.

While we already introduced related transformations in a preliminary version of this paper (Giesl & Middeldorp, 1999), our second (complete) transformation has been simplified compared to its earlier definition and our first transformation has been modified to ease the termination proofs of the resulting transformed TRSs. In Giesl & Middeldorp (1999), instead of Θ_1 the following transformation was proposed.

Definition 47
Let $(\mathscr{R},\mu)$ be a CSRS over a signature $\mathscr{F}$. The TRS $\mathscr{R}_\mu^{1''}$ over the signature $\mathscr{F}_{1''} = \mathscr{F} \cup \{\mathsf{active},\mathsf{mark}\}$ consists of the following rewrite rules:

$$\begin{aligned} \mathsf{active}(l) &\to \mathsf{mark}(r) && \textit{for all } l \to r \in \mathscr{R} \\ \mathsf{mark}(f(x_1,\dots,x_n)) &\to \mathsf{active}\big(f\big([x_1]_1^f,\dots,[x_n]_n^f\big)\big) && \textit{for all } f \in \mathscr{F} \\ \mathsf{active}(x) &\to x \end{aligned}$$

Here $[t]_i^f = \mathsf{mark}(t)$ if $i \in \mu(f)$ and $[t]_i^f = t$ otherwise.

The following theorem states that the TRSs resulting from Θ_1 and Θ_1'' have the same termination behavior. The equivalence proof is given in Appendix C.

Theorem 48
Let $(\mathscr{R},\mu)$ be a CSRS. The TRS $\mathscr{R}_\mu^1$ is terminating if and only if $\mathscr{R}_\mu^{1''}$ is terminating.

However, while Θ_1 is just a different presentation of Θ_1'', termination of $\mathscr{R}_\mu^1$ is often significantly easier to prove than termination of $\mathscr{R}_\mu^{1''}$. For example, termination of the CSRSs in Examples 16 and 17 can easily be verified automatically by traditional simplification orders if Θ_1 is used, whereas Θ_1'' can only rarely be used in combination with such orders (as shown in Borralleras *et al.* 2002), confirming a claim of Giesl & Middeldorp (1999).[6]

Apart from the transformational approach, very recently some standard termination methods for term rewriting have been extended to apply directly to context-sensitive rewriting (Borralleras *et al.*, 2002; Gramlich & Lucas, 2002b). Direct approaches and transformational approaches both have their advantages. Techniques for proving termination of ordinary term rewriting have been extensively studied and with the transformational approach all termination techniques for ordinary term rewriting (including future developments) become available for context-sensitive rewriting as well. In particular, as long as the available techniques for direct termination analysis of context-sensitive rewriting are incomplete or semi-automatic, (complete) transformation methods are also useful since they offer additional possibilities for performing termination proofs. For instance, the method of Borralleras *et al.* (2002) cannot prove termination of the following example, whereas with our first transformation termination is easily proved (automatically).

Example 49

Consider the TRS $\mathscr{R}$

$$\begin{array}{rlrl}
0 - y \to 0 & & 0 \div \mathsf{s}(y) \to 0 & \\
\mathsf{s}(x) - \mathsf{s}(y) \to x - y & & \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{if}(x \geqslant y, \mathsf{s}((x-y) \div \mathsf{s}(y)), 0) & \\
x \geqslant 0 \to \mathsf{true} & & \mathsf{if}(\mathsf{true}, x, y) \to x & \\
0 \geqslant \mathsf{s}(y) \to \mathsf{false} & & \mathsf{if}(\mathsf{false}, x, y) \to y & \\
\mathsf{s}(x) \geqslant \mathsf{s}(y) \to x \geqslant y & & &
\end{array}$$

This example shows that context-sensitive rewriting can also be used to simulate the usual evaluation strategy for " if*". To this end, we define* $\mu(\mathsf{if}) = \{1\}$*. This ensures that in an* if*-term, the condition is evaluated first and depending on the result of the evaluation either the second or the third argument is evaluated afterwards. Moreover, we define* $\mu(\mathsf{s}) = \mu(\div) = \{1\}$ *and* $\mu(f) = \varnothing$ *for all other function symbols* f*. So* μ *is the most restrictive replacement map ensuring that defined symbols on right-hand sides would be on active positions if all arguments of "* if*" were active. This replacement map permits all evaluations which are performed in an eager functional language when starting with a term* $f(t_1, \ldots, t_n)$ *where* f *is applied to "data objects" (i.e., the terms* t_i *are constructor ground terms). In such languages, a term* $f(\cdots)$ *with* $f \neq \mathsf{if}$ *may only be reduced at root position if all its arguments are constructor ground terms. The termination of* $\mathscr{R}_\mu^1$ *is easily proved (see Appendix D).*

In addition to the modifications of the transformations, the present article extends (Giesl & Middeldorp, 1999) by numerous significant new results. While in Giesl & Middeldorp (1999) it remained open whether our first transformation is really more

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[6] The main traditional techniques for automated termination proofs of TRSs are simplification orders like the recursive path order, the Knuth-Bendix order, and (most) polynomial orders. For instance, when using Θ_1'' in Example 16, termination cannot be proved by these techniques and in Example 17, termination cannot even be proved by any simplification order. The reason is that active(zeros) can be reduced to the term active(0 : zeros) in which it is embedded.

powerful than the one of Zantema (1997), we now gave a proof for this claim. We also included a comparison with the technique of Ferreira & Ribeiro (1999), which was developed independently and in parallel to Giesl & Middeldorp (1999). To this end, we showed that already our first transformation is more powerful than the one of Ferreira & Ribeiro (1999). In addition, we proved that Ferreira and Ribeiro's transformation is more powerful than Zantema's transformation. In this way, now the relationship between all existing transformation techniques for context-sensitive rewriting has been clarified. Finally, all observations presented in Sections 6 and 7 are new. In Section 6 we showed that our results also hold for termination of context-sensitive rewriting modulo AC and in Section 7 we prove that in contrast to all other transformation techniques, our transformations behave naturally when restricting the replacement map of context-sensitive rewriting.

As a final remark we mention that, inspired by work of Fokkink *et al.* (2000), recently Lucas (2001a) introduced an extension of context-sensitive rewriting called *on-demand* rewriting which is characterized by two replacement maps. He showed that the two transformations of the preliminary version of this paper (Giesl & Middeldorp, 1999) can also be extended to on-demand rewriting.

Acknowledgements

We thank Salvador Lucas, Hans Zantema, and the anonymous referees for many helpful remarks.

A Proofs for Section 3

Before giving the proofs of Lemmata 9 and 10 we present two useful properties of the mappings Φ and Φ'.

Lemma 50
For all terms $t \in \mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$ *we have* $\Phi'(t) \to_{\mathsf{Z}}^* \Phi(t)$.

