Canad. Math. Bull. Vol. **39** (1), 1996 pp. 10–20

A CONTINUITY-LIKE PROPERTY OF DERIVATIVES

P. S. BULLEN AND D. N. SARKHEL

ABSTRACT. In this paper a refinement of property Z of Zahorski-Weil is defined and shown to be, like the weaker property Z, satisfied by all common derivatives.

1. **Introduction.** Throughout this paper f will be a real-valued function defined on a compact interval $I = [a_0, b_0]$, $a_0 < b_0$. If f is continuous, or, even more generally, approximately continuous and bounded, on I then it is known that f is a derivative on I, (one-sided at a_0, b_0.); see([13], (10.7), p. 132). However the converse is false, even for bounded derivatives; Example 2 in Section 3 provides a counterexample, it originates in [14].

Neugebauer, ([11], p. 842), has shown that f is a derivative if and only if it has a certain property (C_2), consisting of a continuity-like property of f along with the assumption of the existence of an associated additive interval function, which is however very close to the assumption of the existence of an anti-derivative of f.

The search for continuity-like properties of derivatives is of much interest, and the following such properties are already known not only for derivatives but also for approximate derivatives, Peano derivatives, L_p-derivatives, ($p \geq 1$), and approximate Peano derivatives, ([1]–[10], [12], [15]–[18]): Baire-1 property, Darboux property, Denjoy or Denjoy-Clarkson property, ($f^{-1}(T)$ has positive measure for every open interval T intersecting $f(I)$), Zahorski property M_3, Zahorski-Weil property Z.

The property Z was introduced by Weil, [17], who showed that for a function having the Darboux and Denjoy properties the property Z implies the Zahorski property M_3. He also showed, by example ([17], p. 529), that property Z is strictly stronger than property M_3. He then proved that derivatives, approximate derivatives, Peano derivatives, L_p-derivatives, ($p \geq 1$) all have the property Z; subsequently Babcock [1] proved the same for the approximate Peano derivatives; Mařík [8] gave a second proof of this result.

In this paper we introduce a similar continuity-like property Z^*, strictly stronger than the property Z, (Section 3, Example 1), and show that the various derivatives mentioned above share this stronger property. We also show that even for bounded derivatives property Z^* cannot be improved in a manner that parallels property Z, (Section 3, Example 2).

Received by the editors January 9, 1995.

AMS subject classification: 26A24.

Key words and phrases: Derivative, approximate derivative, Peano derivative, approximate Peano derivative, property Z.

2. **Notations and definitions.** Let $|E|$ denote the outer Lebesgue measure of a linear set E. If $\lim_{x\to c+} |E\cap[c,x]|/(x-c)$ exists then this limit is called *the right density of E at c. The left density of E at c* is defined similarly. If E has equal right and left densities at c, then the common value is called *the density of E at c.*

Given a point c and a closed interval J we define

$$\rho(c,J) = \max\{|x-c|\,; x\in J\}.$$

Clearly $|J| \leq 2\rho(c,J)$. Hence, if E has density 0 at c then $|E\cap J|/\rho(c,J) \longrightarrow 0$ as $|J|\longrightarrow 0$, where J is a closed interval not necessarily containing c.

If $c_0 \leq c_1 \leq \cdots \leq c_n$ where n is any positive integer, we agree to say that *the closed interval $[c_0,c_n]$ is partitioned into n sub-intervals $[c_{i-1},c_i]$, $1\leq i\leq n$.*

Let k be a positive integer, $c\in I$: if there are real numbers $f_{(1)}(c), f_{(2)}(c),\ldots, f_{(k)}(c)$ and a measurable set $E\subseteq I$ such that $I\setminus E$ has density 0 at c and such that

$$\frac{[f(x)-\sum_{r=0}^{k} f_{(r)}(c)(x-c)^r/r!]}{(x-c)^k} \longrightarrow 0$$

as $x\longrightarrow c$ over E, where $f_{(0)}(c) = f(c)$, then $f_{(k)}(c)$ is called *the k-th approximate Peano derivative of f at c*; ([4], [5]). We remark that, without loss of generality, the set E here may be assumed to be closed.

As mentioned at the end of [17] a slightly stronger version of property Z, ([17], p. 528; there is a misprint there, $\leq\epsilon$ being written instead of $\geq\epsilon$), can be restated in the following convenient form.

The function f is said to have the property Z on I if for every $c\in I$ and $\epsilon>0$, $\eta>0$ there is a neighbourhood I_c of c in I such that the following conditions Z_+, and Z_- hold.