Proof
We distinguish three cases. If $t = f(t_1,\ldots,t_n)$ with $f \in \mathcal{F}$ or $t = \underline{f}(t_1,\ldots,t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}} \backslash \mathcal{F}_\mu^{\mathsf{Z}}$ then $\Phi'(t) = f(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f) = \Phi(t)$. If $t = \underline{f}(t_1,\ldots,t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{Z}}$ then $\Phi'(t) = f(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f)$ and $\Phi(t) = \underline{f}(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f)$. In this case we obtain $\Phi'(t) \to_{\mathsf{Z}} \Phi(t)$ because $\mathcal{R}_\mu^{\mathsf{Z}}$ contains the rewrite rule $f(x_1,\ldots,x_n) \to \underline{f}(x_1,\ldots,x_n)$ as $\underline{f} \in \mathcal{F}_\mu^{\mathsf{Z}}$. Finally, if $t = \mathsf{a}(t')$ then $\Phi'(t) = \Phi'(t') = \Phi(t)$. □

Lemma 51
For all terms $t \in \mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$ *we have* $\mathsf{a}(\Phi(t)) \to_{\mathsf{Z}} \Phi'(t)$.

Proof
Again we distinguish three cases. If $t = f(t_1,\ldots,t_n)$ with $f \in \mathcal{F}$ or $t = \underline{f}(t_1,\ldots,t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}} \backslash \mathcal{F}_\mu^{\mathsf{Z}}$ then $\Phi(t) = f(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f) = \Phi'(t)$ and thus $\mathsf{a}(\Phi(t)) \to_{\mathsf{Z}} \Phi'(t)$ by applying the rule $\mathsf{a}(x) \to x$. If $t = \underline{f}(t_1,\ldots,t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{Z}}$ then $\Phi(t) = \underline{f}(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f)$ and $\Phi'(t) = f(\langle t_1\rangle_1^f,\ldots,\langle t_n\rangle_n^f)$. Because $\underline{f} \in \mathcal{F}_\mu^{\mathsf{Z}}$, $\mathcal{R}_\mu^{\mathsf{Z}}$ contains the rule $\mathsf{a}(\underline{f}(x_1,\ldots,x_n)) \to f(x_1,\ldots,x_n)$. Hence $\mathsf{a}(\Phi(t)) \to_{\mathsf{Z}} \Phi'(t)$. Finally, if $t = \mathsf{a}(t')$ then $\mathsf{a}(\Phi(t)) = \mathsf{a}(\Phi'(t')) \to_{\mathsf{Z}} \Phi'(t') = \Phi'(\mathsf{a}(t')) = \Phi'(t)$ by applying the rule $\mathsf{a}(x) \to x$. □

Lemma 9
For all terms $s, t \in \mathscr{T}(\mathscr{F}_\mu^{\mathsf{FR}})$, if $s \to_{\mathsf{FR}_1} t$ then $\Phi(s) \to_{\mathsf{Z}}^+ \Phi(t)$.

Proof
Let $s = C[l\sigma] \to_{\mathsf{FR}_1} C[r\sigma] = t$ with $l \to r \in \mathscr{R}_\mu^{\mathsf{FR}_1}$. We have $\Phi(s) = C'[\Phi(l\sigma)]$ or $\Phi(s) = C'[\Phi'(l\sigma)]$ for some context C'. Likewise, $\Phi(t) = C'[\Phi(r\sigma)]$ or $\Phi(t) = C'[\Phi'(r\sigma)]$. Since $\Phi'(l\sigma) \to_{\mathsf{Z}}^* \Phi(l\sigma)$ and $\Phi'(r\sigma) \to_{\mathsf{Z}}^* \Phi(r\sigma)$ by Lemma 50, it is sufficient to prove $\Phi(l\sigma) \to_{\mathsf{Z}}^+ \Phi'(r\sigma)$. Let $l_{\mathsf{Z}} \to r_{\mathsf{Z}}$ be the rewrite rule in $\mathscr{R}_\mu^{\mathsf{Z}}$ corresponding to $l \to r \in \mathscr{R}_\mu^{\mathsf{FR}_1}$. Define the substitution σ_Φ as follows:

$$\sigma_\Phi(x) = \begin{cases} \Phi(\sigma(x)) & \text{if } x \text{ (also) occurs at an inactive position in } l, \\ \Phi'(\sigma(x)) & \text{otherwise.} \end{cases}$$

One might expect that $\Phi(l\sigma) = l_{\mathsf{Z}}\sigma_\Phi$ holds, but if a variable x occurs both at an active and an inactive position in l then in $\Phi(l\sigma)$ the two occurrences of $\sigma(x)$ are replaced by $\Phi'(\sigma(x))$ and $\Phi(\sigma(x))$, respectively, so $\Phi(l\sigma)$ need not be an instance of l_{Z}. However, because $\Phi'(\sigma(x)) \to_{\mathsf{Z}}^* \Phi(\sigma(x))$ by Lemma 50 and because σ_Φ instantiates all occurrences of such variables x in l_{Z} by $\Phi(\sigma(x))$, it follows that

$$\Phi(l\sigma) \to_{\mathsf{Z}}^* l_{\mathsf{Z}}\sigma_\Phi.$$

This can be formally proved as follows. Let us extend Φ and Φ' to terms with variables by defining $\Phi(x) = \Phi'(x) = x$ for every variable x. Note that $l_{\mathsf{Z}} = \Phi(l) = \Phi'(l)$. Hence it suffices to show $\Phi(l\sigma) \to_{\mathsf{Z}}^* \Phi(l)\sigma_\Phi$. This follows from the first part of the following statement, which we prove by induction on the structure of $t \in \mathscr{T}(\mathscr{F}_\mu^{\mathsf{FR}}, \mathscr{V})$:

- $\Phi(t\sigma) \to_{\mathsf{Z}}^* \Phi(t)\sigma_\Phi$ for all non-variable subterms t of l, and
- $\Phi'(t\sigma) \to_{\mathsf{Z}}^* \Phi'(t)\sigma_\Phi$ for all subterms t of l.

If $t \in \mathscr{V}$ then $\Phi'(t)\sigma_\Phi = \sigma_\Phi(t)$. If $\sigma_\Phi(t) = \Phi(t\sigma)$ then we obtain $\Phi'(t\sigma) \to_{\mathsf{Z}}^* \Phi'(t)\sigma_\Phi$ from Lemma 50 and if $\sigma_\Phi(t) = \Phi'(t\sigma)$ then $\Phi'(t\sigma) = \Phi'(t)\sigma_\Phi$. Suppose $t = f(t_1, \ldots, t_n)$ or $t = \underline{f}(t_1, \ldots, t_n)$. We have $\Phi'(t\sigma) = f(\langle t_1\sigma\rangle_1^f, \ldots, \langle t_n\sigma\rangle_n^f)$, $\Phi'(t) = f(\langle t_1\rangle_1^f, \ldots, \langle t_n\rangle_n^f)$, and either $\Phi(t\sigma) = \Phi'(t\sigma)$ and $\Phi(t) = \Phi'(t)$ or $\Phi(t\sigma) = \underline{f}(\langle t_1\sigma\rangle_1^f, \ldots, \langle t_n\sigma\rangle_n^f)$ and $\Phi(t) = \underline{f}(\langle t_1\rangle_1^f, \ldots, \langle t_n\rangle_n^f)$. So it suffices to show that $\langle t_i\sigma\rangle_i^f \to_{\mathsf{Z}}^* \langle t_i\rangle_i^f\sigma_\Phi$. We distinguish two cases. If $i \in \mu(f)$ then $\langle t_i\sigma\rangle_i^f = \Phi'(t_i\sigma)$ and $\langle t_i\rangle_i^f = \Phi'(t_i)$. Hence $\langle t_i\sigma\rangle_i^f \to_{\mathsf{Z}}^* \langle t_i\rangle_i^f\sigma_\Phi$ follows from the second part of the induction hypothesis. If $i \notin \mu(f)$ then $\langle t_i\sigma\rangle_i^f = \Phi(t_i\sigma)$ and $\langle t_i\rangle_i^f = \Phi(t_i)$. If $t_i \notin \mathscr{V}$ then we obtain $\langle t_i\sigma\rangle_i^f \to_{\mathsf{Z}}^* \langle t_i\rangle_i^f\sigma_\Phi$ from the first part of the induction hypothesis. If $t_i \in \mathscr{V}$ then t_i occurs at an inactive position in l since t is a subterm of l and $i \notin \mu(f)$, and thus $\langle t_i\rangle_i^f\sigma_\Phi = \sigma_\Phi(t_i) = \Phi(t_i\sigma)$.

Combining $\Phi(l\sigma) \to_{\mathsf{Z}}^* l_{\mathsf{Z}}\sigma_\Phi$ with $l_{\mathsf{Z}}\sigma_\Phi \to_{\mathsf{Z}} r_{\mathsf{Z}}\sigma_\Phi$ yields $\Phi(l\sigma) \to_{\mathsf{Z}}^+ r_{\mathsf{Z}}\sigma_\Phi$. To conclude the proof it remains to show that $r_{\mathsf{Z}}\sigma_\Phi \to_{\mathsf{Z}}^* \Phi'(r\sigma)$. Let us define r'_{Z} as the term obtained from r_{Z} by replacing every subterm $\mathsf{a}(t)$ by t. Note that $\Phi(r) = \Phi'(r) = r'_{\mathsf{Z}}$. We may write $r'_{\mathsf{Z}} = D[x_1, \ldots, x_n]$ with all occurrences of variables displayed and $r_{\mathsf{Z}} = D[x'_1, \ldots, x'_n]$ with $x'_i = \mathsf{a}(x_i)$ if x_i occurs at an inactive position in l and $x'_i = x_i$ if x_i occurs only at active positions in l. We have $r_{\mathsf{Z}}\sigma_\Phi = D[t_1, \ldots, t_n]$ with $t_i = \mathsf{a}(\Phi(\sigma(x_i)))$ if x_i occurs at an inactive position in l and $t_i = \Phi'(\sigma(x_i))$ if x_i occurs only at active positions in l. Moreover, $\Phi'(r\sigma) = D[u_1, \ldots, u_n]$ with $u_i \in \{\Phi'(\sigma(x_i)), \Phi(\sigma(x_i))\}$. We have $\mathsf{a}(\Phi(\sigma(x_i))) \to_{\mathsf{Z}} \Phi'(\sigma(x_i)) \to_{\mathsf{Z}}^* \Phi(\sigma(x_i))$ by Lemmata 50 and 51, and thus $t_i \to_{\mathsf{Z}}^* u_i$. Hence $r_{\mathsf{Z}}\sigma_\Phi \to_{\mathsf{Z}}^* \Phi'(r\sigma)$ as desired. □

Lemma 10
For all terms $s, t \in \mathcal{T}(\mathcal{F}_\mu^{\mathsf{FR}})$, *if* $s \to_{\mathsf{FR}_2} t$ *then* $\Phi(s) \to_{\mathsf{Z}}^* \Phi(t)$.

Proof
Let $s = C[l\sigma] \to_{\mathsf{FR}_2} C[r\sigma] = t$ with $l \to r \in \mathcal{R}_\mu^{\mathsf{FR}_2}$. As in the proof of Lemma 9, $\Phi(s)$ is $C'[\Phi(l\sigma)]$ or $C'[\Phi'(l\sigma)]$ and $\Phi(t)$ is $C'[\Phi(r\sigma)]$ or $C'[\Phi'(r\sigma)]$ for some context C'. Since $\Phi'(l\sigma) \to_{\mathsf{Z}}^* \Phi(l\sigma)$ and $\Phi'(r\sigma) \to_{\mathsf{Z}}^* \Phi(r\sigma)$ by Lemma 50, it is sufficient to prove $\Phi(l\sigma) = \Phi'(r\sigma)$. We distinguish four cases corresponding to the four different types of rules in $\mathcal{R}_\mu^{\mathsf{FR}_2}$.

(i) If $l \to r = \mathsf{a}(x) \to x$ then $\Phi(l\sigma) = \Phi'(\sigma(x)) = \Phi'(r\sigma)$.

(ii) If $l \to r = f(x_1, \dots, x_n) \to \underline{f}(x_1, \dots, x_n)$ then

$$\Phi(l\sigma) = f\left(\langle \sigma(x_1)\rangle_1^f, \dots, \langle \sigma(x_n)\rangle_n^f\right) = \Phi'(r\sigma).$$

(iii) If $l \to r = \mathsf{a}(\underline{f}(x_1, \dots, x_n)) \to f([\![x_1]\!]_1^f, \dots, [\![x_n]\!]_n^f)$ then

$$\Phi(l\sigma) = f\left(\langle \sigma(x_1)\rangle_1^f, \dots, \langle \sigma(x_n)\rangle_n^f\right)$$

and

$$\Phi'(r\sigma) = f\left(\langle [\![\sigma(x_1)]\!]_1^f\rangle_1^f, \dots, \langle [\![\sigma(x_n)]\!]_n^f\rangle_n^f\right).$$

Note that if $i \in \mu(f)$ then $\langle [\![\sigma(x_i)]\!]_i^f\rangle_i^f = \Phi'(\mathsf{a}(\sigma(x_i))) = \Phi'(\sigma(x_i)) = \langle \sigma(x_i)\rangle_i^f$ and if $i \notin \mu(f)$ then $\langle [\![\sigma(x_i)]\!]_i^f\rangle_i^f = \Phi(\sigma(x_i)) = \langle \sigma(x_i)\rangle_i^f$, so $\Phi(l\sigma) = \Phi'(r\sigma)$.

(iv) If $l \to r = \mathsf{a}(f(x_1, \dots, x_n)) \to f([\![x_1]\!]_1^f, \dots, [\![x_n]\!]_n^f)$ then we obtain $\Phi(l\sigma) = \Phi'(r\sigma)$ exactly as in the previous case.

□

B Proofs for Section 4

Next we turn our attention to Lemmata 20 and 21. We start by proving two useful properties of the mappings Ψ and Ψ'.

Lemma 52
For all terms $t \in \mathcal{T}(\mathcal{F}_1)$ *we have* $\Psi(t) \to_{\mathsf{FR}'}^* \Psi'(t)$.