Z_+ : if $f(x)\geq f(c)$ almost everywhere on a closed interval $J\subset I_c$ then

$$|A|\leq \eta\rho(c,J) \qquad \text{where } A=\{x\in J; f(x)\geq f(c)+\epsilon\}.$$

Z_- : if $f(x)\leq f(c)$ almost everywhere on a closed interval $J\subset I_c$ then

$$|A|\leq \eta\rho(c,J) \qquad \text{where } A=\{x\in J; f(x)\leq f(c)-\epsilon\}.$$

3. **Property Z^*.** We strengthen the above property Z as follows.

DEFINITION. *The function f is said to have the property Z^* on I if* for every $c\in I$ and $\epsilon>0$, $\eta>0$ there is a neighbourhood I_c of c in I such that the following conditions Z^+, and Z^- hold.

Z^+ : if $f(x)\geq f(c)-\epsilon$ almost everywhere on a closed interval $J\subset I_c$ then

$$|A|-|B|\leq\eta\rho(c,J),$$

where $A=\{x\in J; f(x)\geq f(c)+\epsilon\}$, $\quad B=\{x\in J; f(c)-\epsilon\leq f(x)<f(c)\}$.

Z^- : if $f(x)\leq f(c)+\epsilon$ almost everywhere on a closed interval $J\subset I_c$ then

$$|A|-|B|\leq\eta\rho(c,J),$$

https://doi.org/10.4153/CMB-1996-002-6 Published online by Cambridge University Press

where $A = \{x \in J ; f(x) \leq f(c) - \epsilon\}$, $\quad B = \{x \in J ; f(c) < f(x) \leq f(c) + \epsilon\}$.

We observe that f satisfies Z^+ if and only if $-f$ satisfies Z^-, and hence if f satisfies Z^* then so does $-f$. Similar remarks hold for Z_+, Z_- and Z. It is also clear that Z^* implies Z since Z_+ follows from Z^+, and Z_- from Z^-.

It is worth noting that property Z^+ implies that

either: the set $\{x \in I ; f(c) - \epsilon \leq f(x) < f(c)\}$ has positive upper density at c

or: we can write $|A| \leq \eta\rho(c, J)$.

There is a similar comment in the case of property Z^-. Another simple deduction is that since for measurable f, $|B| \leq |J| - |A|$ we can write, in both cases

$$|A| \leq \frac{|J| + \eta\rho(c, J)}{2}.$$

We now present two examples; the first one showing that the property Z^* is strictly stronger than the property Z; the second one showing that even for certain nice bounded derivatives we cannot, in property Z^*, replace $|A| - |B|$ by $|A|$, as in property Z.

EXAMPLE 1. Fix $a < b$ and a positive integer k and define for $n = 1, 2, \ldots$,

$$a_n = a + (b - a)\frac{(k+1)^2}{(k+2)}\Big(\frac{1}{k+n} + \frac{1}{(k+n+1)^2}\Big),$$
$$b_n = a + (b - a)\frac{(k+1)^2}{(k+2)}\Big(\frac{1}{k+n} + \frac{1}{(k+n)^2}\Big).$$

Then for all n, $a < a_{n+1} < b_{n+1} < a_n < b_1 = b$, and $a_n, b_n \longrightarrow a$ as $n \longrightarrow \infty$. In addition, for all n,

$$(b_n - a_{n+1}) < 3\frac{(a_{n+1} - a)}{(k+n)}; \qquad |E| \geq \frac{2}{3}(b - a), \text{ where } E = \cup_{n=1}^{\infty}[b_{n+1}, a_n].$$

Let $c_n = (a_n + b_n)/2$, $n = 1, 2, \ldots$ and first we define a function g as follows:

$$g(a) = g(b) = 0; \qquad g(c_n) = -1/kn, n = 1, 2, \ldots; \qquad g(x) = 1, x \in E.$$

Finally let g be linear on all of the intervals $[a_n, c_n]$, $[c_n, b_n]$. Then g is continuous on $]a, b]$ and lower semi-continuous at a.

Let $a \leq p < q \leq b$; if either $g(x) \geq g(a) = 0$ almost everywhere on $[p, q]$, or, $g(x) \leq g(a) = 0$ almost everywhere on $[p, q]$, then obviously for some n, $a_{n+1} \leq p < q \leq b_n$. Then

$$q - p \leq b_n - a_{n+1} < 3\frac{(a_{n+1} - a)}{(k+n)} < 3\frac{(q-a)}{(k+n)} < 3\frac{(q-a)}{k}.$$

Since $3/(k+n) \longrightarrow 0$ as $n \longrightarrow \infty$, we see that g satisfies property Z in a strong manner at a. Elsewhere g is continuous and so has the property Z on $[a, b]$. In addition we note that g has the properties of Baire-1, Darboux and Denjoy on $[a, b]$.