Proof
We distinguish three cases. If $t = f(t_1, \dots, t_n)$ with $\underline{f} \notin \mathcal{F}_\mu^{\mathsf{FR}}$ or $t = f_{\mathsf{active}}(t_1, \dots, t_n)$ then $\Psi(t) = f(\langle t_1\rangle_1^f, \dots, \langle t_n\rangle_n^f) = \Psi'(t)$. If $t = f(t_1, \dots, t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}}$ then $\Psi(t) = f(\langle t_1\rangle_1^f, \dots, \langle t_n\rangle_n^f)$ and $\Psi'(t) = \underline{f}(\Psi'(t_1), \dots, \Psi'(t_n))$. Because $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}}$, $f(x_1, \dots, x_n) \to \underline{f}(x_1, \dots, x_n) \in \mathcal{R}_\mu^{\mathsf{FR}'}$ and thus $\Psi(t) \to_{\mathsf{FR}'} \underline{f}(\langle t_1\rangle_1^f, \dots, \langle t_n\rangle_n^f)$. Let $i \in \{1, \dots, n\}$. If $i \in \mu(f)$ then $\langle t_i\rangle_i^f = \Psi(t_i) \to_{\mathsf{FR}'}^* \Psi'(t_i)$ by the induction hypothesis. If $i \notin \mu(f)$ then $\langle t_i\rangle_i^f = \Psi'(t_i)$. Hence $\Psi(t) \to_{\mathsf{FR}'}^* \Psi'(t)$ as desired. Finally, if $t = \mathsf{mark}(t')$ then $\Psi(t) = \Psi(t') = \Psi'(t)$. □

Lemma 53
For all terms $t \in \mathcal{T}(\mathcal{F}_1)$ *we have* $\mathsf{a}(\Psi'(t)) \to_{\mathsf{FR}'}^+ \Psi(t)$.

Proof

Again we distinguish three cases. If $t = f(t_1,\dots,t_n)$ with $\underline{f} \notin \mathcal{F}_\mu^{\mathsf{FR}}$ or if $t = f_{\mathsf{active}}(t_1,\dots,t_n)$ then $\Psi'(t) = f(\langle t_1\rangle_1^f,\dots,\langle t_n\rangle_n^f) = \Psi(t)$. We have $\mathsf{a}(\Psi'(t)) \to_{\mathsf{FR}'} \Psi(t)$ by applying the rule $\mathsf{a}(x) \to x$. If $t = f(t_1,\dots,t_n)$ with $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}}$ then $\mathsf{a}(\Psi'(t)) = \mathsf{a}(\underline{f}(\Psi'(t_1),\dots,\Psi'(t_n)))$ and $\Psi(t) = f(\langle t_1\rangle_1^f,\dots,\langle t_n\rangle_n^f)$. Because $\underline{f} \in \mathcal{F}_\mu^{\mathsf{FR}}$, $\mathcal{R}_\mu^{\mathsf{FR}'}$ contains the rule $\mathsf{a}(\underline{f}(x_1,\dots,x_n)) \to f([\![x_1]\!]_1^f,\dots,[\![x_n]\!]_n^f)$. Hence $\mathsf{a}(\Psi'(t)) \to_{\mathsf{FR}'} f([\![\Psi'(t_1)]\!]_1^f,\dots,[\![\Psi'(t_n)]\!]_n^f)$. So it suffices to show that $[\![\Psi'(t_i)]\!]_i^f \to_{\mathsf{FR}'}^* \langle t_i\rangle_i^f$ for all i. If $i \in \mu(f)$ then $[\![\Psi'(t_i)]\!]_i^f = \mathsf{a}(\Psi'(t_i)) \to_{\mathsf{FR}'}^+ \Psi(t_i) = \langle t_i\rangle_i^f$ by the induction hypothesis and if $i \notin \mu(f)$ then $[\![\Psi'(t_i)]\!]_i^f = \Psi'(t_i) = \langle t_i\rangle_i^f$. Finally, if $t = \mathsf{mark}(t')$ then again, $\Psi'(t) = \Psi(t)$ and thus, $\mathsf{a}(\Psi'(t)) \to_{\mathsf{FR}'} \Psi(t)$ by applying the rule $\mathsf{a}(x) \to x$. □

Lemma 20

For all terms $s, t \in \mathcal{T}(\mathcal{F}_1)$, if $s \to_{\mathcal{M}_1} t$ then $\Psi(s) \to_{\mathsf{FR}'}^+ \Psi(t)$.

Proof

Let $s = C[l\sigma] \to C[r\sigma] = t$ with $l \to r \in \mathcal{M}_1$. We have $\Psi(s) = C'[\Psi(l\sigma)]$ or $\Psi(s) = C'[\Psi'(l\sigma)]$ for some context C'. Likewise, $\Psi(t) = C'[\Psi(r\sigma)]$ or $\Psi(t) = C'[\Psi'(r\sigma)]$. Let $l = f_{\mathsf{active}}(l_1,\dots,l_n) \to \mathsf{mark}(r')\downarrow_{\mathcal{M}} = r$ and let $l_{\mathsf{FR}} \to r_{\mathsf{FR}}$ be the corresponding rewrite rule in $\mathcal{R}_\mu^{\mathsf{FR}'}$. We clearly have $\Psi(l\sigma) = \Psi'(l\sigma)$. Lemma 52 yields $\Psi(r\sigma) \to_{\mathsf{FR}'}^* \Psi'(r\sigma)$. Hence, it is sufficient to prove $\Psi(l\sigma) \to_{\mathsf{FR}'}^+ \Psi(r\sigma)$. We prove that $\Psi(r\sigma) = \Psi(r'\sigma)$ by induction on r'. If r' is a variable then $r\sigma = \mathsf{mark}(r'\sigma)$ and thus $\Psi(r\sigma) = \Psi(r'\sigma)$. If $r' = g(r_1,\dots,r_m)$ then $\Psi(r\sigma) = \Psi(g'(u_1,\dots,u_m)) = g(\langle u_1\rangle_1^g,\dots,\langle u_m\rangle_m^g)$, where $g' = g_{\mathsf{active}}$ if $g \in \mathcal{F}_{\mathcal{D}}$ and $g' = g$ if $g \in \mathcal{F}_{\mathcal{C}}$. Here, $u_i = \mathsf{mark}(r_i)\downarrow_{\mathcal{M}}\sigma$ if $i \in \mu(g)$ and $u_i = r_i\sigma$ if $i \notin \mu(g)$. Moreover $\Psi(r'\sigma) = g(\langle r_1\sigma\rangle_1^g,\dots,\langle r_m\sigma\rangle_m^g)$. The induction hypothesis yields $\langle u_i\rangle_i^g = \Psi(\mathsf{mark}(r_i)\downarrow_{\mathcal{M}}\sigma) = \Psi(r_i\sigma) = \langle r_i\sigma\rangle_i^g$ for $i \in \mu(g)$. If $i \notin \mu(g)$ then $\langle u_i\rangle_i^g = \Psi'(r_i\sigma) = \langle r_i\sigma\rangle_i^g$. It follows that $\Psi(r\sigma) = \Psi(r'\sigma)$. We will now show that $\Psi(l\sigma) \to_{\mathsf{FR}'}^+ \Psi(r'\sigma)$. Define the substitution σ_Ψ as follows:

$$\sigma_\Psi(x) = \begin{cases} \Psi'(\sigma(x)) & \text{if } x \text{ (also) occurs at an inactive position in } l, \\ \Psi(\sigma(x)) & \text{otherwise.} \end{cases}$$

Here, we extend μ by defining $\mu(f_{\mathsf{active}}) = \mu(f)$. One might expect that $\Psi(l\sigma) = l_{\mathsf{FR}}\sigma_\Psi$ holds, but if a variable x occurs both at an active and an inactive position in l then in $\Psi(l\sigma)$ the two occurrences of $\sigma(x)$ are replaced by $\Psi(\sigma(x))$ and $\Psi'(\sigma(x))$, respectively, so $\Psi(l\sigma)$ need not be an instance of l_{FR}. (Note that the second case in the definition of $\Psi'(f(t_1,\dots,t_n))$ is never applicable when applied to subterms $f(t_1,\dots,t_n)$ of l during the computation of $\Psi(l)$.) However, because $\Psi(\sigma(x)) \to_{\mathsf{FR}'}^* \Psi'(\sigma(x))$ by Lemma 52 and because σ_Ψ instantiates all occurrences of such variables x in l_{FR} by $\Psi'(\sigma(x))$, it follows that

$$\Psi(l\sigma) \to_{\mathsf{FR}'}^* l_{\mathsf{FR}}\sigma_\Psi.$$

This can be formally proved as follows. Let us extend Ψ and Ψ' to terms with variables by defining $\Psi(x) = \Psi'(x) = x$ for every variable x. Note that $l_{\mathsf{FR}} = \Psi(l)$. Hence it suffices to show $\Psi(l\sigma) \to_{\mathsf{FR}'}^* \Psi(l)\sigma_\Psi$. This follows from the first part of the following statement, which we prove by induction on the structure of $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$:

- $\Psi(t\sigma) \to^*_{\mathsf{FR}'} \Psi(t)\sigma_\Psi$ for all subterms t of l, and
- $\Psi'(t\sigma) \to^*_{\mathsf{FR}'} \Psi'(t)\sigma_\Psi$ for all subterms t at inactive positions in l.

If $t \in \mathcal{V}$ then $\Psi(t)\sigma_\Psi = \sigma_\Psi(t)$. If $\sigma_\Psi(t) \neq \Psi(t\sigma)$ then $\sigma_\Psi(t) = \Psi'(t\sigma)$ and thus we obtain $\Psi(t\sigma) \to^*_{\mathsf{FR}'} \Psi(t)\sigma_\Psi$ from Lemma 52. For the second statement we assume that t appears at an inactive position in l. So $\Psi'(t)\sigma_\Psi = \sigma_\Psi(t) = \Psi'(t\sigma)$. Note that no subterm t of l contains mark. So in the remaining case we have $t = f(t_1,\ldots,t_n)$ or $t = f_{\mathsf{active}}(t_1,\ldots,t_n)$. We obtain $\Psi(t\sigma) = f(\langle t_1\sigma\rangle^f_1,\ldots,\langle t_n\sigma\rangle^f_n)$ and $\Psi(t) = f(\langle t_1\rangle^f_1,\ldots,\langle t_n\rangle^f_n)$, so to conclude the first statement it suffices to show that $\langle t_i\sigma\rangle^f_i \to^*_{\mathsf{FR}'} \langle t_i\rangle^f_i\sigma_\Psi$. We distinguish two cases. If $i \in \mu(f)$ then $\langle t_i\sigma\rangle^f_i = \Psi(t_i\sigma)$ and $\langle t_i\rangle^f_i = \Psi(t_i)$. Hence $\langle t_i\sigma\rangle^f_i \to^*_{\mathsf{FR}'} \langle t_i\rangle^f_i\sigma_\Psi$ follows from the first part of the induction hypothesis. If $i \notin \mu(f)$ then $\langle t_i\sigma\rangle^f_i = \Psi'(t_i\sigma)$ and $\langle t_i\rangle^f_i = \Psi'(t_i)$. Note that t_i occurs at an inactive position in l since t is a subterm of l and $i \notin \mu(f)$. Thus, we obtain $\langle t_i\sigma\rangle^f_i \to^*_{\mathsf{FR}'} \langle t_i\rangle^f_i\sigma_\Psi$ from the second part of the induction hypothesis. For the second statement we reason as follows. Since t appears at an inactive position in l, we have $\underline{f} \in \mathcal{F}^{\mathsf{FR}}_\mu$ and hence $\Psi'(t\sigma) = \underline{f}(\Psi'(t_1\sigma),\ldots,\Psi'(t_n\sigma))$ and $\Psi'(t) = \underline{f}(\Psi'(t_1),\ldots,\Psi'(t_n))$. All subterms of t occur at inactive positions in l and thus $\Psi'(t_i\sigma) \to^*_{\mathsf{FR}'} \Psi'(t_i)\sigma_\Psi$ for all i by the induction hypothesis. Consequently, $\Psi'(t\sigma) \to^*_{\mathsf{FR}'} \Psi'(t)\sigma_\Psi$ as desired.

Combining $\Psi(l\sigma) \to^*_{\mathsf{FR}'} l_{\mathsf{FR}}\sigma_\Psi$ with $l_{\mathsf{FR}}\sigma_\Psi \to_{\mathsf{FR}'} r_{\mathsf{FR}}\sigma_\Psi$ yields $\Psi(l\sigma) \to^+_{\mathsf{FR}'} r_{\mathsf{FR}}\sigma_\Psi$. To conclude the proof of the lemma it remains to show that $r_{\mathsf{FR}}\sigma_\Psi \to^*_{\mathsf{FR}'} \Psi(r'\sigma)$. Let us define r'_{FR} as the term obtained from r_{FR} by replacing every subterm $\mathsf{a}(t)$ by t. Note that $r'_{\mathsf{FR}} = \Psi(r')$. We may write $r'_{\mathsf{FR}} = D[x_1,\ldots,x_n]$ with all occurrences of variables displayed and $r_{\mathsf{FR}} = D[x'_1,\ldots,x'_n]$ with $x'_i = \mathsf{a}(x_i)$ if x_i occurs at an inactive position in l and $x'_i = x_i$ if x_i occurs only at active positions in l. We have $r_{\mathsf{FR}}\sigma_\Psi = D[t_1,\ldots,t_n]$ with $t_i = \mathsf{a}(\Psi'(\sigma(x_i)))$ if x_i occurs at an inactive position in l and $t_i = \Psi(\sigma(x_i))$ if x_i occurs only at active positions in l. Moreover, $\Psi(r'\sigma) = D[u_1,\ldots,u_n]$ with $u_i \in \{\Psi(\sigma(x_i)), \Psi'(\sigma(x_i))\}$. We have $\mathsf{a}(\Psi'(\sigma(x_i))) \to^+_{\mathsf{FR}'} \Psi(\sigma(x_i)) \to^*_{\mathsf{FR}'} \Psi'(\sigma(x_i))$ by Lemmata 52 and 53. Hence $t_i \to^*_{\mathsf{FR}'} u_i$ and thus $r_{\mathsf{FR}}\sigma_\Psi \to^*_{\mathsf{FR}'} \Psi(r'\sigma)$. □

Lemma 21
For all terms $s,t \in \mathcal{T}(\mathcal{F}_1)$, *if* $s \to_{\mathcal{M}_2} t$ *then* $\Psi(s) \to^*_{\mathsf{FR}'} \Psi(t)$.