In particular note that:

(i) $g(a) = g(b) = 0$ and $-1/k \leq g(x) \leq 1$ on $[a, b]$;

(ii) g is continuous on $]a, b]$ and lower semi-continuous at a;

https://doi.org/10.4153/CMB-1996-002-6 Published online by Cambridge University Press

(iii) $|\{x \in [a,b]; g(x) = 1\}| = |E| \geq (2/3)(b-a)$;

(iv) if $a \leq p < q \leq b$ and if either $g(x) \geq 0$ almost everywhere on $[p,q]$, or $g(x) \leq 0$ almost everywhere on $[p,q]$, then $a < p$ and $q - p < 3(q-a)/k$;

(v) g has the property Z on $[a,b]$.

Let us say that such a function g *is of type* (1_k) *on* $[a,b]$.

Now, let $t_1 = b$ and $t_{k+1} = (a+t_k)/2$, and let g_k be a function of type (1_k) on $[t_{k+1}, t_k]$, $k = 1, 2, \ldots$.

We define a function f on $[a,b]$ as follows: $f(a) = 0$, and $f(x) = (-1)^k g_k(x)$ for all x in $[t_{k+1}, t_k]$, $k = 1, 2, \ldots$. The f is bounded on $[a,b]$ and continuous except at the points $a, t_2, t_3, \ldots$. At the points $t_2, t_3, \ldots f$ is continuous on the left and semi-continuous on the right. In addition f has the Baire-1, Darboux and Denjoy properties on $[a,b]$.

Since for each k both of the functions $\pm g_k$ have the property Z on $[t_{k+1}, t_k]$ it follows that f has the property Z at all points of $]a,b]$.

We now show that f also has the property Z at a. Let $a \leq p < q \leq b$ and suppose that either $f(x) \geq f(a) = 0$ almost everywhere on $[p,q]$, or $f(x) \leq f(a) = 0$ almost everywhere on $[p,q]$. Then, obviously, for some k, $t_{k+1} < p < q \leq t_k$. Since g_k is of type (1_k) it follows from (iv) that $(q-p) < 3(q-t_{k+1})/k < 3(q-a)/k$, and so we have that f satisfies property Z at a. Hence f satisfies property Z on $[a,b]$.

However, we now show f does not satisfy property Z^* on $[a,b]$ by showing that f does not satisfy properties Z^+, Z^- at a for any ϵ, $0 < \epsilon < 1$.

Consider any $[a,d]$, $a < d \leq b$ then we can find k, both even and odd, such that $k > 1/\epsilon$, and $a < t_{k+1} < t_k < d$.

Suppose first that k is even. Then $f(x) = g_k(x)$ on $[t_{k+1}, t_k]$ and so by (i), $f(x) \geq -1/k > -\epsilon = f(a) - \epsilon$ on $[t_{k+1}, t_k]$. Further, by (iii), if $A = \{x \in [t_{k+1}, t_k]; f(x) \geq f(a) + \epsilon = \epsilon\}$ then $|A| \geq (2/3)(t_k - t_{k+1})$. So if $B = \{x \in [t_{k+1}, t_k]; f(a) - \epsilon \leq f(x) < f(a)\}$ then $|B| \leq (1/3)(t_k - t_{k+1})$. Since $t_k - t_{k+1} = (t_k - a)/2$ we get that $|A| - |B| \geq (1/6)(t_k - a)$; and so f does not satisfy property Z^+ at a.

If now k is odd then $-f(x) = g_k(x)$ on $[t_{k+1}, t_k]$ and so by the above argument $-f$ does not satisfy property Z^+ at a, or equivalently, f does not satisfy property Z^- at a.

Thus f is a "nice" function having the property Z but failing to have property Z^* in a "bad" way.

EXAMPLE 2. If $0 < t < 1$ and m is a positive integer it is a simple exercise to verify that the set

$$E_0 = \bigcup_{n=1}^{\infty} \left[\frac{m}{m+n-1} - t\left(\frac{m}{m+n-1} - \frac{m}{m+n} \right), \frac{m}{m+n-1} \right],$$

has upper bound 1, lower bound 0, and right density t at 0; further $|E_0 \cap [0,x]| / |[0,x]|$ lies between $t - 1/m$ and $t + 1/m$ for all x in $]0,1]$.

Using these facts, given $a < b$ and positive integers m and k, we easily find a sequence of intervals $\{[a_n, b_n], n = 1, 2\ldots\}$ converging to a, such that

for all n $\quad a < a_{n+1} < b_{n+1} < a_n < b_1 = b$;
the set $\quad E = \cup_{n=1}^{\infty}[b_{2n}, a_{2n-1}]$ has right density $k/(2k+1)$ at a;
the set $\quad N = \cup_{n=1}^{\infty}[a_n, b_n]$ has density 0 at a;
and for all $x \in]a, b]$ we have

$$h(x) = \left|\frac{|E \cap [a,x]|}{|[a,x]|} - \frac{k}{2k+1}\right| + \frac{|N \cap [a,x]|}{|[a,x]|} < \frac{1}{12m}.$$

Then $\lim_{x \to a+} h(x) = 0$; and $h(b) < 1/12$ gives $|E|/(b-a) > k/(2k+1) - 1/12 \geq 1/4$.

We first define a function g on $[a, b]$ as follows:

$$g(a) = g(b) = 0,$$
$$g(x) = 1/k, x \in E,$$
$$g(x) = -1/(k+1), x \in F = \cup_{n=1}^{\infty}[b_{2n+1}, a_{2n}];$$

finally let g be linear on all of the intervals $[a_n, b_n]$. Then g is bounded, and continuous on $]a, b]$. Also we clearly have that, for all $x \in [a, b]$,

$$G(x) = \int_a^x g = \frac{1}{k}|E \cap [a,x]| - \frac{1}{k+1}|F \cap [a,x]| + \int_{N \cap [a,x]} g.$$

Since $|F \cap [a,x]| = |[a,x]| - |E \cap [a,x]| - |N \cap [a,x]|$, and $|g(x)| \leq 1/k$, we have that for all $x \in]a, b]$

$$\frac{|G(x)|}{|[a,x]|} \leq \frac{2k+1}{k(k+1)}h(x) < \frac{2k+1}{k(k+1)}\left(\frac{1}{12m}\right) < \frac{1}{m}.$$

Since $h(x) \longrightarrow 0$ as $x \longrightarrow a+$, it follows that $G'(a)$ exists with value $0 = g(a)$. Also by the continuity of g, $G'(x) = g(x)$ if $x \in]a, b]$. Thus g is a bounded derivative on $[a, b]$ that is continuous on $]a, b]$, but is not even approximately continuous at a since $g(x) = 1/k$ on E. In particular note that:

(i) g is a derivative on $[a, b]$;
(ii) $g(a) = g(b) = 0$ and $-1/(k+1) \leq g(x) \leq 1/k$ on $[a, b]$;
(iii) $|\{x \in [a, b]; g(x) = 1/k\}| = |E| > (1/4)(b-a)$;
(iv) $|\int_a^x g| \leq (1/m)|[a, x]|$, for all $x \in [a, b]$.

Let us call such a function g *a function of type* $(m; k)$ *on* $[a, b]$.

Now let $t_1 = b$, $t_{n+1} = (a + t_n)/2$, $n = 1, 2, \ldots$; and for each n let g_n be a function of type $(n; k_n)$ on $[t_{n+1}, t_n]$, where $k_n = (n+1-3^{i-1})/2$ if n is even but $k_n = (n+2-3^{i-1})/2$ if n is odd, i being the unique positive integer such that $3^{i-1} \leq n < 3^i$.

We define the function f on $[a, b]$ as follows: $f(a) = 0$, and $f(x) = (-1)^n g_n(x)$ if $x \in [t_{n+1}, t_n]$, $n = 1, 2, \ldots$. Since $g_n(t_n) = g_n(t_{n+1}) = 0$ for all n, this is a well-defined

https://doi.org/10.4153/CMB-1996-002-6 Published online by Cambridge University Press

function. From the properties of functions of $(n;k_n)$ type we see that $|f(x)| \leq 1$ for all $x \in [a,b]$ and that if $x \in [t_{n+1}, t_n]$ then

$$\left|\int_a^x f\right| \leq \left|\int_{t_{n+1}}^x g_n\right| + \sum_{m>n}\left|\int_{t_{m+1}}^{t_m} g_m\right|$$
$$\leq \frac{1}{n}|[t_{n+1}, x]| + \sum_{m>n}\frac{1}{m}|[t_{m+1}, t_m]| < \frac{1}{n}|[a,x]|.$$

Hence it follows that $\int_a^x f$ has a derivative $0 = f(a)$ at a, and so from the properties of the functions g_n, f is the derivative of $\int_a^x f$ everywhere on $[a,b]$.

However, let $0 < \epsilon < 1$, and k the integer such that $k < 1/\epsilon \leq k+1$, and consider any $[a,d]$, $a < d \leq b$. Fix an integer i such that $3^{i-1} > 2k$ and $t_n < d$ for all $n \geq 3^{i-1}$.

Then $n = 2k - 1 + 3^{i-1}$ is even and $3^{i-1} < n < 3^i$; hence $k_n = k = (n + 1 - 3^{i-1})/2$. Then $[t_{n+1}, t_n] \subset]a,d[$ and by property (ii) of g_n $f(x) = g_n(x) \geq -1/(k_n+1) = k - 1/(k+1) \geq -\epsilon = f(a) - \epsilon$, for all $x \in [t_{n+1}, t_n]$. Further since $1/k_n = 1/k > \epsilon$ property (iii) of g_n gives that $|A| > \frac{1}{4}(t_n - t_{n+1}) = \frac{1}{8}(t_n - a)$, where $A = \{x \in [t_{n+1}, t_n]; f(x) = g_n(x) \geq \epsilon = f(a) + \epsilon\}$.