Proof
Let $s = C[l\sigma] \to C[r\sigma] = t$ with $l \to r \in \mathcal{M}_2$. As in the proof of Lemma 20, $\Psi(s) = C'[\Psi(l\sigma)]$ or $\Psi(s) = C'[\Psi'(l\sigma)]$ and $\Psi(t) = C'[\Psi(r\sigma)]$ or $\Psi(t) = C'[\Psi'(r\sigma)]$ for some context C'. Since $\Psi(l\sigma) = \Psi'(l\sigma)$ and $\Psi(r\sigma) \to^*_{\mathsf{FR}'} \Psi'(r\sigma)$ by Lemma 52, it is sufficient to prove $\Psi(l\sigma) = \Psi(r\sigma)$. We distinguish two cases corresponding to the different types of rules in $\mathcal{M}_2$.

(i) If $l \to r = f_{\mathsf{active}}(x_1,\ldots,x_n) \to f(x_1,\ldots,x_n)$ then $\Psi(l\sigma) = \Psi(r\sigma)$.

(ii) Let $l \to r = \mathsf{mark}(f(x_1,\ldots,x_n)) \to f'([x_1]^f_1,\ldots,[x_n]^f_i)$ with $f' \in \{f_{\mathsf{active}}, f\}$. We have $\Psi(l\sigma) = f(\langle\sigma(x_1)\rangle^f_1,\ldots,\langle\sigma(x_n)\rangle^f_n)$ and $\Psi(r\sigma) = f(\langle[\sigma(x_1)]^f_1\rangle^f_1,\ldots,\langle[\sigma(x_n)]^f_n\rangle^f_n)$. Note that if $i \in \mu(f)$ then $\langle\sigma(x_i)\rangle^f_i = \Psi(\sigma(x_i)) = \Psi(\mathsf{mark}(\sigma(x_i))) = \langle[\sigma(x_i)]^f_i\rangle^f_i$ and if $i \notin \mu(f)$ then $\langle\sigma(x_i)\rangle^f_i = \Psi'(\sigma(x_i)) = \langle[\sigma(x_i)]^f_i\rangle^f_i$. Hence $\Psi(l\sigma) = \Psi(r\sigma)$.

□

C Proofs for Section 8

Theorem 48

Let $(\mathscr{R},\mu)$ be a CSRS. The TRS $\mathscr{R}^1_\mu$ is terminating if and only if $\mathscr{R}^{1''}_\mu$ is terminating.

Proof

For the “only if” direction we show that if $s \to_{1''} t$ with $s, t \in \mathscr{T}(\mathscr{F}_{1''})$ by an application of a rule $\mathsf{active}(l) \to \mathsf{mark}(r)$ in $\mathscr{R}^{1''}_\mu$ then $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} \to_1^+ t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$. Moreover, if $s \to_{1''} t$ by applying one of the other rules in $\mathscr{R}^{1''}_\mu$ then $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} \to_1^* t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$. Here $\mathscr{A}$ is the (terminating and confluent) rewrite system consisting of the following rules:

$$\begin{array}{rcll}
\mathsf{active}(f(x_1,\dots,x_n)) & \to & f_{\mathsf{active}}(x_1,\dots,x_n) & \text{for all } f \in \mathscr{F}_{\mathscr{D}} \\
\mathsf{active}(f(x_1,\dots,x_n)) & \to & f(x_1,\dots,x_n) & \text{for all } f \in \mathscr{F}_{\mathscr{C}} \\
\mathsf{active}(f_{\mathsf{active}}(x_1,\dots,x_n)) & \to & f_{\mathsf{active}}(x_1,\dots,x_n) & \text{for all } f \in \mathscr{F}_{\mathscr{D}} \\
\mathsf{active}(\mathsf{mark}(x)) & \to & \mathsf{mark}(x) &
\end{array}$$

First suppose that $s|_\pi = \mathsf{active}^m(f(l_1,\dots,l_n))\sigma$ and $t = s[\mathsf{active}^{m-1}(\mathsf{mark}(r))\sigma]_\pi$ for some $m \geqslant 1$, position π, substitution σ, and rule $f(l_1,\dots,l_n) \to r \in \mathscr{R}$, such that there is no active symbol directly above the position π in s. Moreover, let the substitutions σ' and σ'' be defined by $\sigma'(x) = \sigma(x){\downarrow_{\mathscr{A}}}$ and $\sigma''(x) = \sigma'(x){\downarrow_{\mathscr{M}}}$ for all variables x. Then we have

$$\begin{array}{rll}
s{\downarrow_{\mathscr{A}}} & = s[\mathsf{active}^m(f(l_1,\dots,l_n))\sigma]_\pi{\downarrow_{\mathscr{A}}} & \\
& = s[f_{\mathsf{active}}(l_1,\dots,l_n)\sigma]_\pi{\downarrow_{\mathscr{A}}} & \\
& = s{\downarrow_{\mathscr{A}}}[f_{\mathsf{active}}(l_1\sigma{\downarrow_{\mathscr{A}}},\dots,l_n\sigma{\downarrow_{\mathscr{A}}})]_{\pi'} & (\mathsf{active} \text{ is not directly above } \pi) \\
& = s{\downarrow_{\mathscr{A}}}[f_{\mathsf{active}}(l_1\sigma',\dots,l_n\sigma')]_{\pi'} & (l_1,\dots,l_n \text{ do not contain } \mathsf{active}) \\
& = s{\downarrow_{\mathscr{A}}}[f_{\mathsf{active}}(l_1,\dots,l_n)\sigma']_{\pi'} &
\end{array}$$

and thus

$$\begin{array}{rl}
s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} = & s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}[f_{\mathsf{active}}(l_1,\dots,l_n)\sigma'']_{\pi''} \\
\to_1 & s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}[\mathsf{mark}(r){\downarrow_{\mathscr{M}}}\sigma'']_{\pi''} \\
\to^!_{\mathscr{M}} & s{\downarrow_{\mathscr{A}}}[\mathsf{mark}(r)\sigma']_{\pi'}{\downarrow_{\mathscr{M}}}
\end{array}$$

Since

$$\begin{array}{rll}
t{\downarrow_{\mathscr{A}}} & = s[\mathsf{active}^{m-1}(\mathsf{mark}(r))\sigma]_\pi{\downarrow_{\mathscr{A}}} & \\
& = s[\mathsf{mark}(r)\sigma]_\pi{\downarrow_{\mathscr{A}}} & \\
& = s{\downarrow_{\mathscr{A}}}[\mathsf{mark}(r)\sigma{\downarrow_{\mathscr{A}}}]_{\pi'} & (\mathsf{active} \text{ is not directly above } \pi) \\
& = s{\downarrow_{\mathscr{A}}}[\mathsf{mark}(r)\sigma']_{\pi'} & (r \text{ does not contain } \mathsf{active})
\end{array}$$

we obtain $t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} = s{\downarrow_{\mathscr{A}}}[\mathsf{mark}(r)\sigma']_{\pi'}{\downarrow_{\mathscr{M}}}$ and thus $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} \to_1^+ t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$.