Again $n = 2k - 2 + 3^{i-1}$ is odd and $3^{i-1} \leq n < 3^i$; hence $k_n = k = (n+2-3^{i-1})/2$. Then $[t_{n+1}, t_n] \subset]a,d[$ and now $f(x) = -g_n(x) \leq 1/(k_n+1) = 1/(k+1) \leq \epsilon = f(a) + \epsilon$, for all $x \in [t_{n+1}, t_n]$. As before $|A| > \frac{1}{8}(t_n - a)$, where $A = \{x \in [t_{n+1}, t_n]; f(x) = -g_n(x) \leq -\epsilon = f(a) - \epsilon\}$.

Thus we see that, even for such a "nice" derivative as f, for the point a and any ϵ, $0 < \epsilon < 1$, neither in property Z^+, nor in property Z^- can we replace $|A| - |B|$ by $|A|$ alone, as was done in the properties Z_+, Z_-.

4. **The Main Result.** We now turn attention to our main result.

THEOREM. *If f has a k-th approximate Peano derivative $f_{(k)}$ everywhere on I, then $f_{(k)}$ has the property Z^* on I.*

We first prove a lemma which extends the lemmas in ([17], p. 532; [1], p. 291); in addition our method of proof is simpler and shorter.

LEMMA. *Hypotheses: $a < b$; $J = [a,b]$; $\epsilon > 0$; the function g has a finite k-th derivative $g^{(k)} \geq -\epsilon$ throughout J;*

$$A = \{x \in J; g^{(k)}(x) \geq \epsilon\}, \qquad B = \{x \in J; -\epsilon \leq g^{(k)}(x) < 0\};$$
$$F(t) = |A \cap [a,t]| - |B \cap [a,t]| \ m \text{ for } a \leq t \leq b.$$

Conclusion: there is a $[p,q] \subseteq J$ with $F(q) - F(p) \leq 0$ such that each of $[a,p]$ and $[q,b]$ can be partitioned into 2^{k-1} sub-intervals on each of which each of the functions $g^{(k-1)}$, $g^{(k-2)}, \ldots, g^{(0)} = g$ is of constant sign; and for every further sub-interval $[x,y]$

$$|g(y) - g(x)|^{1/k} \geq \left(\frac{\epsilon}{k!}\right)^{1/k}\big(F(y) - F(x)\big). \tag{1}$$

https://doi.org/10.4153/CMB-1996-002-6 Published online by Cambridge University Press

NOTE. For all $[\alpha,\beta] \subseteq J$ we have that $F(\beta) - F(\alpha) = |A \cap [\alpha,\beta]| - |B \cap [\alpha.\beta]|$.

PROOF. If $g^{(k-1)}$ has constant sign on J then we take any $p = q \in J$. If $g^{(k-1)}$ changes sign on J, then, being continuous, it vanishes somewhere on J and we take

$$p = \inf\{x \in J; g^{(k-1)}(x) = 0\}, \qquad q = \sup\{x \in J; g^{(k-1)}(x) = 0\},$$

Then, by continuity, $g^{(k-1)}(p) = g^{(k-1)}(q) = 0$, and $g^{(k-1)}$ is of constant sign on each of $[a,p]$, $[q,b]$. Since $g^{(k)} \geq 0$ on $J \setminus (A \cup B)$ we have, for all $[\alpha,\beta] \subseteq J$,

$$g^{(k-1)}(\beta) - g^{(k-1)}(\alpha) = \int_\alpha^\beta g^{(k)} \geq \epsilon[F(\beta) - F(\alpha)]. \tag{2}$$

So in both cases $F(q) - F(p) \leq 0$. Further in the case $k = 1$ (2) implies (1), and this case is proved.

Suppose that $k \geq 2$; then $g^{(k-2)}$ is monotonic and continuous on each of $[a,p]$, $[q,b]$, and vanishes at some point of each of these intervals, unless it is of constant sign on that interval. So we can partition each of $[a,p]$, $[q,b]$ into two sub-intervals on each of which $g^{(k-2)}$ is of constant sign. If $k \geq 3$, the argument applies to $g^{(k-3)}$ on each of these sub-intervals. Proceeding in this manner, we obtain after $k-1$ steps a partitioning of each of $[a,p]$, $[q,b]$ into 2^{k-1} sub-intervals, on each of which the functions $g^{(k-1)}, g^{(k-2)}, \dots, g^{(0)}$ are of constant sign.

We now verify (1) by proving, by an induction on k, that the following holds:

(*) if $g^{(k)} \geq -\epsilon$ and each of $g^{(k-1)}$, $g^{(k-2)}, \dots, g^{(0)}$ is of constant sign on some $[c,d] \subseteq J$, then (1) holds for every $[x,y] \subseteq [c,d]$.

To this end fix $[x,y] \subseteq [c,d]$ and put

$$u = \sup\{t \in [x,y]; F(t) - F(x) \leq 0\}, \qquad v = \inf\{t \in [u,y]; F(y) - F(t) \leq 0\}.$$

Since F is continuous, we have

$$F(u) - F(x) \leq 0 \text{ and } F(y) - F(v) \leq 0, \tag{3}$$

and so

$$F(y) - F(x) \leq F(v) - F(u).$$

Further since g' is of constant sign on $[c,d]$, g is monotonic there and so

$$|g(y) - g(x)| \geq |g(v) - g(u)|.$$

It follows that to prove (*) it is sufficient to show that

$$|g(v) - g(u)| \geq \frac{\epsilon}{k!}[F(v) - F(u)]^k. \tag{4}$$

Now write $G(t) = F(t) - F(u)$ and $H(t) = F(v) - F(t)$. From the definition of u, v and by (3) we have for all $t \in [u,v]$,

$$G(t) = [F(t) - F(x)] + [F(x) - F(u)] \geq 0$$

https://doi.org/10.4153/CMB-1996-002-6 Published online by Cambridge University Press

and

$$H(t) = [F(v) - F(y)] + [F(y) - F(t)] \geq 0.$$

In addition, F is absolutely continuous, and since for all $\alpha \leq \beta$, $F(\beta) - F(\alpha) \leq \beta - \alpha$ we have that $F'(t) \leq 1$ almost everywhere on $[u, v]$. So, for every positive integer n, and almost everywhere on $[u, v]$ we have

$$\frac{d}{dt}\left[\frac{1}{n+1}G^{n+1}(t)\right] = G^n(t)G'(t) = G^n(t)F'(t) \leq G^n(t),$$
$$\frac{d}{dt}\left[\frac{-1}{n+1}H^{n+1}(t)\right] = -H^n(t)H'(t) = H^n(t)F'(t) \leq H^n(t).$$

Consequently we have

(5) $$\int_u^v G^n \geq \frac{1}{n+1}[G^{n+1}(v) - G^{n+1}(u)] = \frac{1}{n+1}[F(v) - F(u)]^{n+1},$$

(6) $$\int_u^v H^n \geq \frac{-1}{n+1}[H^{n+1}(v) - H^{n+1}(u)] = \frac{1}{n+1}[F(v) - F(u)]^{n+1}.$$

Now let $k = 2$. First suppose that $g^{(k-1)} = g'$ is non-negative on $[c, d]$. Then from (2) with $\alpha = u$ and $\beta = t$, $u \leq t \leq v$, we get

$$g'(t) \geq \epsilon[F(t) - F(u)] + g'(u) \geq \epsilon G(t) \geq 0.$$

So, by (5),

$$g(v) - g(u) = \int_u^v g' \geq \epsilon \int_u^v G \geq \frac{\epsilon}{2}[F(v) - F(u)]^2.$$

Next suppose that g' is non-positive on $[c, d]$; then from (2) with $\beta = v$ and $\alpha = t$, $u \leq t \leq v$, we get

$$-g'(t) \geq \epsilon[F(v) - F(t)] - g'(v) \geq \epsilon H(t) \geq 0.$$

So, by (6),

$$g(u) - g(v) = -\int_u^v g' \geq \epsilon \int_u^v H \geq \frac{\epsilon}{2}[F(v) - F(u)]^2.$$

Thus we have (4) and (*) is proved in the case $k = 2$.

Next let $k \geq 3$ and assume (*) for derivatives of order $k - 1$. Writing $h = g'$ we easily see that $h^{(k-1)} \geq -\epsilon$ and each of $h^{(k-2)}, h^{(k-3)}, \ldots, h^{(0)}$ is of constant sign on $[c, d]$, and moreover

$$A = \{x \in J; h^{(k-1)}(x) \geq \epsilon\}, \qquad B = \{x \in J; -\epsilon \leq h^{(k-1)}(x) < 0\}.$$

So by the induction hypothesis (1) holds with $h = g'$ in place of g. So for all $t \in [u, v]$ we have

(7) $$|g'(t) - g'(u)|^{1/(k-1)} \geq \left(\frac{\epsilon}{(k-1)!}\right)^{1/(k-1)}[F(t) - F(u)],$$

(8) $$|g'(v) - g'(t)|^{1/(k-1)} \geq \left(\frac{\epsilon}{(k-1)!}\right)^{1/(k-1)}[F(v) - F(t)].$$

Since $g^{(2)}$ has constant sign on $[c,d]$, g' is monotone on $[c,d]$.