Next let $s|_\pi = \mathsf{active}^{m-1}(\mathsf{mark}(f(t_1,\dots,t_n)))$ and $t = s[\mathsf{active}^m(f([t_1]^f_1,\dots,[t_n]^f_n))]_\pi$ for some $m \geqslant 1$, position π, terms $t_1,\dots,t_n$, and $f \in \mathscr{F}$, such that there is no active symbol directly above the position π in s. Let $f' = f_{\mathsf{active}}$ if $f \in \mathscr{F}_{\mathscr{D}}$ and $f' = f$ if

$f \in \mathscr{F}_{\mathscr{C}}$. Then we have

$$\begin{aligned}
s{\downarrow_{\mathscr{A}}} &= s[\mathsf{active}^{m-1}(\mathsf{mark}(f(t_1,\ldots,t_n)))]_\pi{\downarrow_{\mathscr{A}}} \\
&= s{\downarrow_{\mathscr{A}}}[\mathsf{mark}(f(t_1{\downarrow_{\mathscr{A}}},\ldots,t_n{\downarrow_{\mathscr{A}}}))]_{\pi'} && (\mathsf{active}\text{ is not directly above }\pi) \\
&\to_{\mathscr{M}} s{\downarrow_{\mathscr{A}}}\big[f'\big([t_1{\downarrow_{\mathscr{A}}}]_1^f,\ldots,[t_n{\downarrow_{\mathscr{A}}}]_n^f\big)\big]_{\pi'} \\
&= s{\downarrow_{\mathscr{A}}}\big[f'\big([t_1]_1^f,\ldots,[t_n]_n^f\big){\downarrow_{\mathscr{A}}}\big]_{\pi'} \\
&= s{\downarrow_{\mathscr{A}}}\big[\mathsf{active}\big(f\big([t_1]_1^f,\ldots,[t_n]_n^f\big)\big){\downarrow_{\mathscr{A}}}\big]_{\pi'} \\
&= s\big[\mathsf{active}^m\big(f\big([t_1]_1^f,\ldots,[t_n]_n^f\big)\big)\big]_\pi{\downarrow_{\mathscr{A}}} && (\mathsf{active}\text{ is not directly above }\pi) \\
&= t{\downarrow_{\mathscr{A}}}
\end{aligned}$$

and hence $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} = t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$.

Finally, let $s|_\pi = \mathsf{active}^m(f(t_1,\ldots,t_n))$ and $t = s[\mathsf{active}^{m-1}(f(t_1,\ldots,t_n))]_\pi$ for some $m \geqslant 1$, position π, some $f \in \mathscr{F} \cup \{\mathsf{mark}\}$, and terms $t_1,\ldots,t_n$, such that there is no active symbol directly above the position π in s. We distinguish three cases. First assume that $f \in \mathscr{F}_{\mathscr{C}} \cup \{\mathsf{mark}\}$. Then we have

$$\begin{aligned}
s{\downarrow_{\mathscr{A}}} &= s[\mathsf{active}^m(f(t_1,\ldots,t_n))]_\pi{\downarrow_{\mathscr{A}}} \\
&= s{\downarrow_{\mathscr{A}}}[f(t_1{\downarrow_{\mathscr{A}}},\ldots,t_n{\downarrow_{\mathscr{A}}})]_{\pi'} && (\mathsf{active}\text{ is not directly above }\pi) \\
&= s[\mathsf{active}^{m-1}(f(t_1,\ldots,t_n))]_\pi{\downarrow_{\mathscr{A}}} && (\mathsf{active}\text{ is not directly above }\pi) \\
&= t{\downarrow_{\mathscr{A}}}
\end{aligned}$$

and thus $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} = t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$. Similarly, if $f \in \mathscr{F}_{\mathscr{D}}$ and $m \geqslant 2$ then

$$\begin{aligned}
s{\downarrow_{\mathscr{A}}} &= s[\mathsf{active}^m(f(t_1,\ldots,t_n))]_\pi{\downarrow_{\mathscr{A}}} &&= s{\downarrow_{\mathscr{A}}}[f_{\mathsf{active}}(t_1{\downarrow_{\mathscr{A}}},\ldots,t_n{\downarrow_{\mathscr{A}}})]_{\pi'} \\
&= s[\mathsf{active}^{m-1}(f(t_1,\ldots,t_n))]_\pi{\downarrow_{\mathscr{A}}} &&= t{\downarrow_{\mathscr{A}}}
\end{aligned}$$

and thus again $s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} = t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}$. Otherwise, we have $f \in \mathscr{F}_{\mathscr{D}}$, $m = 1$, and thus

$$\begin{aligned}
s{\downarrow_{\mathscr{A}}} &= s[\mathsf{active}(f(t_1,\ldots,t_n))]_\pi{\downarrow_{\mathscr{A}}} \\
&= s{\downarrow_{\mathscr{A}}}[f_{\mathsf{active}}(t_1{\downarrow_{\mathscr{A}}},\ldots,t_n{\downarrow_{\mathscr{A}}})]_{\pi'} && (\mathsf{active}\text{ is not directly above }\pi)
\end{aligned}$$

which implies that

$$\begin{aligned}
s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} &= s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}[f_{\mathsf{active}}(t_1{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}},\ldots,t_n{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}})]_{\pi''} \\
&\to_1 s{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}[f(t_1{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}},\ldots,t_n{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}})]_{\pi''} \\
&\to_{\mathscr{M}}^{!} s{\downarrow_{\mathscr{A}}}[f(t_1{\downarrow_{\mathscr{A}}},\ldots,t_n{\downarrow_{\mathscr{A}}})]_{\pi'}{\downarrow_{\mathscr{M}}} \\
&= s[f(t_1,\ldots,t_n)]_\pi{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}} && (\mathsf{active}\text{ is not directly above }\pi) \\
&= t{\downarrow_{\mathscr{A}}}{\downarrow_{\mathscr{M}}}
\end{aligned}$$