There are four cases to be considered.

(I) Suppose g' is monotone increasing and non-negative on $[c,d]$. Then, by (7), for all $t \in [u,v]$ we have

$$g'(t) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)G^{k-1}(t) + g'(u) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)G^{k-1}(t).$$

So, by (5),

$$g(v) - g(u) = \int_u^v g' \geq \Big(\frac{\epsilon}{(k-1)!}\Big)\int_u^v G^{k-1} \geq \Big(\frac{\epsilon}{k!}\Big)[F(v) - F(u)]^k.$$

(II) Suppose g' is monotone decreasing and non-positive on $[c,d]$. Then, by (7), for all $t \in [u,v]$ we have

$$-g'(t) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)G^{k-1}(t) - g'(u) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)G^{k-1}(t).$$

So as above, by (5),

$$-\big(g(v) - g(u)\big) \geq \Big(\frac{\epsilon}{k!}\Big)[F(v) - F(u)]^k.$$

(III) Suppose g' is monotone decreasing and non-negative on $[c,d]$. Then, by (8), for all $t \in [u,v]$ we have

$$g'(t) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)H^{k-1}(t) + g'(v) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)H^{k-1}(t).$$

So, by (6),

$$g(v) - g(u) = \int_u^v g' \geq \Big(\frac{\epsilon}{(k-1)!}\Big)\int_u^v H^{k-1} \geq \Big(\frac{\epsilon}{k!}\Big)[F(v) - F(u)]^k.$$

(IV) Finally suppose g' is monotone increasing and non-positive on $[c,d]$. Then, by (8), for all $t \in [u,v]$ we have

$$-g'(t) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)H^{k-1}(t) - g'(v) \geq \Big(\frac{\epsilon}{(k-1)!}\Big)H^{k-1}(t).$$

So as above, by (6),

$$-\big(g(v) - g(u)\big) \geq \Big(\frac{\epsilon}{k!}\Big)[F(v) - F(u)]^k.$$

So in all cases we have (4), proving (*) for all k.

This complete the induction and the proof of the Lemma. ∎

PROOF OF THE THEOREM. Fix $c \in I$ and define g on I by

$$g(x) = f(x) - \sum_{r=0}^{k} \frac{(x-c)^r}{r!} f_{(r)}(c).$$

By the definition of $f_{(k)}(c)$ there is a closed set $E \subseteq I$ such that, given $\epsilon > 0$ and $\eta > 0$ there is a neighbourhood I_c of c such that

$$|g(x)| \leq \left(\frac{\epsilon}{k!}\right)\left(\frac{\eta}{2^{k+2}}\right)^k |x - c|^k, \tag{9}$$

for all x in $E \cap I_c$; further, for all $[x, y] \subset I_c$,

$$|[x, y] \setminus E| \leq \frac{\eta}{2^{k+1}} \rho(c, [x, y]). \tag{10}$$

We ignore the trivial case of $a = b$ and suppose that $f_{(k)}(x) \geq f_{(k)}(c) - \epsilon$ almost everywhere on an interval $[a, b] \subset I_c$. Since $f_{(k)}$ has the Denjoy property, ([1], Corollary 5.1, p. 291), $f^{(k)}$ exists and equals $f_{(k)}$ on $[a, b]$, ([1], Theorem 4.1, p. 283). So we have

$$g^{(k)}(x) = f^{(k)}(x) - f^{(k)}(c) = f_{(k)}(x) - f_{(k)}(c) \geq -\epsilon \text{ on } [a, b].$$

Let

$$A = \{x \in [a, b]; f_{(k)}(x) - f_{(k)}(c) = g^{(k)}(x) \geq \epsilon\},$$
$$B = \{x \in [a, b]; -\epsilon \leq f_{(k)}(x) - f_{(k)}(c) = g^{(k)}(x) < 0\}.$$

By the Lemma there is a $[p, q] \subseteq [a, b]$ with

$$|A \cap [p, q]| - |B \cap [p, q]| \leq 0,$$

and such that each of the intervals $[a, p]$, $[q, b]$ can be partitioned into 2^{k-1} sub-intervals such that, if J is any of these 2^k closed sub-intervals the inequality (11) of the Lemma holds for every $[x, y] \subseteq J$.

If $E \cap J \neq \emptyset$, then taking $[x, y]$ to be the largest interval with end points on the closed set $E \cap J$, we have by (1) and (9),

$$\begin{aligned} |A \cap J| - |B \cap J| &= |A \cap [x, y]| - |B \cap [x, y]| + |A \cap J \setminus [x, y]| - |B \cap J \setminus [x, y]| \\ &\leq \left(\frac{k!}{\epsilon}\right)^{1/k} |g(y) - g(x)|^{1/k} + |J \setminus [x, y]| \\ &\leq \left(\frac{k!}{\epsilon}\right)^{1/k} [|g(y)|^{1/k} + |g(x)|^{1/k}] + |J \setminus E| \\ &\leq \left(\frac{\eta}{2^{k+2}}\right)[|y - c| + |x - c|] + |J \setminus E| \\ &\leq \left(\frac{\eta}{2^{k+1}}\right)\rho(c, J) + |J \setminus E|. \end{aligned}$$

On the other hand if $E \cap J = \emptyset$ then $|A \cap J| - |B \cap J| \leq |J| = |J \setminus E|$. Hence in both cases, using (10) and writing $J_0 = [a, b]$ we have

$$|A \cap J| - |B \cap J| \leq \left(\frac{\eta}{2^{k+1}}\right)\rho(c, J) + \left(\frac{\eta}{2^{k+1}}\right)\rho(c, J) \leq \left(\frac{\eta}{2^k}\right)\rho(c, J_0).$$

Then, recalling (11), and summing over the 2^k intervals J,

$$\begin{aligned} |A| - |B| &\leq |A \cap ([a, p] \cup [q, b])| - |B \cap ([a, p] \cup [q, b])| \\ &= \sum_J (|A \cap J| - |B \cap J|) \leq 2^k \left(\frac{\eta}{2^k}\right)\rho(c, J_0) = \eta\rho(c, J_0). \end{aligned}$$

Thus $f_{(k)}$ has property Z^+ on I. So $-f_{(k)} = (-f)_{(k)}$ also has the property Z^+ on I; in other words $f_{(k)}$ has property Z^- on I, and so property Z^*.

This completes the proof of the Theorem. ∎

REMARK. An ordinary derivative is an approximate derivative, which is the first approximate Peano derivative; a k-th Peano derivative is a k-th approximate Peano derivative, and a k-th L_p-derivative, $p \geq 1$, is a $(k+1)$-th Peano derivative, ([5], Theorem 1, p. 382); so every such derivative has property Z^*.

REFERENCES

1. B. S. Babcock, *On properties of the approximate Peano derivatives*, Trans. Amer. Math. Soc. **212**(1975), 279–294.

2. J. A. Clarkson, *A property of derivatives*, Bull. Amer. Math. Soc. **53**(1947), 124–125.

3. A. Denjoy, *Sur une propriété des fonctions dérivées exactes*, Enseignement Math. **18**(1916), 320–328.

4. A. Denjoy, *Sur l'intégration des coefficients différentiels d'ordre supérieur*, Fund. Math. **25**(1935), 273–326.

5. M. J. Evans, *L_p derivatives and approximate Peano derivatives*, Trans. Amer. Math. Soc. **165**(1972), 381–388.

6. C. Goffman and C. J. Neugebauer, *On approximate derivatives*, Proc. Amer. Math. Soc. **11**(1960), 962–966.

7. A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. **9**(1927), 217–279.

8. J. Mařík, *On generalized derivatives*, Real Anal. Exchange **3**(1977–1978), 87–92.

9. S. Marcus, *On a theorem of Denjoy and on approximate derivative*, Monatsh. Math. **66**(1962), 435–440.

10. S. N. Mukhopadhyay, *On the approximate Peano derivatives*, Fund. Math. **88**(1975), 133–143.

11. C. J. Neugebauer, *Darboux functions of Baire class one and derivatives*, Proc. Amer. Math. Soc. **13**(1962), 838–843.

12. H. W. Oliver, *The exact Peano derivatives*, Trans. Amer. Math. Soc. **76**(1954), 444–456.

13. S. Saks, *Theory of the Integral*, Dover, New York, 1964.

14. D. N. Sarkhel, *The pointwise characterization of derivatives of integrals*, Proc. Amer. Math. Soc. **63**(1977), 125–128.

15. G. P. Tolstoff, *Sur la dérivée approximative exacte*, Rec. Math. (Mat. Sb.), (N.S.) **4**(1938), 499–504.

16. C. E. Weil, *On properties of derivatives*, Trans. Amer. Math. Soc. **114**(1965), 363–376.

17. ———, *A property for certain derivatives*, Indiana Univ. Math. J. **23**(1973/74), 527–536.

18. Z. Zahorski, *Sur la prémière dérivée*, Trans. Amer. Math. Soc. **69**(1950), 1–54.

Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Z2
e-mail: bullen@math.ubc.ca

Department of Mathematics
University of Kalyani
Kalyani, W.B.
India 741235