The “if” direction can be proved in a similar way, cf. (Giesl & Middeldorp, 2002). Here, one has to show that if $s \to_1 t$ for $s, t \in \mathscr{T}(\mathscr{F}_1)$, then $s{\downarrow_{\mathscr{B}}} \to_{1''}^{+} t{\downarrow_{\mathscr{B}}}$, where $\mathscr{B}$ is the confluent and terminating TRS consisting of the rules

$$f_{\mathsf{active}}(x_1,\ldots,x_n) \to \mathsf{active}(f(x_1,\ldots,x_n))$$

for all $f \in \mathscr{F}_{\mathscr{D}}$. The key observation is that $\mathsf{mark}(u) \to_{1''}^{*} \mathsf{mark}(u){\downarrow_{\mathscr{M}}}{\downarrow_{\mathscr{B}}}$ for all terms $u \in \mathscr{T}(\mathscr{F},\mathscr{V})$. $\square$

D Example 49

Let $(\mathscr{R},\mu)$ be the CSRS of Example 49. Our transformation Θ_1 generates the following TRS $\mathscr{R}^1_\mu$:

$$\begin{array}{rclrcl}
\mathsf{0} -_{\mathsf{active}} y & \to & \mathsf{0} & \mathsf{mark}(\mathsf{0}) & \to & \mathsf{0} \\
\mathsf{s}(x) -_{\mathsf{active}} \mathsf{s}(y) & \to & x -_{\mathsf{active}} y & \mathsf{mark}(\mathsf{s}(x)) & \to & \mathsf{s}(\mathsf{mark}(x)) \\
x \geqslant_{\mathsf{active}} \mathsf{0} & \to & \mathsf{true} & \mathsf{mark}(x - y) & \to & x -_{\mathsf{active}} y \\
\mathsf{0} \geqslant_{\mathsf{active}} \mathsf{s}(y) & \to & \mathsf{false} & \mathsf{mark}(x \geqslant y) & \to & x \geqslant_{\mathsf{active}} y \\
\mathsf{s}(x) \geqslant_{\mathsf{active}} \mathsf{s}(y) & \to & x \geqslant_{\mathsf{active}} y & \mathsf{mark}(x \div y) & \to & \mathsf{mark}(x) \div_{\mathsf{active}} y \\
\mathsf{0} \div_{\mathsf{active}} \mathsf{s}(y) & \to & \mathsf{0} & \mathsf{mark}(\mathsf{if}(x,y,z)) & \to & \mathsf{if}_{\mathsf{active}}(\mathsf{mark}(x),y,z) \\
\mathsf{s}(x) \div_{\mathsf{active}} \mathsf{s}(y) & \to & \multicolumn{4}{l}{\mathsf{if}_{\mathsf{active}}(x \geqslant_{\mathsf{active}} y, \mathsf{s}((x-y) \div \mathsf{s}(y)), \mathsf{0})} \\
\mathsf{if}_{\mathsf{active}}(\mathsf{true},x,y) & \to & \mathsf{mark}(x) & x -_{\mathsf{active}} y & \to & x - y \\
\mathsf{if}_{\mathsf{active}}(\mathsf{false},x,y) & \to & \mathsf{mark}(y) & x \geqslant_{\mathsf{active}} y & \to & x \geqslant y \\
\mathsf{if}_{\mathsf{active}}(x,y,z) & \to & \mathsf{if}(x,y,z) & x \div_{\mathsf{active}} y & \to & x \div y
\end{array}$$

We prove termination with the dependency pair method. There are 13 dependency pairs, where $f^\sharp$ denotes the tuple symbol corresponding to f:

$$\begin{array}{rclrcl}
\mathsf{s}(x) -^{\sharp}_{\mathsf{active}} \mathsf{s}(y) & \to & x -^{\sharp}_{\mathsf{active}} y & \mathsf{mark}^\sharp(\mathsf{s}(x)) & \to & \mathsf{mark}^\sharp(x) \\
\mathsf{s}(x) \geqslant^{\sharp}_{\mathsf{active}} \mathsf{s}(y) & \to & x \geqslant^{\sharp}_{\mathsf{active}} y & \mathsf{mark}^\sharp(x - y) & \to & x -^{\sharp}_{\mathsf{active}} y \\
\mathsf{s}(x) \div^{\sharp}_{\mathsf{active}} \mathsf{s}(y) & \to & \multicolumn{4}{l}{\mathsf{if}^{\sharp}_{\mathsf{active}}(x \geqslant_{\mathsf{active}} y, \mathsf{s}((x-y) \div \mathsf{s}(y)), \mathsf{0})} \\
\mathsf{s}(x) \div^{\sharp}_{\mathsf{active}} \mathsf{s}(y) & \to & x \geqslant^{\sharp}_{\mathsf{active}} y & \mathsf{mark}^\sharp(x \geqslant y) & \to & x \geqslant^{\sharp}_{\mathsf{active}} y \\
\mathsf{if}^{\sharp}_{\mathsf{active}}(\mathsf{true},x,y) & \to & \mathsf{mark}^\sharp(x) & \mathsf{mark}^\sharp(x \div y) & \to & \mathsf{mark}(x) \div^{\sharp}_{\mathsf{active}} y \\
\mathsf{if}^{\sharp}_{\mathsf{active}}(\mathsf{false},x,y) & \to & \mathsf{mark}^\sharp(y) & \mathsf{mark}^\sharp(x \div y) & \to & \mathsf{mark}^\sharp(x) \\
\mathsf{mark}^\sharp(\mathsf{if}(x,y,z)) & \to & \mathsf{if}^{\sharp}_{\mathsf{active}}(\mathsf{mark}(x),y,z) & \mathsf{mark}^\sharp(\mathsf{if}(x,y,z)) & \to & \mathsf{mark}^\sharp(x)
\end{array}$$

Since the pairs $\mathsf{s}(x) \div^{\sharp}_{\mathsf{active}} \mathsf{s}(y) \to x \geqslant^{\sharp}_{\mathsf{active}} y$, $\mathsf{mark}^\sharp(x - y) \to x -^{\sharp}_{\mathsf{active}} y$, and $\mathsf{mark}^\sharp(x \geqslant y) \to x \geqslant^{\sharp}_{\mathsf{active}} y$ are not on cycles of the (estimated) dependency graph, we can ignore them. Moreover, it suffices if dependency pairs of the form $\mathsf{mark}^\sharp(\cdot) \to f(\cdots)$ with $f \neq \mathsf{mark}^\sharp$ are only weakly decreasing (since they do not form a cycle on their own). By using an argument filtering which maps $x - y$, $x -_{\mathsf{active}} y$, $\mathsf{mark}(x)$, and $\mathsf{mark}^\sharp(x)$ to x, the resulting constraints are satisfied by the recursive path order induced by the quasi-precedence $f \sim f_{\mathsf{active}} \sim f^\sharp_{\mathsf{active}}$ for all $f \in \mathscr{F}_{\mathscr{D}} \backslash \{-\}$ and "$\div$" $\succ$ if, "$\geqslant$", s, $\mathsf{0}$ and "$\geqslant$" $\succ$ true, false. Thus, termination of the original CSRS can easily be proved automatically using our transformation Θ_1.

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